# ASYMPTOTIC FORMULAS IN DISCRETE TIME RISK MODEL WITH DEPENDENCE OF FINANCIAL AND INSURANCE RISKS

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## ABSTRACT

Asymptotic formulas for a ruin probability in discrete time risk model with a dependence of financial and insurance risks are obtained. These formulas are constructed in a suggestion which is adequate to economical crisis: the larger is a financial risk the larger is an insurance risk.

## 1. INTRODUCTION

In this paper we obtain asymptotic formulas for a ruin probability in discrete time risk model with a dependence of financial and insurance risks. Earlier simple asymptotic formulas for the ruin probability in a case of independent financial and insurance risks have been obtained in [1]. More complicated cases with special restrictions on insurance risks dependence are considered for example in [2,3]. Nevertheless until recently an asymptotic analysis of risk model with a dependence of insurance and financial risks is not made. But in modern period of strong economical crisis such dependence may be recognized easily in different large anthropogenic catastrophes. So a problem to analyze asymptotically this dependence is actual now.

In this paper we consider special model of insurance and financial risks dependence based on suggestion that a financial risk has a finite number of meanings and for each meaning an insurance risk has its own distribution. Then Pareto-tailed and Weibull-tailed asymptotics of insurance risks distributions are considered. In frames of this suggestion we assume that the larger is the financial risk the larger is the insurance risk. This stochastic modeling approach allows to obtain new asymptotic formulas for ruin probability in risk models.

## 2 PRELIMINARIES

**Classes of distributions.** Throughout, for a given random variable (r.v.) $X$ concentrated on $(-\infty,\infty)$ with a distribution function (d.f.) $F$ then its right tail $\overline{F}(x)=P(X>x)$. For two d.f.'s $F_1$ and $F_2$ concentrated on $(-\infty,\infty)$ we write by $F_1 * F_2(x)$ the convolution of $F_1$ and $F_2$ and write by $F_1^{*2}=F_1*F_1$ the convolution of $F_1$ with itself. All limiting relationships, unless otherwise stated, are for $x\to\infty$. Let $a(x)\ge 0$ and $b(x)>0$ be two infinitesimals, satisfying

$$l^- \le \liminf_{x\to\infty}\frac{a(x)}{b(x)} \le \limsup_{x\to\infty}\frac{a(x)}{b(x)} \le l^+ .$$

We write $a(x)=O(b(x))$, if $l^+<\infty$ and $a(x)\sim b(x)$ if $l^+=l^-=1$.

Introduce the following classes of d.f.`s concentrated on $[0,\infty)$:

$$\mathcal{S}=\left\{F(x): \lim_{x\to\infty}\frac{\overline{F^{*2}}(x)}{\overline{F}(x)}=2\right\},\ \mathcal{L}=\left\{F(x): \forall t \lim_{x\to\infty}\frac{\overline{F}(x-t)}{\overline{F}(x)}=1\right\},$$

$\mathcal{R}_{-\alpha}=\left\{F(x): \forall\theta>0 \lim_{x\to\infty}\frac{\overline{F}(\theta x)}{\overline{F}(x)}=\theta^{-\alpha}\right\}$, $0<\alpha<\infty$, $\mathcal{R}=\bigcup_{0<\alpha<\infty}\mathcal{R}_{-\alpha}$. $\mathcal{S}$ is called the class of subexponential d.f.`s. $\mathcal{L}$ is called the class of long tailed d.f.`s. $\mathcal{R}$ (or $\mathcal{R}_{-\alpha}$) is called the class of regular varying d.f.`s (with index $\alpha$). More generally, d.f. $F$ concentrated on $(-\infty,\infty)$ is also said to belong to these classes if its right-hand distribution $\overline{F}(x) = F(x)\mathbb{I}(x>0)$ does.

**Proposition 1.** The classes $\mathcal{R}, \mathcal{S}, \mathcal{L}$ satisfy the formula [4] $\mathcal{R} \subset \mathcal{S} \subset \mathcal{L}$. If for some $a, b, 0 < a, 0 < b < 1$ d.f. $F$ satisfies the equivalence $\overline{F}(t) \sim \exp\left(-at^b\right)$, $t \to \infty$ then $F \in \mathcal{S}$.

**Proposition 2.** Let $F_1$ and $F_2$ be two d.f.'s concentrated on $(-\infty,\infty)$. If $F_2 \in \mathcal{L}$, $F_1 \in \mathcal{S}$ and $\overline{F}_2(x) = O\left(\overline{F}_1(x)\right)$, then [1, Lemma 3.2] $F_1 * F_2 \in \mathcal{S}$ and $\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$.

## 3 DISCRETE TIME RISK MODEL AND ITS PROPERTIES

Consider discrete time risk model (with annual step) with initial capital $x, x \geq 0$ and nonnegative losses

$$Z_n,\ n = 1,2,\ldots,\ P(Z_n < t) = F(t).$$

Suppose that income $A_n$, $n = 1,2,\ldots$ to end of $n$-th year is defined as difference between unit premium sum and loss $A_n = 1 - Z_n$. Assume that $R_n > 1$ is inflation factor from $n-1$ to $n$ year, $n = 1,2,\ldots$. In [5] $X_n = -A_n$ is called insurance risk and $Y_n = R_n^{-1}$ is called financial risk.

Suppose that the following condition is true:

**(A).** $\{(A_n, R_n), n \geq 1\}$ is sequence of independent and identically distributed random vectors (i.i.d.r.) vectors

$$S_0 = x,\ S_n = R_n S_{n-1} + A_n,\ n = 1,2\ldots \tag{1}$$

In this model with initial capital $x$ ruin time is defined by formula

$$\tau(x) = \inf\{n = 1,2,\ldots : S_n \leq 0 | S_0 = x\},$$

and finite time ruin probability $\psi(x,n)$ - by formula

$$\psi(x,n) = P(\tau(x) \leq n).$$

So the sum $S_n$ money accumulated by insurance company to $n$- th year end satisfies recurrent formula

$$S_0 = x\ ,S_n = x\prod_{j=1}^{n} B_j + \sum_{i=1}^{n} A_i \prod_{j=i+1}^{n} B_j\ ,\ \ n = 1,2\ldots, \tag{2}$$

where $\prod_{j=n+1}^{n} = 1$ by convention.

According to the notation above, we can rewrite the discounted value of the surplus $S_n$ in (2) as

$$\tilde{S}_0 = x, \qquad \tilde{S}_n = S_n \prod_{j=1}^{n} Y_j = x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j = x - W_n.$$

Hence, we easily understand that, for each $n = 0,1,\ldots$,

$$\psi(x,n) = \mathrm{P}(U_n > x), \tag{3}$$

where

$$U_n = \max\left\{0, \max_{1\leq k\leq n} W_k\right\}, \text{ with } U_0 = 0. \tag{4}$$

Define another Markov chain as

$$V_0 = 0,\quad V_n = Y_n(0, X_n + V_{n-1}),\quad n = 1,2\ldots. \tag{5}$$

**Theorem 1.** Suppose that the condition **(A)** is true.

1. Random variables $U_n$ and $V_n$ coincide by distribution

$$U_n \overset{d}{=} V_n, . \ n = 0,1,... \tag{6}$$

2. Equality

$$\psi(x,n) = P(V_n \geq x) \tag{7}$$

is true.

**Proof:** The result (6) is trivial for the case when $n = 0$. Now we aim at (6) for each $n = 1,2...$. Let $n \geq 1$ be fixed. It is easy to obtain the equality

$$U_n = \max\left\{0, \max_{1\leq k\leq n} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_i\right\} = T_n((X_1,Y_1),...,(X_n,Y_n)).$$

Here $T_n$ is a deterministic function. From the condition **(A)** we obtain that

$$((X_1,Y_1),...,(X_n,Y_n)) \overset{(d)}{=} ((X_n,Y_n),...,(X_1,Y_1)).$$

and consequently

$$U_n = T_n((X_1,Y_1),...,(X_n,Y_n)) \overset{(d)}{=} T_n((X_n,Y_n),...,(X_1,Y_1)) =$$

$$= \max\left\{0, \max_{1\leq k\leq n} \sum_{i=1}^{k} X_{n+1-i} \prod_{j=1}^{i} Y_{n+1-j}\right\} = \max\left\{0, \max_{1\leq k\leq n} \sum_{i^*=n+1-k}^{n} X_{i_*} \prod_{j^*=i^*}^{n} Y_{j^*}\right\} =$$

$$= \max\left\{0, \max_{1\leq k^*\leq n} \sum_{i^*=k^*}^{n} X_{i_*} \prod_{j^*=i^*}^{n} Y_{j^*}\right\} = \tilde{V}_n. \tag{8}$$

Here

$$\tilde{V}_n = Y_n\left(0, X_n + \tilde{V}_{n-1}\right)^+, \ n = 1,2...,$$

which is just the same as (5). So we immediately conclude that $\tilde{V}_n = V_n$ for each $n = 1,2...$. Finally, it follows from (8) that (6) holds for each $n = 1,2...$. The formula (7) is a sequence of the formulas (3), (6). This ends the proof of Theorem 1.

**Remark 1.** Theorem 1 proof practically repeats the proof of [1,Theorem 2.1]. A single difference is that the condition of r.v.\`s $X_1,Y_1,...,X_n,Y_n$ independence is replaced by more weak condition **(C).**

Introduce the finite set $Q = \{1,...,m\}$ and for any $q \in Q$ define d.f.\`s $F_q(t)$ and i.i.d.r.v.\`s $X_{n,q}$, $P\left(X_{n,q} > t\right) = \overline{F}_q(t)$ and positive constants $R_q$, $R_1 < R_q$, $q \neq 1$, $q \in Q$. Suppose that $0 < p_q < 1$, $\sum_{q\in Q} p_q = 1$.

**(B).** Random vector $(X_n,Y_n)$ satisfies the condition

$$P\left((X_n,Y_n) = \left(X_n^{(q)}, \frac{1}{R^{(q)}}\right)\right) = p_q, \ q \in Q. \tag{9}$$

From the formula (2) and the condition (9) we have

$$P(V_n > t) = \sum_{q\in Q} p_q P\left(X_n^{(q)} + V_{n-1} > R^{(q)} t\right), \ t > 0. \tag{10}$$

**(C).** Suppose that $F_q(t) \in S$, $q \in Q$ and for any $q_1, q_2 \in Q$, $q_1 \neq q_2$ one of the following equalities is true

$$\overline{F_{q_1}}(t) = O\left(\overline{F_{q_2}}(t)\right) \text{ or } \overline{F_{q_2}}(t) = O\left(\overline{F_{q_1}}(t)\right) \tag{11}$$

## 4 ASYMPTOTIC ANALYSIS OF RUIN PROBABILITY

**Theorem 2.** If the conditions **(A), (B), (C)** are true then for $t \to \infty$

$$P(V_n > t) \sim \sum_{q_1 \in Q} p_{q_1} \overline{F}_{q_1}\left(R^{(q_1)}t\right) + \sum_{q_1, q_2 \in Q} p_{q_1} p_{q_2} \overline{F}_{q_1}\left(R^{(q_1)}R^{(q_2)}t\right) + \ldots + \\ + \sum_{q_1,\ldots,q_n \in Q} p_{q_1} \cdot \ldots \cdot p_{q_n} \overline{F}_{q_1}\left(R^{(q_1)} \cdot \ldots \cdot R^{(q_n)}t\right). \quad (12)$$

**Proof:** Suppose that $n=1$ then

$$P(V_1 > t) = \sum_{q \in Q} p_q P\left(X_1^{(q)} > R^{(q)}t\right) = \sum_{q \in Q} p_q \overline{F}_{q_1}\left(R^{(q)}t, t > 0\right).$$

So for $n=1$ the asymptotic formula (12) is true. Suppose that the formula (12) takes place for fixed $n$. Then from the formula (10) we obtain

$$P(V_{n+1} > t) = \sum_{q_{n+1} \in Q} p_{q_{n+1}} P\left(X_{n+1}^{(q_{n+1})} + V_n > R^{(q_{n+1})}t\right), t > 0 .$$

So from the formula 12) and from Propositions 1, 2 and from the conditions **A), (B), (C)** we have for $t \to \infty$ that

$$P(V_{n+1} > t) \sim \sum_{q_{n+1} \in Q} p_{q_{n+1}} \left[ \sum_{q_1 \in Q} p_{q_{n+1}} \overline{F}_{q_1}\left(R^{(q_{n+1})}R^{(q)_1}t\right) + \sum_{q_1, q_2 \in Q} p_{q_1} p_{q_2} \overline{F}_{q_1}\left(R^{(q_{n+1})}R^{(q_1)}R^{(q_2)}t\right) + \ldots + \right.$$

$$\left. + \sum_{q_1,\ldots,q_n \in Q} p_{q_1} \cdot \ldots \cdot p_{q_n} \overline{F}_{q_1}\left(R^{(q_{n+1})}R^{(q_1)} \cdot \ldots \cdot R^{(q_n)}t\right) + F_{q_{n+1}}\left(R^{(q_{n+1})}t\right) \right] =$$

$$= \sum_{q_{n+1}, q_1 \in Q} p_{q_{n+1}} p_{q_1} \cdot \overline{F}_{q_1}\left(R^{(q_{n+1})}R^{(q_1)}t\right) + \sum_{q_{n+1}, q_1, q_2 \in Q} p_{q_{n+1}} p_{q_1} p_{q_2} \cdot \overline{F}_{q_1}\left(R^{(q_{n+1})}R^{(q_1)}R^{(q_2)}t\right) + \ldots +$$

$$+ \sum_{q_{n+1}, q_1, \ldots, q_n \in Q} p_{q_{n+1}} p_{q_1} \cdot \ldots \cdot p_{q_n} \cdot \overline{F}_{q_1}\left(R^{(q_{n+1})}R^{(q_1)} \cdot \ldots \cdot R^{(q_n)}t\right) + \sum_{q_{n+1} \in Q} p_{q_{n+1}} F_{q_{n+1}}\left(R^{(q_{n+1})}t\right) =$$

$$= \sum_{q_1 \in Q} p_{q_1} \overline{F}_{q_1}\left(R^{(q_1)}t\right) + \sum_{q_1, q_2 \in Q} p_{q_1} p_{q_2} \overline{F}_{q_1}\left(R^{(q_1)}R^{(q_2)}t\right) + \ldots +$$

$$+ \sum_{q_1, \ldots, q_n, q_{n+1} \in Q} p_{q_1} \cdot \ldots \cdot p_{q_{n+1}} \cdot \overline{F}_{q_1}\left(R^{(q_1)} \cdot \ldots \cdot R^{(q_{n+1})}t\right).$$

Last equality is obtained by a replacement of indexes $1,\ldots,n+1$ in its summands. So the formula (12) is proved for index $n+1$ also.

Consider the following asymptotic conditions for $t \to \infty$.

**(D₁)**. There are positive numbers $c_q, \alpha_q$, $q \in Q$, $\alpha_1 < \alpha_q$, $1 < q \le m$ so that $\overline{F}_q(t) \sim c_q t^{-\alpha_q}$

**(D₂)**. There are positive numbers $c_q$, $q \in Q$, $\alpha$ so that $\overline{F}_q(t) \sim c_q t^{-\alpha}$

**(D₃)**. There are positive numbers $c_q, \beta_q$, $q \in Q$, $\beta_1 < \beta_q$, $1 < q \le m$ so that $\overline{F}_q(t) \sim \exp\left(-c_q t^{\beta_q}\right)$.

**(D₄)**. There are positive numbers $c_q$, $q \in Q$, $\beta$ so that $\overline{F}_q(t) \sim\sim \exp\left(-c_q t^{\beta}\right)$

It is easy to prove that the family $F_q(t)$, $q \in Q$ under each of the conditions **(D₁)**, **(D₂)**, **(D₃)**, **(D₄)** satisfies the condition **(C)**.

In the condition **(D₁)** the formula (12) may be represented in the following form

$$\psi(t,n) \sim \sum_{k=1}^{n} p_1^k \overline{F}_1\left(R_1^k t\right) \sim c_1 t^{-\alpha_1} \sum_{k=1}^{n} \frac{p_1^k}{R_1^{k\alpha}}$$

and consequently

$$\psi(t,n) \sim \begin{cases} c_1 t^{-\alpha_1} \dfrac{1-p_1^{n+1} R_1^{-(n+1)\alpha}}{1-p_1 R_1^{-\alpha}}, & p_1 R_1^{-\alpha} \neq 1 \\ n c_1 t^{-\alpha_1}, & p_1 R_1^{-\alpha} = 1. \end{cases} \quad (13)$$

In the condition **($D_2$)** the formula (12) may be represented in the following form

$$\begin{aligned} \psi(t,n) \sim t^{-\alpha} \Bigg[ & \sum_{q_1 \in Q} c_{q_1} p_{q_1} R_{q_1}^{-\alpha} + \sum_{q_1, q_2 \in Q} c_{q_1} p_{q_1} p_{q_2} R_{q_1}^{-\alpha} R_{q_2}^{-\alpha} + \\ & + \sum_{q_1, \ldots, q_n \in Q} c_{q_1} p_{q_1} \cdot \ldots \cdot p_{q_n} R_{q_1}^{-\alpha} \cdot \ldots \cdot R_{q_n}^{-\alpha} + \Bigg] = t^{-\alpha} S_1 \frac{1-S_2^n}{1-S_2}. \end{aligned} \quad (14)$$

with

$$S_1 = \sum_{q_1 \in Q} c_{q_1} p_{q_1} R_{q_1}^{-\alpha}, \; S_1 = \sum_{q_1 \in Q} c_{q_1} p_{q_1} R_{q_1}^{-\alpha}$$

In the condition **($D_3$)** the formula (12) may be represented in the following form

$$\psi(t,n) \sim \sum_{k=1}^{n} p_1^k F_1\left(R_1^k t\right) = \sum_{k=1}^{n} p_1^k \exp\left(-c_1 \left(R_1^k t\right)^{\beta_1}\right)$$

and so

$$\psi(t,n) \sim \begin{cases} p_1^n \left(-c_1 \left(R_1^n t\right)^{\beta_1}\right), R_1 < 1, \\ p_1 \left(-c_1 \left(R_1 t\right)^{\beta_1}\right), R_1 > 1, \\ \exp\left(-c_1 \left(R_1 t\right)^{\beta_1}\right) \dfrac{1-p_1^{n+1}}{1-p_1}, R_1 = 1. \end{cases} \quad (15)$$

In the condition **($D_4$)** the formula (12) may be represented in the following form

$$\psi(t,n) \sim \sum_{k=1}^{n} \left[ \sum_{q_1, \ldots, q_k \in Q} p_{q_1} \cdot \ldots \cdot p_{q_k} \exp\left(-c_{q_1} \left(R_{q_1} \cdot \ldots \cdot R_{q_k} t\right)^{\beta}\right) \right]. \quad (16)$$

Suppose that there is the constant $q'$ satisfying the inequalities

$$c_{q'} R_{q'}^{\beta} < c_{q'} R_q^{\beta}, \; q \neq q', \; q \in Q.$$

The equivalences (16) may be rewritten as follows

$$\psi(t,n) \sim p_{q'} \sum_{k=1}^{n} p_1^{k-1} \exp\left(-c_{q'} \left(R_{q'} R_1^{k-1} t\right)^{\beta}\right).$$

and so

$$\psi(t,n) \sim \begin{cases} p_{q'} p_1^{n-1} \exp\left(-c_{q'} \left(R_{q'} R_1^{n-1} t\right)^{\beta}\right), R_1 < 1, \\ p_{q'} \exp\left(-c_{q'} \left(R_{q'} t\right)^{\beta}\right), R_1 < 1, \\ p_{q'} \dfrac{1-p_1^n}{1-p_1} \exp\left(-c_{q'} \left(R_{q'} t\right)^{\beta}\right), R_1 = 1. \end{cases} \quad (17)$$

## 5 CONCLUSION

A comparison of the asymptotic formulas (13), (14), (15), (17) with the results of [1] shows that a dependence of financial and insurance risks introduces significant changes into asymptotic formulas for ruin probability of discrete time risk model.

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