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RANDOM FUZZY CONTINUOUS-TIME MARKOV JUMP PROCESSES

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ABSTRACT

Continuous-time Markov chains are an important subclass in stochastic processes, which have facilitated many applications in business decisions, investment risk analysis, insurance policy making and reliability modeling. One should be fully aware that the existing continuous-time Markov chains theory is merely a framework under which the random uncertainty governs the phenomena. However, the real world phenomena often reveal a reality in which randomness and vagueness co-exist, and thus probabilistic continuous-time Markov chains modeling practices may be not wholly adequate. In this paper, we define random fuzzy continuous-time Markov chains, explore the related average chance distributions, and propose both a scheme for parameter estimation and a simulation scheme. It is expected that a foundational base can be established for reliability modeling and risk analysis, particularly, repairable system modeling.

1 INTRODUCTION

One should be fully aware that vagueness is an intrinsic feature of today's diversified business environments. As Carvalho and Machado (2006) commented, "In a global market, companies must deal with a high rate of changes in business environment. … The parameters, variables and restrictions of the production system are inherently vagueness." Therefore a co-existence of random uncertainty and fuzzy uncertainty is inevitable aspect of safety and reliability analysis and modelling.

It is obvious that probabilistic modeling is a good approximation to real world problem only when random uncertainty governs the phenomenon. Philosophically, if fuzziness and randomness both appear then probabilistic modeling alone may be questionable or inadequate. Therefore, it is logical to develop appropriate models for co-existent fuzziness and randomness.

Markov processes have been applied to large and complex system modeling and analysis in the reliability literature, for example, in recent work of Kolowrocki (2007), Love et al. (2000), Soszynska (2007), and Tamura (2004), etc.

We may also note that in recent year researchers in repairable system modeling, particularly in Asian reliability communities, proposed repair impact scenario models, which assume that the repair impacts to a repairable system may be classified into several states: no improvement, minor improvement, medium improvement, and major improvement. Hence one may utilize Kijima's age models (Kijima, 1989) to estimate those repair effects on the system repair states for optimal maintenance policy decision making, see Chan and Shaw (1993), Dohi et al. (2006), Lim and Lie (2000), Love et al. (2000), Wang, H. and Pham, H. (1996), Sheu et al. (2004), and Zhang (2002). However, less attention has been paid to the repair effect estimation, except for a few authors, Guo and Love (1992, 2004), Lim and Lie (2000), Yun et al. (2004), etc.

In this paper, we will give a systematic treatment for random fuzzy continuous-time Markov chains not only in the mathematical sense (building models based on postulates and definitions), but also in the statistical sense (estimation and hypothesis testing based on sample data).

2 PROBABILISTIC CONTINUOUS-TIME MARKOV JUMP PROCESSES

Grimmett and Stirzaker (1992) and also Guo (2009) describe continuous-time Markov jump processes by focusing the stochastic semigroup and the rate matrix.

Let $X=\{X_t,t\geq 0\}$ be a Markov chain with state space $\mathbb{S}=\{0,1,2,\cdots,N-1\}$. Further, let

$$p_{ij}(s,t)=\Pr\{X_t=j\mid X_s=i\} \tag{1}$$

be the transition probabilities. For the stationary Markov chain

$$p_{ij}(0,t-s)=p_{ij}(s,t),\ \forall s<t \tag{2}$$

Definition 1: (Grimmett and Stirzaker (1992)) A stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$, with $\mathrm{P}_t=\left(p_{ij}(t)\right)_{N\times N}$ satisfies the following properties:

(a) $\mathrm{P}_0=\mathrm{I}$, an $N\times N$ identity matrix;

(b) For $\forall t$, $0\leq p_{ij}(t)\leq 1$, $\sum_j p_{ij}(t)=1$;

(c) The Chapman-Kolmogorov equations, for any $s,t>0$, $\mathrm{P}_{t+s}=\mathrm{P}_t\mathrm{P}_s$.

A stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$ is standard if $\lim_{t\downarrow 0}\mathrm{P}_t=\mathrm{I}$. The characterization of a stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$ can be stated as a theorem.

Theorem 1: For a standard stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$, the limit

$$\lim_{h\downarrow 0}\frac{p_{ii}(h)-1}{h}=-q_i\left(=q_{ii}\right) \tag{3}$$

exists (maybe $-\infty$), while the limit

$$\lim_{h\downarrow 0}\frac{p_{ij}(h)}{h}=q_{ij} \tag{4}$$

exists and is finite. Guo (2009) detailed the proof of Theorem 1.

Definition 2: The matrix Q

$$\mathrm{Q}=\begin{bmatrix} -q_0 & q_{01} & \cdots & q_{0,N-1} \\ q_{10} & -q_{11} & \cdots & q_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N-1,1} & q_{N-1,2} & \cdots & -q_{N-1,N-1} \end{bmatrix} \tag{5}$$

where

$$\lim_{h\downarrow 0}\frac{p_{ij}(h)-\delta_{ij}}{h}=q_{ij} \tag{6}$$

with $\delta_{ij}=1,\ i=j$, *0* otherwise.

Lemma 1: In the rate matrix Q,

$$q_i = \sum_{j=0,\, j\neq i}^{N-1} q_{ij},\ i, j = 0,1,2,\cdots, N-1 \tag{7}$$

The rate matrix Q characterizes the movements of the continuous-time Markov chain $X = \{X_t, t \geq 0\}$. The following theorem reveals that fundamental fact.

Theorem 2: If the process $X = \{X_t, t \geq 0\}$ is currently halted at state i, it halts in state i during a time exponentially distributed with parameter q_i, independently of how the process reached state i and of how long it takes to get there. Furthermore, The process $X = \{X_t, t \geq 0\}$ leaves state i, and moves to state j with probability q_{ij}/q_i $(i \neq j)$.

Theorem 3. A standard stochastic semigroup $\mathrm{P} = \{\mathrm{P}_t, t \geq 0\}$ satisfies Kolmogorov equations:

$$\begin{aligned} \frac{d}{dt}\mathrm{P}_t &= \mathrm{P}_t\mathrm{Q} \text{ (Forward)} \\ \frac{d}{dt}\mathrm{P}_t &= \mathrm{Q}\mathrm{P}_t \text{ (Backward)} \end{aligned} \tag{8}$$

Corollary 1. A standard stochastic semigroup $\mathrm{P} = \{\mathrm{P}_t, t \geq 0\}$ satisfies

$$\mathrm{P}_t = e^{\mathrm{Q}t} \tag{9}$$

where matrix

$$e^{\mathrm{Q}t} = \sum_{i=0}^{\infty} \frac{1}{i!}(\mathrm{Q}t)^i \tag{10}$$

It is well-established fact that every entry of P_t, say $p_{ij}(t)$, can be expressed by a linear combination of $e^{\rho_l t}$ with appropriate coefficient $c(l)$, where ρ_l is the l^{th} eigenvalue of Q or of an appropriate minor matrix of Q, i.e.,

$$p_{ij}(t) = \sum_{l=0}^{N-1} c(l) e^{\rho_l t} \tag{11}$$

Example 1: Two-state continuous-time Markov chain. Let the rate matrix

$$\mathrm{Q} = \begin{bmatrix} -\nu & \nu \\ \lambda & -\lambda \end{bmatrix} \tag{12}$$

The eigenvalues are $(\rho_1, \rho_2) = (0, -(\nu+\lambda))$, thus

$$\mathrm{P}_t = \begin{bmatrix} \dfrac{\lambda}{\lambda+\nu} + \dfrac{\nu}{\lambda+\nu} e^{-(\lambda+\nu)t} & \dfrac{\nu}{\lambda+\nu} - \dfrac{\nu}{\lambda+\nu} e^{-(\lambda+\nu)t} \\ \dfrac{\lambda}{\lambda+\nu} - \dfrac{\lambda}{\lambda+\nu} e^{-(\lambda+\nu)t} & \dfrac{\nu}{\lambda+\nu} + \dfrac{\lambda}{\lambda+\nu} e^{-(\lambda+\nu)t} \end{bmatrix} \tag{13}$$

which confirms the formal result Equation (11).

3 FOUNDATION OF RANDOM FUZZY PROCESSES

Without a solid understanding of the intrinsic feature of random fuzzy processes, there is no base for exploring the modelling of random fuzzy continuous-time Markov chains. Liu's (2004, 2007) hybrid variable theory established on the axiomatic credibility measure and probability measure foundations provides the mathematical foundation.

Guo et al. (2009) gave a systematic review on random fuzzy variable theory. In order to shorten the current paper, we keep only contents necessary for notational clarity, for details, see Guo and Guo (2009), or directly Liu's books (2004, 2007).

First let us review the credibilistic fuzzy variable theory. Let Θ be a nonempty set, and $\mathfrak{P}(\Theta)$ the power set on Θ.

Definition 3: Any set function $\mathrm{Cr}:\mathfrak{P}(\Theta)\to[0,1]$ which satisfies Liu's four Axioms (2004, 2007) is called a credibility measure. The triple $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ is called the credibility measure space.

Definition 4: A fuzzy variable ξ is a measurable mapping, i.e., $\xi:(\Theta,\mathfrak{P}(\Theta))\to(\mathbb{R},\mathfrak{B}(\mathbb{R}))$.

A fuzzy variable is not a fuzzy set in the sense of Zadeh's fuzzy theory (1965, 1978), in which a fuzzy set is defined by a membership function.

Definition 5: (Liu (2004, 2007)) The credibility distribution $\Lambda:\mathbb{R}\to[0,1]$ of a fuzzy variable ξ on $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ is

$$\Lambda(x)=\mathrm{Cr}\left\{\theta\in\Theta\,\middle|\,\xi(\theta)\le x\right\} \tag{14}$$

Liu (2004, 2007) defines a random fuzzy variable as a mapping from the credibility space $(\Theta,2^{\Theta},\mathrm{Cr})$ to a set of random variables.

Definition 6: (Guo et al, (2007)) A random fuzzy variable, denoted as $\xi=\left\{X_{\beta(\theta)},\theta\in\Theta\right\}$, is a set of random variables X_{β} defined on the common probability space $(\Omega,\mathfrak{A},\Pr)$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta,2^{\Theta},\mathrm{Cr})$.

Definition 7: (Liu (2004, 2007)) Let ξ be a random fuzzy variable, then the average chance measure denoted by $\mathrm{ch}\{\cdot\}$, of a random fuzzy event $\{\xi\le x\}$, is

$$\mathrm{ch}\{\xi\le x\}=\int_0^1\mathrm{Cr}\left\{\theta\in\Theta|\Pr\{\xi(\theta)\le x\}\ge\alpha\right\}\mathrm{d}\alpha \tag{15}$$

Then function $\Psi(\cdot)$ is called as average chance distribution if and only if

$$\Psi(x)=\mathrm{ch}\{\xi\le x\} \tag{16}$$

Definition 8: A random fuzzy process is a family of random fuzzy variables defined on the common Product measure space $(\Theta,2^{\Theta},\mathrm{Cr})\times(\Omega,\mathfrak{A},\Pr)$, denoted by $\xi=\{\xi_t,t\in\mathbb{T}\}$, where $\mathbb{T}$ is an index set.

Theorem 4: Let ζ be a fuzzy variable defined on the credibility space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ and τ be a random variable defined on the probability space $(\Omega,\mathfrak{A}(\Omega),P)$, then

(1) Let $\oplus$ be an arithmetic operator, which can be "$+$", "$-$", "$\times$" or "$\div$" operations, such that $\zeta\oplus\tau$ maps from $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ to a collection of random variables on $(\Omega,\mathfrak{A}(\Omega),P)$, denoted by ξ. Then ξ is a random fuzzy variable defined on hybrid product space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})\times(\Omega,\mathfrak{A}(\Omega),P)$.

(2) Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous mapping, such that $f(\zeta,\tau)$ maps from $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ to a collection of random variables on $(\Omega,\mathfrak{A}(\Omega),P)$, denoted by ξ. Then $\xi=f(\zeta,\tau)$ is a random fuzzy variable defined on hybrid product space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})\times(\Omega,\mathfrak{A}(\Omega),P)$.

(3) Let $F(x;\theta)$ be the probability distribution of random variable τ with parameter θ (possibly vector-valued), then $F(x;\zeta)$ defines a random fuzzy variable ξ on the hybrid product space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})\times(\Omega,\mathfrak{A}(\Omega),P)$.

Note that Theorem 4 merely repeats facts stated in Liu's books, (2004, 2007).

4 STATIONARY RANDOM FUZZY CONTINUOUS-TIME MARKOV CHAIN

Let $X=\{X_t,t\geq 0\}$ be a Markov process with a standard stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$ having a fuzzy rate matrix Q defined on credibility space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ with credibility distribution function matrix $\Lambda=(\Lambda_{ij})_{N\times N}$. Then by a direct application of Theorem 4, Item (3), a random fuzzy continuous-time Markov chain can be obtained.

Definition 9: A process is called a random fuzzy continuous-time Markov chain $\xi=\{\xi_t,t\geq 0\}$ taking values in set $\mathbb{S}=\{0,1,2,\cdots,N-1\}$, if

(a) $\xi=\{\xi_t,t\geq 0\}$ satisfies a Markov property:

$$\begin{aligned}&\Pr\left\{\xi_t=j\,|\,\xi_{t_1}=i_1,\xi_{t_2}=i_2,\cdots,\xi_s=i\right\}\\&=\Pr\left\{\xi_t=j\,|\,\xi_s=i\right\}\end{aligned}\tag{17}$$

for all $t_1<t_2<\cdots<s<t$ and any $i_1,i_2,\cdots,i,j\in\mathbb{S}$.

(b) the stochastic semigroup $\mathrm{P}=\{\mathrm{P}_t,t\geq 0\}$ is standard;

(c) and the fuzzy rate matrix

$$\mathrm{Q}=(q_{ij})_{N\times N}=\lim_{t\downarrow 0}\frac{\mathrm{P}_t-\mathrm{I}}{t}\tag{18}$$

is defined on credibility space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$ with credibility distribution function matrix $\Lambda=(\Lambda_{ij})_{N\times N}$.

It is obvious that in Definition 9 for a given value of matrix $\mathrm{Q}=\mathrm{Q}_0$, $\xi=\{\xi_t,t\geq 0\}$ is a probabilistic continuous-time Markov chain. However, if Q is a fuzzy matrix, then for any given time t, the count ξ_t is a random fuzzy variable according to Theorem 5. Therefore, Definition 9 defines a stationary random fuzzy Poisson process.

Theorem 5: If the process $\xi=\{\xi_t,t\geq 0\}$ is currently halted at state i, it halts in state i during a a time interval which is exponentially distributed with fuzzy parameter q_i, independently of how and when the process reached state i and of how long it has been there. Furthermore, The process $\xi=\{\xi_t,t\geq 0\}$ leaves state i, and moves to state j with a fuzzy probability q_{ij}/q_i $(i\neq j)$.

Proof: A straightforward application of Definition 9 and Theorem 2.

Corollary 2: If q_{ij} $(i\neq j)$, $i,j=0,1,\cdots,N-1$, follow piecewise linear credibility distributions

$$\Lambda_{ij}(x)=\begin{cases} 0 & x<a_{ij} \\ \dfrac{x-a_{ij}}{2\left(b_i-a_{ij}\right)} & a_{ij}\le x<b_{ij} \\ \dfrac{x+c_{ij}-2b_{ij}}{2\left(c_{ij}-b_{ij}\right)} & b_{ij}\le x<c_{ij} \\ 1 & x\ge c_{ij} \end{cases},\ i\neq j \tag{19}$$

The halting times, denoted by T_i, $i=0,1,\cdots,N-1$, are independent random fuzzy exponential variables with fuzzy parameter $q_i=\sum_j q_{ij}$ following a piecewise linear credibility distribution

$$\Lambda_i(x)=\begin{cases} 0 & x<a_i \\ \dfrac{x-a_i}{2\left(b_i-a_i\right)} & a_i\le x<b_i \\ \dfrac{x+c_i-2b_i}{2\left(c_i-b_i\right)} & b_i\le x<c_i \\ 1 & x\ge c_i \end{cases}, \tag{20}$$

where

$$\begin{cases} a_i=\sum\limits_{j=1,j\neq i}^{N-1} a_{ij} \\ b_i=\sum\limits_{j=1,j\neq i}^{N-1} b_{ij} \\ c_i=\sum\limits_{j=1,j\neq i}^{N-1} c_{ij} \end{cases} \tag{21}$$

Thus the average chance distributions (for holding times) are

$$\begin{aligned}\Psi_i(t)&=\int_0^1 \mathrm{Cr}\left\{\theta:q_i(\theta)\ge-\ln(1-\alpha)/t\right\}\mathrm{d}\alpha \\ &=1+\frac{e^{-b_it}-e^{-a_it}}{2\left(b_i-a_i\right)t}+\frac{e^{-c_it}-e^{-b_it}}{2\left(b_i-c_i\right)t}\end{aligned} \tag{22}$$

Proof: Note that

$$\Pr\left\{T\left(q_i\right)\le t\right\}=1-e^{-q_it} \tag{23}$$

Therefore the event $\left\{\theta:\Pr\left\{T\left(q_i(\theta)\right)\le t\right\}\ge\alpha\right\}$ is a fuzzy event and is equivalent to the fuzzy event $\left\{\theta:q_i(\theta)\ge-\ln(1-\alpha)/t\right\}$. As a critical part of the derivation of the average chance distribution, it is necessary to calculate the credibility measure for fuzzy event $\left\{\theta:q_i(\theta)\ge-\ln(1-\alpha)/t\right\}$, i.e., to obtain the expression for

$$\mathrm{Cr}\left\{\theta:q_i(\theta)\ge-\ln(1-\alpha)/t\right\} \tag{24}$$

Recall that for the credibilistic fuzzy variable, $q_i=\sum_{j\neq i}q_{ij}$, the credibility measure takes the form

$$\mathrm{Cr}\left\{\theta : q_i(\theta) > x\right\} = \begin{cases} 1 & x < a_i \\ \dfrac{2b_i - a_i - x}{2(b_i - a_i)} & a_i \le x < b_i \\ \dfrac{c_i - x}{2(c_i - b_i)} & b_i \le x < c_i \\ 0 & x \ge c_i \end{cases}, \tag{25}$$

Accordingly, the range for integration over α can be determined as shown in Table 1. Recall that the expression of $x = -\ln(1-\alpha)/t$ appears in Equation (25), which constitutes the link between intermediate variable α and average chance measure.

The average chance distribution for the exponentially distributed random fuzzy lifetime is then derived by splitting the integration into five terms according to the range of α and the corresponding mathematical expression for the credibility measure $\mathrm{Cr}\left\{\theta : q_i(\theta) \ge -\ln(1-\alpha)/t\right\}$, which is detailed in the following table.

Table 1. Range analysis for α

x	α and credibility measure expression	
$-\infty < x < a$	Range for α	$0 \le \alpha \le 1 - e^{-at}$
	$\mathrm{Cr}\{\lambda(\theta) \ge -\ln(1-\alpha)/t\}$	1
$a \le x < b$	Range for α	$1 - e^{-at} < \alpha \le 1 - e^{-bt}$
	$\mathrm{Cr}\{\lambda(\theta) \ge -\ln(1-\alpha)/t\}$	$1 - (x-a)/(2(b-a))$
$b \le x < c$	Range for α	$1 - e^{-bt} < \alpha \le 1 - e^{-ct}$
	$\mathrm{Cr}\{\lambda(\theta) \ge -\ln(1-\alpha)/t\}$	$(c-x)/2(c-b)$
$c \le x < +\infty$	Range for α	$1 - e^{-ct} < \alpha \le 1$
	$\mathrm{Cr}\{\lambda(\theta) \ge -\ln(1-\alpha)/t\}$	0

Then the exponential random fuzzy lifetime has an average chance distribution function:

$$\begin{aligned}\Psi(t) &= \int_0^1 \mathrm{Cr}\left\{\theta : \lambda(\theta) \ge -\ln(1-\alpha)/t\right\} \mathrm{d}\alpha \\ &= 1 + \frac{e^{-bt} - e^{-at}}{2(b-a)t} + \frac{e^{-bt} - e^{-ct}}{2(c-b)t}\end{aligned} \tag{26}$$

and the average chance density is

$$\begin{aligned}\psi(t) = &\frac{e^{-at} - e^{-bt}}{2(b-a)t^2} + \frac{be^{-bt} - ae^{-at}}{2(b-a)t} \\ &+ \frac{e^{-bt} - e^{-bt}}{2(c-b)t^2} + \frac{ce^{-ct} - be^{-bt}}{2(c-b)t}\end{aligned} \tag{27}$$

This expression concludes the proof.

Similarly to the probabilistic reliability theory, we define a reliability function or survival function for a random fuzzy lifetime and accordingly name it as the average chance reliability function, which is defined accordingly as

$$\bar{\Psi}(t) = 1 - \Psi(t) \tag{28}$$

Then, for an exponential random fuzzy lifetime, the average chance reliability function is

$$\bar{\Psi}(t)=\frac{e^{-at}-e^{-bt}}{2(b-a)t}+\frac{e^{-bt}-e^{-ct}}{2(c-b)t} \tag{29}$$

Remark 1: The average chance distributions of jump probabilities q_{ij}/q_j do not have closed forms, which require the application of Zadeh's extension theorem (1978). However, the values of fuzzy probability q_{ij}/q_j fall in intervals

$$\left[\min\left(\frac{a_{ij}}{\sum_{j\neq i}a_{ij}},\frac{c_{ij}}{\sum_{j\neq i}c_{ij}}\right),\max\left(\frac{a_{ij}}{\sum_{j\neq i}a_{ij}},\frac{c_{ij}}{\sum_{j\neq i}c_{ij}}\right)\right] \tag{30}$$

which will inform the explorations of the process $\xi=\{\xi_t,t\geq 0\}$.

5 NON-STATIONARY RANDOM FUZZY CONTINUOUS-TIME MARKOV CHAIN

The probabilistic non-stationary continuous-time Markov chain is an extension to the stationary case except that the rate matrix is function of time, i.e., time-dependent. Hence, a non-stationary random fuzzy continuous-time Markov chain can be defined as follows.

Definition 10: A process is called as random fuzzy continuous-time non-stationary Markov chain $\xi=\{\xi_t,t\geq 0\}$ taking values in state space $\mathbb{S}=\{0,1,2,\cdots,N-1\}$, if:

(a) $\xi=\{\xi_t,t\geq 0\}$ satisfies Markov property:

$$\begin{aligned}&\Pr\left\{\xi_t=j\,|\,\xi_{t_1}=i_1,\xi_{t_2}=i_2,\cdots,\xi_s=i\right\}\\&=\Pr\left\{\xi_t=j\,|\,\xi_s=i\right\}\end{aligned} \tag{31}$$

for all $t_1<t_2<\cdots<s<t$ and any $i_1,i_2,\cdots,i,j\in\mathbb{S}$.

(b) for $\forall s<t$, $p_{ij}(s,t)=\Pr\{\xi_t=j\,|\,\xi_s=i\}$, the transitional probabilities satisfy:

(i) for a small time-increment h, $\xi=\{\xi_t,t\geq 0\}$ moves from state i to state j with (fuzzy) probability:

$$p_{ij}(t,t+h)=q_{ij}(t)h+o(h)\ h\downarrow 0(i\neq j) \tag{32}$$

(ii) for a small time-increment h, $\xi=\{\xi_t,t\geq 0\}$ remaining in state i with (fuzzy) probability:

$$p_{ii}(t,t+h)=1-q_i(t)h+o(h)\ h\downarrow 0 \tag{33}$$

where the rate functions are given by

$$q_i(t)=\sum_{j=0,j\neq i}^{N-1}q_{ij}(t),\ i=0,1,\cdots,N-1 \tag{34}$$

(c) The parameters of rate functions, i.e., the entries of the fuzzy rate matrix $\mathrm{Q}(t)=\left(q_{ij}(t)\right)_{N\times N}$ are credibilistic fuzzy variables defined on the common credibility measure space $(\Theta,\mathfrak{P}(\Theta),\mathrm{Cr})$.

Theorem 6: If the process $\xi=\{\xi_t,t\geq 0\}$ is currently halted at state i, it halts in state i during a time interval that is exponentially distributed with fuzzy parameter $q_i(t)$, independently of how and when the process reached state i and of how long it has been there. Furthermore, the process $\xi=\{\xi_t,t\geq 0\}$ leaves state i, and moves to state j with a fuzzy probability $q_{ij}(t)/q_i(t)\ (i\neq j)$.

Corollary 3: The probability distribution of halting times given the current state $\xi_{w_{l-1}}=x_{l-1}\in\mathbb{S}$, is

$$\begin{aligned}&\Pr\left\{W_l - w_{l-1} > t, \xi_{w_{l-1}} = x_{l-1}\right\} = \\ &\exp\left(-\left(m_{x_{l-1}}\left(w_{l-1}+t\right)\right) - m_{x_{l-1}}\left(w_{l-1}+t\right)\right)\end{aligned} \tag{35}$$

where

$$m_i\left(t\right) = \int_0^t q_i\left(u\right) du \tag{36}$$

is called the i^{th} integrated rate function.

Example 2: Assume a linear rate function:

$$q_{ij}\left(t\right) = \beta_{0,ij} + \beta_{1,ij} t,\ \left(j \neq i\right), \beta_{0,ij} > 0,\ \beta_{1,ij} > 0 \tag{37}$$

Further, we assume that β_0 and β_1 both have piecewise linear credibility distribution:

$$\Lambda_{ij}^{(k)}\left(x\right) = \begin{cases} 0 & x < a_{ij}^{(k)} \\ \dfrac{x - a_{ij}^{(k)}}{2\left(b_{ij}^{(k)} - a_{ij}^{(k)}\right)} & a_{ij}^{(k)} \le x < b_{ij}^{(k)} \\ \dfrac{x + c_{ij}^{(k)} - 2b_{ij}^{(k)}}{2\left(c_{ij}^{(k)} - b_{ij}^{(k)}\right)} & b_{ij}^{(k)} \le x < c_{ij}^{(k)} \\ 1 & x \ge c_{ij}^{(k)} \end{cases},\ k = 0,1 \tag{38}$$

Then the diagonal entries $q_i\left(t\right)$, $i = 0,1,\cdots,N-1$, have credibility distributions

$$\Lambda_i\left(x\right) = \begin{cases} 0 & x < a_i \\ \dfrac{x - a_i}{2\left(b_i - a_i\right)} & a_i \le x < b_i \\ \dfrac{x + c_i - 2b_i}{2\left(c_i - b_i\right)} & b_i \le x < c_i \\ 1 & x \ge c_i \end{cases},\ i = 0,1,\cdots,N-1 \tag{40}$$

where

$$\begin{cases} a_i = \sum_{j=0}^{N-1}\left(a_{ij}^{(0)} + a_{ij}^{(1)} t\right) \\ b_i = \sum_{j=0}^{N-1}\left(b_{ij}^{(0)} + b_{ij}^{(1)} t\right) \\ c_i = \sum_{j=0}^{N-1}\left(c_{ij}^{(0)} + c_{ij}^{(1)} t\right) \end{cases} \tag{41}$$

The integrated diagonal entries of $\mathrm{Q}\left(t\right)$:

$$m_i\left(t\right) = \beta_{0,i} t + \beta_{1,i} t^2 \tag{42}$$

will have credibility distributions:

$$\Lambda_{m_i(t)}(y)=\begin{cases} 0 & y<A_i \\ \dfrac{y-A_i}{2(B_i-A_i)} & A_i\le y<B_i \\ \dfrac{y+C_i-2B_i}{2(C_i-B_i)} & B_i\le y<C_i \\ 1 & y\ge C_i \end{cases} \tag{43}$$

where

$$\begin{cases} A_i=\sum_{j=0}^{N-1}\left(a_{ij}^{(0)}t+a_{ij}^{(1)}t^2\right) \\ B_i=\sum_{j=0}^{N-1}\left(b_{ij}^{(0)}t+b_{ij}^{(1)}t^2\right) \\ C_i=\sum_{j=0}^{N-1}\left(c_{ij}^{(0)}t+c_{ij}^{(1)}t^2\right) \end{cases} \tag{44}$$

In general, to obtain the credibility distribution denoted as $\Lambda_{m_i(t)}$, of the integrated intensity function $m(t)$, it is necessary to apply Zadeh's extension principle (1978), but for the piecewise linear credibility distribution case, the mathematical arguments are relatively simple.
Now let us derive the average chance distribution for the first halting times at i^{th} state (the initial state).

$$\Psi_T(t)=\int_0^1 \mathrm{Cr}\left(\theta:\Pr\left\{T_1(\theta)\le t\right\}\ge\alpha\right)d\alpha \tag{45}$$

Note that for the first arrival time,

$$\begin{aligned} &\left\{\theta:\Pr\left\{T_1(\theta)\le t\right\}\ge\alpha\right\} \\ &=\left\{\theta:1-\exp\left(-\int_0^t(\beta_0+\beta_1 u)\,du\right)\ge\alpha\right\} \\ &=\left\{\theta:1-e^{-m(t)}\ge\alpha\right\} \\ &=\left\{\theta:m(t)\ge-\ln(1-\alpha)\right\} \end{aligned} \tag{46}$$

Therefore, the average chance distribution for T_1, the first halting at state *i*, is

$$\begin{aligned} &\Psi_{T_i}(t) \\ &=\int_0^1 \mathrm{Cr}\left(\theta:\Pr\left\{T_1(\theta)\le t\right\}\ge\alpha\right)d\alpha \\ &=\int_0^1 \mathrm{Cr}\left(\theta:m(t)\ge-\ln(1-\alpha)\right)d\alpha \end{aligned} \tag{47}$$

We observe that $y=-\ln(1-\alpha)$, therefore,

$$\mathrm{Cr}\left\{m(t) > y\right\} = \begin{cases} 1 & y < A_i \\ \dfrac{2B_i - 2 - y}{2\left(B_i - A_i\right)} & A_i \le y < B_i \\ \dfrac{C_i - y}{2\left(C_i - B_i\right)} & B_i \le y < C_i \\ 0 & y \ge C_i \end{cases} \tag{48}$$

Hence,

$$\begin{aligned} &\Psi_{T_i}(t) = \\ &1 - e^{-m(A_i)} + \frac{2B_i - A_i - 1}{2\left(B_i - A_i\right)}\left(e^{-m(A_i)} - e^{-m(B_i)}\right) \\ &+ \frac{1}{2(B_i - A_i)}\left(-m\left(B_i\right)e^{-m(B_i)} + m\left(A_i\right)e^{-m(A_i)}\right) \\ &+ \frac{C_i - 1}{2\left(C_i - B_i\right)}\left(e^{-m(B_i)} - e^{-m(C_i)}\right) \\ &+ \frac{1}{2(C_i - B_i)}\left(-m\left(C_i\right)e^{-m(C_i)} + m\left(B_i\right)e^{-m(B_i)}\right) \end{aligned} \tag{49}$$

6 A PARAMETER ESTIMATION SCHEME

Here parameter estimation is essentially a problem of estimation of credibility distributions from fuzzy observations. Guo and Guo (2009) recently proposed a maximally compatible random variable to a credibilistic fuzzy variable and thus the fuzzy estimation problem is converted into estimating the distribution function of the maximally compatible random variable. The following scheme is for estimating a piecewise linear credibility distribution.

Definition 11: Let X be a random variable defined in $\left(\mathbb{R}, \mathfrak{B}(\mathbb{R})\right)$ such that

$$\mu^c = \mathrm{Cr} \circ \xi^{-1} = \mu = P \circ X^{-1} \tag{50}$$

Then X is called a maximally compatible to fuzzy variable ξ.

In other words, a random variable X can take all the possible real-values the fuzzy variable ξ may take and the distribution of X , $F_X(r)$ equals the credibility distribution of ξ, $\Lambda_\xi(r)$ for all $r \in \mathbb{R}$.

It is observed that the induced measure $\mu^c = \mathrm{Cr} \circ \xi^{-1}$ and measure $\mu = P \circ X^{-1}$ are defined on the same measurable space $\left(\mathbb{R}, \mathfrak{B}(\mathbb{R})\right)$. Furthermore, we note that the pre-image $\xi^{-1}(B) \in \mathfrak{P}(\Theta)$, but the pre-image $X^{-1}(B) \in \mathfrak{A}(\Theta) \subset \mathfrak{P}(\Theta)$, which implies that for the same Borel set $B \in \mathfrak{B}(\mathbb{R})$, the pre-images under fuzzy variable ξ and random variable X are not the same. It is expected that

$$\left\{\theta \in \Theta : X(\theta) \le r\right\} \subseteq \left\{\theta \in \Theta : \xi(\theta) \le r\right\} \tag{51}$$

but

$$\Pr\left\{\theta \in \Theta : X(\theta) \le r\right\} = \mathrm{Cr}\left\{\theta \in \Theta : \xi(\theta) \le r\right\} \tag{52}$$

The statistical estimation scheme for parameters (a, b, c) of the credibility distribution based on fuzzy observations $\{x_1, x_2, \cdots, x_n\}$ can be stated as:

Estimation Scheme 1:

Step 1: Rank fuzzy observations $\{x_1, x_2, \cdots, x_n\}$ to obtain "order" statistics $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ in ascending order;

Step 2: Set $\hat{a} = x_{(1)}$ and $\hat{c} = x_{(n)}$;

Step 3: Set a tentative estimator for b,

$$\hat{b}_e = \frac{4\overline{x}_n - x_{(1)} - x_{(n)}}{2} \tag{53}$$

where

$$\overline{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{54}$$

Step 4: Identify $x_{(i_0)}$ from $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ such that $x_{(i_0)} \le \hat{b}_e < x_{(i_1)}$ and $1 < i_0 < i_1$, then we may see $\{x_{(1)}, x_{(2)}, \cdots, x_{(i_0)}\}$ as a set of order statistics from uniform [a,b]. Hence the "sufficient" statistic for parameter b is $x_{(i_0)}$.

Then $\left(\hat{a}, \hat{b}, \hat{c}\right) = \left(x_{(1)}, x_{(i_0)}, x_{(n)}\right)$ is the parameter estimator for the piecewise linear credibility distribution.

$$\hat{\Lambda}(x) = \begin{cases} 0 & x < \hat{a} \\ \dfrac{x-\hat{a}}{2\left(\hat{b}-\hat{a}\right)} & \hat{a} \le x < \hat{b} \\ \dfrac{x+\hat{c}-2\hat{b}}{2\left(\hat{c}-\hat{b}\right)} & \hat{b} \le x < \hat{c} \\ 1 & x \ge \hat{c} \end{cases} \tag{55}$$

The next issue is how to extract the information on matrix rate Q in the stationary random fuzzy continuous-time Markov chain. Basawa and Prakasa Rao (1980) developed a maximum likehood procedure for estimating the entries q_{ij} in Q.

It is noted that for a given random fuzzy continuous-time Markov chain $\xi = \{\xi_t, t \ge t\}$, if we fix the fuzzy rate matrix at a given value Q_0, then $\xi = \{\xi_t, t \ge t\}$ becomes a probabilistic continuous Markov chain, Obtain the sample of the process: $K_\tau = \left\{N_\tau, X(0), W_1, X(W_1 +), W_2, \cdots, W_{N_\tau} X\left(W_{N_\tau} +\right)\right\}$, which is sufficient. Then an MLE estimator for Q_0, denoted as $\hat{Q}_0$ is obtained. Repeat the sampling procedure from the random fuzzy continuous-time Markov chain as many times as possible, say, m times, then the fuzzy rate matrix "observation" sequence is

$$\left\{\hat{Q}_1, \hat{Q}_2, \cdots, \hat{Q}_m\right\} = \left\{\left(\hat{q}_{ij}^{(1)}\right), \left(q_{ij}^{(2)}\right), \cdots, \left(\hat{q}_{ij}^{(m)}\right)\right\} \tag{56}$$

Apply the Estimation Scheme 1 to the estimated observations at $(i, j)^{th}$ entry of rate matrix Q $\left\{\hat{q}_{ij}^{(1)}, q_{ij}^{(2)}, \cdots, \hat{q}_{ij}^{(m)}\right\}$, then the piecewise linear credibility distribution shown in Equation (55) for q_{ij}.

For the non-stationary random fuzzy continuous-time Markov chain, the parameters specifying the rate matrix $Q(t;\underline{\beta})$, we may use a maximum likelihood procedure for estimating the parameters that define fuzzy parameters $\underline{\beta}$. Therefore the idea is similar to that of stationary case but the credibility distribution treatments involved may be very complicated, since Zadeh's extension principle (1978) must be applied. And mean measure involves two linear piecewise credibility distributions for fuzzy parameters β_0 and β_1 respectively.

7 A SIMULATION SCHEME

Simulation of a random fuzzy continuous-time Markov chain is intrinsically two-stage procedure: a fuzzy parameter simulation for generating realizations $\left\{\left(q_{ij}^{(1)}\right),\left(q_{ij}^{(2)}\right),\cdots,\left(q_{ij}^{(m)}\right)\right\}$ from a matrix of credibility distribution functions $\left(\Lambda_{ij}\right)$ and then for each realization of $\left(q_{ij}\right)$, a probabilistic continuous-time Markov chain is simulated. Repeat this procedure until all the $\left(q_{ij}\right)$ realizations are complete.

As to the fuzzy parameter simulation, following Guo and Guo (2009), we utilize the concept of a maximally compatible random variable to a fuzzy variable and the inverse transformation of the probability distribution function approach to generate fuzzy variable realizations. An algorithm is stated as follows:

Simulation scheme 1:

Step 1: Simulate a uniform random variable U[0,1], and denote the simple random sample as $\{u_1,u_2,\cdots,u_n\}$;

Step 2: Set $\Lambda(x_i)=u_i,\ (k=1,2,\cdots,n)$;

Step 3: Set $x_i,\ (i=1,2,\cdots,n)$:

$$x_i=\begin{cases} a+2(b-a)u_i & \text{if } 0\le u_i\le 0.5 \\ 2b-c+2(c-b)u_i & \text{if } 0.5\le u_i\le 1 \end{cases} \tag{57}$$

Then $\{x_1,x_2,\cdots,x_n\}$ is a sample from the fuzzy variable ξ with a piecewise linear credibility distribution Λ.

Step 4: Repeat Step 1 to Step 3, until m realizations of fuzzy rate matrix $\{Q_1,Q_2,\cdots,Q_m\}$ are obtained.

Step 5: For each rate matrix, say, Q_i, simulate a probabilistic continuous-time Markov chain, until m set of realizations of random fuzzy continuous-time Markov chain are obtained.

It should be mentioned that simulating a probabilistic continuous-time Markov chain is well-established in the literature.

8 CONCLUSION

In this paper, we give a systematic treatment of random fuzzy continuous-time Markov chains not only for the stationary one, and then for the non-stationary case, but also propose a parameter estimation scheme and a simulation scheme. In this way, the foundation is provided for the random fuzzy continuous-time Markov chains, although in its early stage. The applications to reliability engineering fields and the risk analysis now can extend from case with only random uncertainty to case with both co-existing randomness and fuzziness. It is expected that this development will assist reliability and risk analysis researchers as well as reliability analysts and engineers.

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