ALGORITHMIC PROBLEMS IN DISCRETE TIME RISK MODEL

G.Tsitsiashvili

•

640041, Russia, Vladivostok, Radio str. 7, IAM, FEB RAS,
e-mail: guram@iam.dvo.ru,

ABSTRACT

In this paper we consider some algorithmic problems which appear in a calculation of a ruin probability in discrete time risk models with an interest force which creates stationary and reversible Markov chain. These problems are connected as with a generation of the Markov chain by its stationary distribution so with a calculation of the ruin probability.

Keywords: ruin probability, transportation problem, asymptotic formula, enumeration problem

1. INTRODUCTION

In this paper we consider some algorithmic problems which occur in a calculation of a ruin probability in discrete time risk model with an interest force which creates stationary and reversible Markov chain. Such model of the interest force is suggested by A.A. Novikov. Algorithmic problems are connected as with accuracy calculation of the ruin probability so when we deal with its asymptotic analysis. First numerical experiments show that without a solution of these algorithmic problems it is impossible to construct programs of numerical calculation of the ruin probability.

Markov chain generation reduces to a definition of permissible solutions of appropriate transportation problems. The ruin probability calculation is connected as with a definition of special sums of exponents so with a calculation of ruin probability asymptotic. These procedures also need to solve some auxiliary algorithmic problems: of convenient designations and symbolic calculations, some enumeration problem and so on. A specific of these problems is that primitive variants of their solution have very high complexity. Moreover some calculation procedures can not be realized without a solution of these problems.

2. PRELIMINARIES

Consider recurrent discrete time risk model (with annual step) with initial capital x, $x \ge 0$ and nonnegative losses Z_n, $n = 1, 2, \ldots$, $P(Z_n < t) = F(t)$:

$$S_0 = x, \; S_n = B_n S_{n-1} + A_n, \; n = 1, 2\ldots, \tag{1}$$

Here annual income A_n, $n = 1, 2, \ldots$ to end of n-th year is defined as difference between unit premium sum and loss $A_n = 1 - Z_n$. Assume that $B_n > 1$ is inflation factor from $n-1$ to n year, $n = 1, 2, \ldots$. In [1] $X_n = -A_n$ is called insurance risk and $Y_n = B_n^{-1}$ is called financial risk. In this model with initial capital x ruin time is defined by formula

$$\tau(x) = \inf\{n = 1, 2, \ldots : S_n \le 0 | S_0 = x\}$$

and finite time ruin probability $\psi(x,n)$ - by formula

$$\psi(x,n) = P(\tau(x) \le n).$$

So the sum S_n money accumulated by insurance company to *n*- th year end satisfies recurrent formula

$$S_0 = x \ , S_n = x\prod_{j=1}^{n} B_j + \sum_{i=1}^{n} A_i \prod_{j=i+1}^{n} B_j \ , \quad n = 1,2... , \tag{2}$$

where $\prod_{j=n+1}^{n} = 1$ by convention. According to the notation above, we can rewrite the discounted value of the surplus S_n in (2) as

$$\tilde{S}_0 = x \, , \qquad \tilde{S}_n = S_n \prod_{j=1}^{n} Y_j = x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j = x - W_n \, .$$

Hence, we easily understand that, for each $n = 0,1,...$,

$$\psi(x,n) = \mathrm{P}(U_n > x) \, , \ U_n = \max\left\{0, \max_{1 \le k \le n} W_k\right\}, \ U_0 = 0 \, . \tag{3}$$

Suppose that the sequence $\{Y_n, n \ge 0\}$ is stationary and reversible Markov chain with state set $\{r_q^{-1}, q \in Q\}$, $Q = \{1,...,m\}$ consisting of different positive numbers and transition matrix $\|\pi_{q',q}\|_{q',q \in Q}$. It means that the following formulas are true

$$P\left(Y_n = r_q^{-1}\right) = p_q \, , \ 0 < p_q \, , \ \sum_{q \in Q} p_q = 1 \, , \ p_{q'} \pi_{q'q} = p_q \pi_{qq'} \, , \ q, q' \in Q \, , \ n \ge 0$$

and consequently [2,Theorem 2.4]

$$(Y_1,...,Y_n) \overset{(d)}{=} (Y_n,...,Y_1) \, , \ n \ge 1 \, . \tag{4}$$

Assume that the random sequence $\{\omega_n, n \ge 0\}$ consists of independent and identically distributed random variables (i.i.d.r.v.\`s) with uniform distribution on interval [0,1]. Suppose that random sequences $\{Y_n, n \ge 0\}, \{\omega_n, n \ge 0\}$ are independent. Introduce distribution functions (d.f.\`s) $F_q, q \in Q$ and designate $F^{-1}(\omega)$, $0 \le \omega \le 1$, inverse function to distribution function $F(t)$, $-\infty < t < \infty$. Denote $Z_n = F_{Y_n}^{-1}(\omega_n)$, $n \ge 0$, then from the formula (4) we obtain the formula

$$((X_1,Y_1),...,(X_n,Y_n)) \overset{(d)}{=} ((X_n,Y_n),...,(X_1,Y_1)) \, , \ n \ge 1 \, . \tag{5}$$

In such a way it is possible to introduce dependence between financial and insurance risks provided financial risks create stationary and reversible Markov chain.
Define another random sequence

$$V_0 = 0 \, , \ V_n = Y_n \max(0, X_n + V_{n-1}) \, , \ n = 1,2,.... \tag{6}$$

Using recurrent formula (6) we introduce Markov chain (Y_n, V_n), $n = 1,2,...$ and designate

$$\psi_{n,q}(x) = P\left(Y_n = r_q^{-1}, V_n > x\right), \ q \in Q \, , \ x \ge 0 \, , \ n \ge 0 \, .$$

Theorem 1. The formula

$$\psi_n(x) = \sum_{q \in} \psi_{n,q}(x) \, , \ n = 0,1,... \, , \ x \ge 0 \, , \tag{7}$$

is true.

Proof . The result (7) is trivial for the case when $n=0$. Now we aim at (7) for each $n=1,2,\dots$. Let $n\geq 1$ be fixed. In view of the equality (5) we replace X_i and Y_j in U_n respectively by X_{n+1-i} and Y_{n+1-j} in deriving the following relations:

$$U_n=\max\left\{0,\max_{1\leq k\leq n}\sum_{i=1}^{k}X_i\prod_{j=1}^{i}Y_i\right\}\overset{(d)}{=}\max\left\{0,\max_{1\leq k\leq n}\sum_{i=1}^{k}X_{n+1-i}\prod_{j=1}^{i}Y_{n+1-j}\right\}=$$

$$=\max\left\{0,\max_{1\leq k\leq n}\sum_{i^*=n+1-k}^{n}X_{i^*}\prod_{j^*=i^*}^{n}Y_{j^*}\right\}=\max\left\{0,\max_{1\leq k^*\leq n}\sum_{i^*=k^*}^{n}X_{i^*}\prod_{j^*=i^*}^{n}Y_{j^*}\right\}. \quad (8)$$

If we write the right-hand side of (8) as $\tilde{V}_n$, which satisfies the recurrence equation

$$V_n^*=Y_n\max\left(0,X_n+V_{n-1}^*\right),\ n=1,\dots,$$

which is just the same as (6). So we immediately conclude that $V_n^*=V_n$ for each $n=1,\dots$. Finally, it follows from (8) that (7) holds for each $n=1,\dots$. This ends the proof of Theorem 1.

3. RECURRENT ALGORITHMES OF RUIN PROBABILITY CALCULATIONS

Introduce m -dimensional vectors $1_q=\left(\delta_{1,q},\dots,\delta_{m,q}\right)$ where $\delta_{i,j}$ is Kronecker symbol and

$$R=(r_1,\dots,r_m),\ K=(k_1,\dots,k_m),\ r_i>0,\ k_i\in\{0,1,\dots\},\ i=1,\dots,m,$$

and denote

$$R^K=\prod_{q\in Q}r_q^{k_q},\ |K|=\sum_{q\in Q}k_q.$$

Redefine the function $\exp(-t)$ so that for $t<0$ we have $\exp(-t)=1$ and for $t\geq 0$ this function is defined as usual. Introduce the function

$$E(t)=\begin{cases}1, t\leq 0,\\ 0, t>0.\end{cases}$$

Suppose that

$$\overline{F}_q(t)=\sum_{i=1}^{l}a_{q,i}\exp(-\lambda_i t),\ n\geq 1,\ t\geq 0,$$

with

$$-\infty<a_i<\infty,\ i=1,\dots,l,\ \sum_{i=1}^{l}a_{q,i}=1,\ q\in Q.$$

Theorem 2. Suppose that

$$R^K\lambda_i\neq\lambda_j, 1\leq i,\ j\leq l,\ 1\leq|K|. \quad (9)$$

Then there are real numbers $B_{n,i,q}^K$, $i=1,\dots,l$, $1\leq|K|\leq n$, which satisfy for $n\geq 1$, $i=1,\dots,l,$ initial conditions

$$B_{1,i,q}^{1_q}=p_q a_{q,i\exp(-\lambda_i)},\ B_{1,i,q}^{1_{q'}}=0, q,q'\notin Q,\ q\neq q'. \quad (10)$$

and recurrent formulas for $q\in Q$:

$$B_{n+1,i,q}^{1_q}=\sum_{q'\in Q}p_{q'}\pi_{q',q}\left[\sum_{1\leq|K|\leq n}\sum_{j=1}^{l}\frac{B_{n,j,q'}^K a_{q,i}}{R^K\lambda_j-\lambda_i}R^K\lambda_j\exp(-\lambda_i)+B_{n,q'}^0 a_{q,i}\exp(-\lambda_i)\right],\ q=q',$$

$$B_{n+1,i,q}^{1_{q'}} = 0\,,\; q \neq q'\,, \tag{11}$$

$$B_{n,j,q}^{K} = -\sum_{q'\in Q} p_{q'}\pi_{q',q} I\left(k_q > 0\right) \sum_{j=1}^{l} \frac{B_{n,j,q'}^{K-1_q} a_{q,j}\lambda_j}{R^{K-1_q}\lambda_i - \lambda_j} \exp\left(-R^{K-1_q}\lambda_i\right),\; 1 < |K| \le n+1\,, \tag{12}$$

so that

$$\psi_{s,q(t)} = \sum_{1\le|K|\le s} \sum_{i=1}^{l} B_{s,i,q}^{K} \exp\left(-R^{K}\lambda_i t\right) + B_{s,q}^{0} E(t)\,,\; s>0\,, \tag{13}$$

where

$$B_{s,q}^{0} = p_q - \sum_{1\le|K|\le s} \sum_{i=1}^{l} B_{s,i,q}^{K}\,. \tag{14}$$

Proof . If positive random variables ξ,η are independent and

$$P(\xi > t) = \exp(-\mu t),\; P(\eta > t) = \exp(-\lambda t)\,,\; \lambda,\mu > 0\,,\; \lambda \neq \mu\,,$$

then it is easy to obtain that

$$P(\xi + \eta > t) = \frac{\mu\exp(-\lambda t) - \lambda\exp(-\mu t)}{\mu - \lambda}\,. \tag{15}$$

Calculating for $t > 0$

$$P\left(Y_1 = r_q^{-1}, Y_1(Z_1 - 1) > t\right)\Big| = p_q P\left(F_q^{-1}(\omega_1) - 1 > R^{1_q} t\right) = p_q P\Big|\left(F_q^{-1}(\omega_1) >\Big| R^{1_q} t + 1\right) =$$

$$p_q \sum_{i=1}^{l} a_{q,i} \exp\left(-\lambda_i\left(R^{1_q} t + 1\right)\right) = p_q \sum_{i=1}^{l} a_{q,i} \exp(-\lambda_i)\exp\left(-\lambda_i R^{1_q} t\right)$$

we obtain that

$$\psi_{1,q}(t) = p_q \sum_{i=1}^{l} a_{q,i} \exp(-\lambda_i)\exp\left(-\lambda_i R^{1_q} t\right) + B_{1,q}^{0} E(t) =$$

$$= \sum_{i=1}^{l} B_{1,i,q}^{1_q} \exp\left(-\lambda_i R^{1_q} t\right) + B_{1,q}^{0} E(t)\,.$$

So the formula (13) is true for $s = 1$ with the initial conditions (10) and the equality (14). Suppose that the formula (13) takes place for $s = n$ and using the formula (15) calculate

$$\psi_{n+1,q}(t) = P\left(Y_{n+1} = r_q^{-1}, Y_{n+1}(V_n + Z_{n+1} - 1) > t\right) =$$

$$= P\left(Y_{n+1} = r_q^{-1}, V_n + F_q^{-1}(\omega_{n+1}) > r_q t + 1\right) = \sum_{q'\in Q} p_{q'}\pi_{q',q} P\left(Y_{n+1} = r_{q'}^{-1}, V_n + F_q^{-1}(\omega_{n+1}) > r_q t + 1\right).$$

As for $x > 0$

$$P\left(Y_n = r_{q'}^{-1}, V_n + F_q^{-1}(\omega_{n+1}) > x\right) =$$

$$= \sum_{1\le|K|\le n} \sum_{i=1}^{l} \sum_{j=1}^{l} \frac{B_{n,i,q'}^{K} a_{q',j}}{R^{K}\lambda_i - \lambda_j}\left(R^{K}\lambda_i \exp\left(-|\lambda_j x\right) - \lambda_j \exp\left(-R^{K}\lambda_i x\right)\right) + B_{n,q'}^{0} \sum_{i=1}^{l} a_{q,i} \exp(-\lambda_i x)\,,$$

so for $t > 0$

$$P\left(Y_n = r_{q'}^{-1}, V_n + F_q^{-1}(\omega_{n+1}) > r_q t + 1\right) = \sum_{1\le|K|\le n} \sum_{i=1}^{l} \sum_{j=1}^{l} \frac{B_{n,i,q'}^{K} a_{q,j}}{R^{K}\lambda_i - \lambda_j} A_{i,j,q,n}^{k}(t) +$$

$$+B^0_{n,q'}\sum_{i=1}^{l}a_{q,i}\exp(-\lambda_i x)\exp\left(-R^{1_q}\lambda_i t\right)$$

with

$$A^K_{i,j,q,n}(t)=R^K\lambda_i\exp\left(-\lambda_j\left(r_q t+1\right)\right)-\lambda_j\exp\left(-R^K\lambda_i\left(r_q t+1\right)\right)=$$

$$=R^K\lambda_i\exp\left(-\lambda_j\right)\exp\left(-\lambda_j r_q t\right)-\lambda_j\exp\left(-R^K\lambda_i\right)\exp\left(-R^K\lambda_i r_q t\right)=$$

$$=R^K\lambda_i\exp\left(-\lambda_j\right)\exp\left(-\lambda_j R^{1_q}t\right)-\lambda_j\exp\left(-R^K\lambda_i\right)\exp\left(-R^{K+1_q}\lambda_i t\right)=$$

Consequently we obtain for $t>0$

$$\psi_{n+1,q}(t)=\sum_{q'\in Q}\Big[p_{q'}\pi_{q',q}\sum_{1\le|K|\le n}\sum_{i=1}^{l}\sum_{j=1}^{l}\frac{B^K_{n,i,q'}a_{q,j}}{R^K\lambda_i-\lambda_j}\Big[R^K\lambda_i\exp\left(-\lambda_j\right)\exp\left(-\lambda_j R^{1_q}t\right)-$$

$$-\lambda_j\exp\left(-R^K\lambda_i\right)\exp\left(-R^{K+1_q}\lambda_i t\right)\Big]+B^0_{n,q'}\sum_{i=1}^{l}a_{q,i}\exp(-\lambda_i)\exp\left(-R^{1_q}\lambda_i t\right)\Big]=$$

$$=\sum_{q'\in Q}p_{q'}\pi_{q',q}\sum_{1\le|K|\le n}\sum_{i=1}^{l}\sum_{j=1}^{l}\frac{B^K_{n,i,q'}a_{q,i}}{R^K\lambda_j-\lambda_i}R^K\lambda_j\exp(-\lambda_i)\exp\left(-\lambda_i R^{1_q}t\right)-$$

$$-\sum_{q'\in Q}p_{q'}\pi_{q',q}\sum_{1\le|K|\le n}\sum_{i=1}^{l}\sum_{j=1}^{l}\frac{B^{K'}_{n,i,q'}a_{q,i}}{R^K\lambda_i-\lambda_j}\lambda_j\exp\left(-R^{K'}\lambda_i\right)\exp\left(-R^{K'+1_q}\lambda_i t\right)+$$

$$+\sum_{q'\in Q}p_{q'}\pi_{q',q}+B^0_{n,q'}\sum_{i=1}^{l}a_{q,i}\exp(-\lambda_i)\exp\left(-R^{1_q}\lambda_i t\right)\Big]=\sum_{1\le|K|\le n}\sum_{i=1}^{l}B^K_{n+1,i,q}\exp\left(-R^K\lambda_i t\right)=$$

$$=\sum_{q'\in Q}\sum_{i=1}^{l}B^{1_{q'}}_{n+1,i,q}\exp\left(-R^{1_{q'}}\lambda_i t\right)+\sum_{2\le|K|\le n+1}\sum_{i=1}^{l}B^K_{n+1,i,q}\exp\left(-R^K\lambda_i t\right).$$

So the formula (13) is true for $s=n+1$. Here for $i=1,...,l$, $1<|K|\le n+1$ we have the recurrent formulas (12) and for $i=1,...,l$, $|K|=1$ the recurrent formulas (11) and $B^0_{n+1,q}$ the equality (14).

4. ASYMPTOTIC FORMULAS

Using the complete probability formula we obtain

$$P(V_n>t)=\psi_n(t)=\sum_{q_1,...,q_n\in Q}\psi_{n,q_1,...,q_n}(t)\,p_{q_1}\pi_{q_1,q_2}\cdot...\cdot\pi_{q_{n-1},q_n} \qquad (16)$$

with

$$\psi_{n,q_1,...,q_n}(t)=P\left(V_n>t/Y_1=r^{-1}_{q_1},...,Y_n=r^{-1}_{q_n}\right).$$

(C) Suppose that $F_q(t)\in\mathcal{S}$, $q\in Q$, where $\mathcal{S}$ is the class of subexponential distributions. Assume that for any $q_1,q_2\in Q$, $q_1\ne q_2$ and for any positive a one of the following equalities is true

$$\overline{F}_{q_1}(t)=O\left(\overline{F}_{q_2}(at)\right)\text{ or }\overline{F}_{q_2}(at)=O\left(\overline{F}_{q_1}(t)\right),\ t>0. \qquad (17)$$

Then using [3, Lemma 3.2] it is possible to obtain that

$$F_{q_1}(t)*F_{q_2}(at)\in\mathcal{S}\ ,\ \overline{F_{q_1}(t)*F_{q_2}(at)}\sim\overline{F}_{q_1}(t)+\overline{F}_{q_2}(at)\Big|,\ t\to\infty. \qquad (18)$$

Here $F*G$ is a conjuncture of distributions F,G. Further we consider equivalences “~” only for $t\to\infty$.

Theorem 3. If the condition **(C)** is true then

$$\psi_n(t) \sim \sum_{k=1}^{n} \sum_{q_1,\dots,q_n \in Q} \overline{F}_{q_1}\left(t \prod_{i=1}^{n-k+1} r_{q_i}\right) p_{q_1}\pi_{q_1,q_2}\cdot\ldots\cdot\pi_{q_{n-k},q_{n-k+1}}. \quad (19)$$

Proof . It is obvious that

$$\psi_{1,q_1}(t) = P\left(Y_1X_1 > t / Y_1 = r_{q_1}^{-1}\right) = P\left(X_1 > r_{q_1}t\right) = \overline{F}_{q_1}\left(r_{q_1}t\right).$$

Using the condition **(C)** and the formula (18) we obtain for $n>1$ that

$$\psi_{n,q_1,\dots,q_n}(t) \sim P\left(V_{n-1} > r_{q_n}t / Y_1 = r_{q_1}^{-1},\dots,Y_n = r_{q_n}^{-1}\right) + P\left(X_n > r_{q_n}t / Y_1 = r_{q_1}^{-1},\dots,Y_n = r_{q_n}^{-1}\right) =$$

$$= \psi_{n-1,q_1,\dots,q_{n-1}}\left(r_{q_n}t\right) + \overline{F}_{q_n}\left(r_{q_n}t\right).$$

So an induction by n and the formula (16) give the equivalence

$$\psi_{n,q_1,\dots,q_n}(t) \sim \sum_{k=1}^{n} \overline{F}_{q_k}\left(t\prod_{i=k}^{n} r_{q_i}\right).$$

Consequently using the formula (16) it is easy to obtain the equivalence

$$\psi_n(t) \sim \sum_{q_1,\dots,q_n \in Q} p_{q_1}\pi_{q_1,q_2}\cdot\ldots\cdot\pi_{q_{n-1},q_n} \sum_{k=1}^{n} \overline{F}_{q_k}\left(t\prod_{i=k}^{n} r_{q_i}\right).$$

So the formula (19) is true.

Consider the following conditions.

1) There are positive numbers c_q, α_q, $q \in Q$, $\alpha_1 < \alpha_q$, $1 < q \le m$, so that $\overline{F_q}(t) \sim c_q t^{-\alpha_q}$.

2) There are positive numbers c_q, $q \in Q$, α, so that $\overline{F_q}(t) \sim c_q t^{-\alpha}$.

3) There are positive numbers c_q, β_q, $q \in Q$, $\beta_1 < \beta_q$, $1 < q \le m$, so that $\overline{F_q}(t) \sim \exp\left(-c_q t^{\beta_q}\right)$.

4) There are positive numbers c_q, $q \in Q$, β, so that $\overline{F_q}(t) \sim \exp\left(-c_q t^{\beta}\right)$ and $c_1 r_1^{\beta} < c_q r_q^{\beta}$, $q \in Q$, $q \ne 1$.

It is easy to prove that the family $F_q(t)$, $q \in Q$ under each of the conditions 1) - 4) satisfies the condition **(C)**.

Using Theorem 3 it is possible to obtain the following statements. If the condition 1) is true then

$$\psi_n(t) \sim c_1 p_1 (r_1 t)^{-\alpha_1} \sum_{k=1}^{n} S_{n-k+1},$$

with

$$S_1 = 1,\ S_2 = \sum_{q_2 \in Q} S_{2,q_2},\ S_{2,q_2} = \pi_{1,q_2} r_{q_2}^{-\alpha_1},$$

$$S_1 = \sum_{q_i \in Q} S_{i,q_i},\ S_{i,q_i} = \sum_{q_{i-1} \in Q} S_{i-1,q_{i-1}}\pi_{q_{i-1},q_i} r_{q_i}^{-\alpha_1},\ 2 < i. \quad (20)$$

If the condition 2) is true then

$$\psi_n(t) \sim t^{-\alpha} \sum_{k=1}^{n} T_{n-k+1}$$

with

$$T_1 = \sum_{q_1,q_1 \in Q} T_{1,q_1,q_1},\ T_{1,q_1,q_1} = c_{q_1} p_{q_1} r_{q_1}^{-\alpha},$$

$$T_i = \sum_{q_1,q_i \in Q} T_{i,q_1,q_i},\ T_{i,q_1,q_i} = \sum_{q_1,q_{i-1} \in Q} T_{i-1,q_1,q_{i-1}}\pi_{q_{i-1},q_i} r_{q_i}^{-\alpha},\ 1 < i. \quad (21)$$

The formulas (20), (21) show that to find asymptotic constants in the conditions 1), 2) it is necessary to use number of arithmetical operations proportional to n. If the condition 3) is true then

$$\psi_n(t) \sim p_1 \exp\left(-c_1 (r_1 t)^{-\beta_1}\right).$$

If the condition 4) is true then

$$\psi_n(t) \sim p_1 \exp\left(-c_1 (r_1 t)^{\beta}\right).$$

5. GENERATION OF TRANSITION MATRICES FOR STATIONARY AND REVERSIBLE MARKOV CHAINS

Consider stationary and reversible Markov chain Y_n, $n| \ge 0$, with stationary distribution p_q, $q \in Q$. Then its transition matrix $\|\pi_{i,q}\|_{i,q=1}^{m}$ satisfies the equalities:

$$A_{i,j} = A_{j,i} > 0\,,\ \sum_{j=1}^{m} A_{i,j} = p_i = \sum_{j=1}^{m} A_{j,i}\,,\ 1 \le i, j \le m\ . \tag{22}$$

where $A_{i,j} = p_i \pi_{i,j}$. So symmetric matrix $\|A_{i,j}\|_{i,j=1}^{m}$ with positive elements is a permissible solution of the transportation problem (22) with n sources and n consumers. If we have the problem (22) solution $\|A_{i,j}\|_{i,j=1}^{m}$ then it is possible to find the transition matrix $\|\pi_{i,j}\|_{i,j=1}^{m}$ using the formula $\pi_{i,j} = A_{i,j} / p_i$.

Each permissible solution of the transportation problem (2) may be found by the following sequence of algorithms.

The algorithm $\{p_1,...,p_m\}$ generates $A_{1,1},...,A_{1,m}$ so that

$$0 < A_{1,1} < p_1, ..., 0 < A_{1,m} < p_m\,,\ \sum_{k=1}^{m} A_{1,k} = p_1$$

and put $A_{2,1} = A_{1,2}, ..., A_{m,1} = A_{1,m}$ and redefines $p_1 := p_1 - p_1 = 0$, $p_2 := p_2 - A_{1,2}, ..., p_m := p_m - A_{1,m}$.

As a result the transportation problem (2) with n sources and n consumers is transformed into the transportation problem

$$A_{i,j} = A_{j,i} > 0\,,\ \sum_{j=2}^{m} A_{i,j} = p_j\,,\ 2 \le i, j \le m\ . \tag{23}$$

with $n-1$ sources and $n-1$ consumers. So the algorithm $\{p_1,...,p_m\},\{p_2,...,p_m\},...,\{p_{m-1},...,p_m\}$ generates arbitrary solution of the transportation problem (22).

The algorithm $\{p_1,...,p_m\}$ consists of m steps.

Step 1. Define $A_{1,1}$ from the inequalities $0 < A_{1,1} < p_1$, $p_1 - A_{1,1} < p_2 + ... + p_m$ and put $p_1 := p_1 - A_{1,1}$.

Step 2. Define $A_{1,2}$ from the inequalities $0 < A_{1,2} < p_1$, $0 < A_{1,2} < p_2$, $p_1 - A_{1,2} < p_3 + ... + p_m$ and put $p_1 := p_1 - A_{1,2}$.

Step m-1. Define $A_{1,m-1}$ from the inequalities $0 < A_{1,m-1} < p_1$, $0 < A_{1,m-1} < p_{m-1}$, $p_1 - A_{1,m-1} < p_m$ and put $p_1 := p_1 - A_{1,m-1}$.

Step m. Define $A_{1,m-1} = p_1$

The algorithm $\{p_1,...,p_m\},\{p_2,...,p_m\},...,\{p_{m-1},...,p_m\}$ is a modification of the algorithm for constructing the routing matrix for an open network [4, p.177].

6. ENUMERATION PROBLEM

Assume that vector K consists of components 0,1,… and its dimension is dim $\dim K$. Introduce the sets of vectors $K_i^j = \{K : \dim K = j, |K| = i\}$ and designate $|K_i^j|$ number of vectors in the set $K_i^j, i \geq 0, j \geq 1$. Our purpose is to enumerate all vectors K of the set

$$\{K : \dim K = m, 1 \leq |K| \leq n\} = \bigcup_{i=1}^{n} K_i^m .$$

It is easy to construct algorithm to define the set K_i^m from the set K_{i-1}^m as follows: $K_i^m = \{K + 1_q : K \in K_{i-1}^m, q = 1,...,m\}$. But a complexity of this algorithm is proportional to m^n and it may generate coinciding vectors. So it is worthy to construct more efficient algorithm for example with power by n complexity.

It is obvious that K_i^{j+1} is a union of nonintersecting sets

$$K_i^{j+1} = \bigcup_{k_{j+1}=0}^{i} \{(K, k_{j+1}) : K \in K_{i-k_{j+1}}^j\} \tag{24}$$

and consequently

$$|K_i^{j+1}| = \sum_{t=0}^{i} |K_{i-t}^j| = \sum_{t=0}^{i} |K_t^j| \text{ where } j \geq 1, i \geq 0. \tag{25}$$

Here K_0^j consists of single j- dimensional vector with zero components, $|K_0^j| = 1$ and $K_0^j \leq K_1^j \leq ... \leq K_i^j$. From the formula (25) we have by induction that

$$|K_i^{j+1}| = \sum_{t=0}^{i} |K_t^j| \leq (i+1) K_i^j \leq (i+1)^{j+1}. \tag{26}$$

As K_i^1 consists of single one dimensional vector i so to find the set K_n^m using the formula (24) we construct the sequence of the sets

$$K_0^1, K_0^2, ..., K_0^m;$$
$$K_1^1, K_1^2, ..., K_1^m;$$
$$......................;$$
$$K_n^1, K_n^2, ..., K_n^m.;$$

Consequently complexity of this algorithm is not larger than $(n+1)^{m+1}$.

7. SOLUTION OF SMALL DENOMINATORS PROBLEM

Consider how to find $s_1,...,s_m,\nu_1,...,\nu_l$ so that for any fixed $\varepsilon > 0$ the inequalities

$$|s_1 - r_1| < \varepsilon, ..., |s_m - r_m| < \varepsilon, |\nu_1 - \lambda_1| < \varepsilon, ..., |\nu_l - \lambda_l| < \varepsilon \tag{27}$$

are true and for any K we have an analogy of the condition (9):

$$S^K \nu_i \neq \nu_j \ 1 \leq i, j \leq l, 1 \leq |K|, \tag{28}$$

where $S = (s_1,...,s_m)$. For this aim we take integer b so that $2^b < \varepsilon$ and choose fractions

$s_i = \frac{v_i}{2^b}, \ i = 1,...,m, \ \nu_j = \frac{u_j}{2^b}, \ j = 1,...,l$, with odd numerators v_i so that the formula (27) is true. Then the formula (28) takes place.

REFERENCES

1. Norberg R. 1999. Ruin problems with assets and liabilities of diffusion type. *Stochastic Process. Appl.* Vol. 81. No. 2.
2. Serfozo R. Introduction to Stochastic Networks. *Springer Verlag.* New York. 1999.
3. Tang Q.; Tsitsiashvili G. 2003. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stochast.Process. Appl.* Vol. 108, no. 2, 299-325.
4. Tsitsiashvili G.Sh., Osipova M.A. New Product Theorems for Queuing Networks// *Problems of Information Transmission.* 2005. Vol. 41. No. 2. P. 171-181.