ACCURACY FORMULAS OF RUIN PROBABILITY CALCULATIONS FOR DISCRETE TIME RISK MODEL WITH DEPENDENCE OF FINANCIAL AND INSURANCE RISKS

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ABSTRACT

In this article discrete time risk model with heavy tailed losses distribution and dependence between financial and insurance risks is considered. It is shown that known asymptotic formulas work with good accuracy for sufficiently large arguments. Direct methods based on calculation of ruin probability by solution of appropriate integral equations demand large volumes of calculations and so work for sufficiently small arguments. Fast and accuracy algorithms, based on approximation of loss distribution by mixture of exponential ones, to calculate ruin probability in this interval are developed. This approximation of considered model is based on continuity theorems and analog of Bernstein theorem in L_1 metrics.

INTRODUCTION

Discrete time risk model with dependence between financial and insurance risks is considered. In modern period of strong economical crisis such dependence may be recognized easily in different large anthropogenic catastrophes. So a problem to analyze asymptotically this dependence influence on the ruin probability is actual now.

This problem is discussed in risk theory and in queueing theory (Asmussen & Bladt 1996, Asmussen 2000, Feldmann & Whitt 1998, Dufresne 2005, Albrecher, Teugels & Tichy 2001). For heavy tailed distributions of losses it is shown that known asymptotic formulas (Embrechts & Veraverbeke 1982, Embrechts, Kluppelberg & Mikosch 1997, Tang 2004) work with good accuracy for sufficiently large arguments (Asmussen 2000, Kalashnikov 1997, Kalashnikov 1999). Direct methods based on calculation of ruin probability by solution of appropriate integral equations demand large volumes of calculations (Skvarnik 2004) and so work for sufficiently small arguments. As a result an interval of mean arguments appears. This interval is interesting practically and in it asymptotic formulas still do not work and direct methods already do not work. So it is interesting to process sufficiently fast and accuracy algorithms to calculate ruin probability in this interval.

To solve this problem an analogy with the queueing theory with an approximation of heavy tailed distribution by a mixture of exponential ones is used. It is known that waiting times in an one server queueing system which creates the Lindley chain coincide by a distribution with maximums of sequential sums and are continuous for fluctuations of distributions (Borovkov 1971, Zolotarev 1976). Analogously in discrete time risk model with dependent financial and insurance risks a finite interval ruin probability coincides with tail of distribution of some Markov chain.

In this paper finite interval ruin probability is represented by sum of exponents with unknown coefficients. To find these coefficients some recurrent procedure is suggested. It allows to consider risk model with exponential, Pareto, Weibull and some other loss distributions. We consider special model of insurance and financial risks dependence based on suggestion that a financial risk has a finite number of meanings and for each meaning an insurance risk has its own distribution.

Asymptotic formulas for the ruin probability in a case of independent financial and insurance risks have been obtained in (Tang & Tsitsiashvili 2003 (Stochastic Processes Applied)). More complicated cases with special restrictions on insurance risks dependence are considered for example in (Tang & Tsitsiashvili 2003 (Extremes), Tang & Wang 2010). First asymptotic formulas for dependent financial and insurance risks are obtained in (Tsitsiashvili 2010).

1 PRELIMINARIES

Consider recurrent discrete time risk model (with annual step) with initial capital $x, x \geq 0,$ and nonnegative losses $Z_n, n = 1,2..., P(Z_n < t) = F(t)$:

$$S_0 = x,\ S_n = B_n\ S_{n\text{-}1} + A_n,\ n = 1,2,... \tag{1}$$

Here income $A_n, n = 1,2,...,$ to end of n-th year is defined as difference between unit premium sum and loss $A_n = 1 - Z_n$. Assume that $B_n > 1$ is inflation factor from $n-1$ to n year, $n = 1,2,...$ In (Norberg 1999) $X_n = \text{-}A_n$ is called insurance risk and $Y_n = B_n^{-1}$ is called financial risk. Suppose that $\{(A_n, B_n),\ n \geq 1\}$ is sequence of independent and identically distributed random vectors (i.i.d.r.v.`s). In this model with initial capital x ruin time is defined by formula

$$\tau(x) = \inf\ \{n = 1,2,...:\ S_n \leq 0 \mid\ S_0 = x\}$$

and finite time ruin probability by formula

$$\psi_n(x) = P(\tau(x) \leq n).$$

So the sum S_n money accumulated by insurance company to n-th year end satisfies recurrent formula

$$S_0 = x,\ S_n = x\prod_{j=1}^{n} B_j + \sum_{i=1}^{n} A_i \prod_{j=i+1}^{n} B_j,\ n = 1,2,..., \tag{2}$$

where $\prod_{j=n+1}^{n} B_j = 1$ by convention. According to the notation above we can rewrite the discounted value of the surplus S_n in (2) as

$$\tilde{S}_0 = x,\ \tilde{S}_n = S_n \prod_{j=1}^{n} Y_j = x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j = x - W_n.$$

Hence we easily understand that for each n=0,1,...

$$\psi_n(x) = P(U_n > x),\ U_n = \max\left\{0, \max_{1\leq k\leq n} W_k\right\},\ U_0 = 0. \tag{3}$$

Define another Markov chain as

$$V_0 = 0,\ V_n = Y_n \max(0, X_n + V_{n-1}),\ n = 1,2,... \tag{4}$$

In (Tang & Tsitsiashvili 2003 (Stochastic Processes Applied), Tsitsiashvili 2010) the following statement is proved.

Theorem 1. The formula $\psi_n(x) = P(V_n > x),\ n = 1,2,...$ is true.

Suppose that $Q = \{1,...,m\}$ and introduce m-dimensional vectors $1_q = \{\delta_{1q},...,\delta_{mq}\}$ where δ_{ij} is Kroneker symbol and

$$R = (r_1,...,r_m),\ K = (k_1,...,k_m),\ r_i > 0,\ k_i \in \{0,1,...\},\ i = 1,...,m,$$

and denote

$$R^K = \prod_{q\in Q} r_q^{k_q},\ |K| = \sum_{q\in Q} k_q.$$

Redefine the function e^{-t} so that for $t<0$ we have $e^{-t}=1$ and for $t\geq 0$ this function is defined as usual. Introduce the function $E(t)=\begin{cases}0, & t>0,\\ 1, & t\leq 0.\end{cases}$ Suppose that i.i.d.r.vectors $(Y_n, Z_n),\ n\geq 1,$ have following distributions

$$(Y_n^{-1}=r_q, Z_n=Z_n^q)=p_q,\ P(Z_n^q>t)=\overline{F}_q(t),\ n\geq 1,\ q\in Q.$$

Consider disturbed Markov chain $\tilde{V}_n,\ n\geq 0,$ so that

$$\tilde{V}_0=0,\ \tilde{V}_n=Y_n\max(0, \tilde{X}_n+\tilde{V}_{n-1}),\ n=1,2,... \tag{5}$$

have following distributions

$$(Y_n^{-1}=r_q, \tilde{Z}_n=\tilde{Z}_n^q)=p_q,\ P(\tilde{Z}_n^q>t)=\overline{G}_q(t),\ n\geq 1,\ q\in Q.$$

Denote $\varphi_n(x)=P(\tilde{V}_n>x).$

2 RECURRENT ALGORITHMS OF RUIN PROBABILITY CALCULATIONS

Theorem 2. Suppose that there are real numbers $a_{qi},\ q\in Q,\ i=1,...,l,$ and $p_q>0,\ q\in Q,$ $\sum_{q\in Q}p_q=1,$ so that $\overline{G}_q(t)=\sum_{i=1}^{l}a_{qi}\exp(-\lambda_i t),\ q\in Q, n\geq 1,$ and

$$R^K\lambda_i\neq\lambda_j,\ 1\leq i,j\leq l,\ |K|\geq 1. \tag{6}$$

Then there are real numbers $B_{n,i}^K,\ i=1,...,l,\ 1\leq |K|\leq n,$ which satisfy for $n\geq 1, i=1,...,l$ initial conditions

$$B_{1,i}^{1_q}=p_q\ a_{qi}\exp(-\lambda_i),\ q\in Q, \tag{7}$$

and recurrent formulas

$$B_{n+1,i}^{1_q}=p_q\sum_{1\leq|K|\leq n}\sum_{j=1}^{l}\frac{B_{n,j}^K\,a_{qi}\,R^K\lambda_j}{R^K\lambda_j-\lambda_i}\exp(-\lambda_i)+p_qB_n^0a_{qi}\exp(-\lambda_i),\ q\in Q, \tag{8}$$

$$B_{n+1,i}^K=-\sum_{q\in Q}I(k_q>0)p_q\sum_{j=1}^{l}\frac{B_{n,i}^{K-1_q}\,a_{qj}\,\lambda_j}{R^{K-1_q}\lambda_i-\lambda_j}\exp(-R^{K-1_q}\lambda_i),\ 1<|K|\leq n+1, \tag{9}$$

so that

$$\varphi_s(t)=\sum_{1\leq|K|\leq s}\sum_{i=1}^{l}B_{s,i}^K\exp(-R^K\lambda_i\,t)\ +B_s^0E(t),\ s>0, \tag{10}$$

where

$$B_s^0=1-\sum_{1\leq|K|\leq s}\sum_{i=1}^{l}B_{s,i}^K\ . \tag{11}$$

Proof. If random variables ξ,η are independent and

$$P(\xi>t)=\backslash\exp(-\mu t), P(\eta>t)=\backslash\exp(-\lambda t),\ \lambda,\mu>0,\ \lambda\neq\mu,$$

then it is easy to obtain that

$$P(\xi+\eta>t)=\frac{\mu\exp(-\lambda t)-\lambda\exp(-\mu t)}{\mu-\lambda}. \tag{12}$$

Calculating

$$P(Y_1(\tilde{Z}_1-1)>t)=\sum_{q\in Q}p_q\ P(\tilde{Z}_1^q-1>R^{1_q}t)=\sum_{q\in Q}p_q\ P(\tilde{Z}_1^q>R^{1_q}t+1)=$$

$$=\sum_{q\in Q} p_q \sum_{i=1}^{l} a_{qi}\exp(-\lambda_i(R^{1_q}t+1))=\sum_{q\in Q} p_q \sum_{i=1}^{l} a_{qi}\exp(-\lambda_i)\exp(-\lambda_i R^{1_q}t)$$

we obtain that

$$\varphi_1(t)=\sum_{q\in Q} p_q \sum_{i=1}^{l} a_{qi}\exp(-\lambda_i)\exp(-\lambda_i R^{1_q}t)+B_1^0E(t)=\sum_{q\in Q}\sum_{i=1}^{l} B_{1,i}^{1_q}\exp(-\lambda_i R^{1_q}t)+B_1^0E(t).$$

So the formula (10) is true for s=1 with the initial conditions (7) and the equality (11).

Suppose that the formula (10) takes place for $s=n$ and using the formula (12) calculate $P(Y_{n+1}(\tilde{V}_n+\tilde{Z}_{n+1}-1)>t)=\sum_{q\in Q} p_q P(\tilde{V}_n+\tilde{Z}^q_{n+1}>r_q t+1).$ As

$$P(\tilde{V}_n+\tilde{Z}^q_{n+1}>x)=\sum_{1\le|K|\le n}\sum_{i=1}^{l}\sum_{j=1}^{l}\frac{B_{n,i}^K a_{qj}}{R^K\lambda_i-\lambda_j}(R^K\lambda_i\exp(-\lambda_j x)-\lambda_j\exp(-R^K\lambda_i x))+B_n^0\sum_{i=1}^{l} a_{qi}\exp(-\lambda_i x)$$

so

$$P(\tilde{V}_n+\tilde{Z}^q_{n+1}>r_q t+1)=\sum_{1\le|K|\le n}\sum_{i=1}^{l}\sum_{j=1}^{l}\frac{B_{n,i}^K a_{qj}}{R^K\lambda_i-\lambda_j}[R^K\lambda_i\exp(-\lambda_j)\exp(-\lambda_j R^{1_q}t)-\lambda_j\exp(-R^K\lambda_i)\exp(-R^{K+1_q}\lambda_i t)]+$$

$$+B_n^0\sum_{i=1}^{l} a_{qi}\exp(-\lambda_i)\exp(-\lambda_i R^{1_q}t).$$

Consequently we obtain

$$\varphi_{n+1}(t)=\sum_{q\in Q} p_q\sum_{1\le|K|\le n}\sum_{i=1}^{l}\sum_{j=1}^{l}\frac{B_{n,i}^K a_{qj}}{R^K\lambda_i-\lambda_j}[R^K\lambda_i\exp(-\lambda_j)\exp(-\lambda_j R^{1_q}t)-\lambda_j\exp(-R^K\lambda_i)\exp(-R^{K+1_q}\lambda_i t)]+$$

$$+\sum_{q\in Q} p_q B_n^0\sum_{i=1}^{l} a_{qi}\exp(-\lambda_i)\exp(-\lambda_i R^{1_q}t)+B_{n+1}^0E(t)=$$

$$=\sum_{q\in Q} p_q\sum_{i=1}^{l}\sum_{1\le|K|\le n}\sum_{j=1}^{l}\frac{B_{n,j}^K a_{qi}}{R^K\lambda_j-\lambda_i}R^K\lambda_j\exp(-\lambda_i)\exp(-\lambda_i R^{1_q}t)-$$

$$-\sum_{q\in Q} p_q\sum_{i=1}^{l}\sum_{1\le|K'|\le n}\sum_{j=1}^{l}\frac{B_{n,i}^{K'} a_{qj}}{R^{K'}\lambda_i-\lambda_j}\lambda_j\exp(-R^{K'}\lambda_i)\exp(-R^{K'+1_q}\lambda_i t)]+$$

$$+\sum_{q\in Q} p_q B_n^0\sum_{i=1}^{l} a_{qi}\exp(-\lambda_i)\exp(-\lambda_i R^{1_q}t)+B_{n+1}^0E(t)=\sum_{1\le|K|\le n+1}\sum_{i=1}^{l} B_{n+1,i}^K\exp(-R^K\lambda_i t)+B_{n+1}^0E(t)=$$

$$=\sum_{q\in Q}\sum_{i=1}^{l} B_{n+1,i}^{1_q}\exp(-R^{1_q}\lambda_i t)+\sum_{2\le|K|\le n+1}\sum_{i=1}^{l} B_{n+1,i}^K\exp(-R^K\lambda_i t)+B_{n+1}^0E(t).$$

So the formula (10) is true for $s=n+1$. Here for $i=1,...,l,\ 1<|K|\le n+1$ we have the recurrent formulas (9) and for $i=1,...,l,\ |K|=1$ the recurrent formulas (8) and for B_{n+1}^0 the equality (11). The theorem is proved.

3 CONTINUITY OF RISK MODEL IN L$_1$ METRICS

In the sequel we assume that

$$(Y_n,Z_n)=\sum_{q=1}^{m} I(i_n=q)(r_q^{-1},\,F_q^{-1}(\omega_n))\ ,\ (Y_n,\tilde{Z}_n)=\sum_{q=1}^{m} I(i_n=q)(r_q^{-1},\,G_q^{-1}(\omega_n)).\qquad(13)$$

Here i.i.d.r.v\`s $\omega_n, n\ge 1,$ are uniformly distributed on interval [0, 1], $F^{-1}(\omega), 0\le\omega\le 1,$ is inverse to d.f. $F(t)$ function. Then using uniform metrics

$$\rho(F,G)=\sup_{x\ge 0}|F(x)-G(x)|$$

and results on stability of queueing systems (Zolotarev 1976) it is simple to obtain following statement.

Theorem 3. For fixed $n \geq 0$ inequality

$$\rho(\psi_n, \varphi_n) \leq n\rho(F, G) \tag{14}$$

is true.

Say that distribution density $f(t)$ concentrated on $[0, \infty)$, is absolutely monotone if it has derivatives of all orders and $(-1)^k f^{(k)}(t) \geq 0$ for all $t > 0$ and $k \geq 1$. Example of such distribution is Pareto distribution satisfying equality $\overline{F}(x) = (1+bx)^{-\alpha}$, $x > 0$. From Bernstein theorem (Feldmann 998) it is known that for d.f. F with absolutely monotone density there is sequence of d.f.`s represented as sums of exponents

$$F_s(x) = \sum_{i=1}^{l_s} p_{si}(1 - \exp(-\lambda_{si}x)) \ , \ x \geq 0, \ s > 0,$$

where $0 < \lambda_{si}, p_{si} < \infty, p_{s1} + ... + p_{sl_s} = 1$ and $\rho(F, F_s) \to 0, \ s \to \infty.$

Theorem 3 and Bernstein theorem allow to construct approximative algorithm for a calculation of ruin probability. But linear by n upper bound in (14) is not convenient for this aim. So we begin to reformulate these results in terms of L_1 metrics. Denote $EY_n = a^{-1}$ and introduce the metrics $L_1(F, G)$ between d.f.`s *F, G* as follows

$$L_1(F, G) = \int_{-\infty}^{\infty} | F(t) - G(t) | \, dt. \tag{15}$$

Theorem 4. If $\delta = \max_{q \in Q} L_1(F_q, G_q)$ and a >1 then

$$L_1(\varphi_n, \psi_n) \leq \frac{\delta}{a - 1}. \tag{16}$$

Proof. From the formulas (4), (5) we have that $E | V_0 - \tilde{V}_0 | = 0$ and

$$E | V_n - \tilde{V}_n | = E \sum_{q \in Q} I(i_n = q) r_q^{-1} | \max(0, \ V_{n-1} + F_q^{-1}(\omega_n) - 1) - \max(0, \ \tilde{V}_{n-1} + G_q^{-1}(\omega_n) - 1) | \leq$$

$$\leq E \sum_{q \in Q} I(i_n = q) r_q^{-1} (| V_{n-1} - \tilde{V}_{n-1} | + | F_q^{-1}(\omega_n) - G_q^{-1}(\omega_n) |) =$$

$$= \sum_{q \in Q} r_q^{-1} p_q (E | V_{n-1} - \tilde{V}_{n-1} | + E | F_q^{-1}(\omega_n) - G_q^{-1}(\omega_n) |) = \frac{E | V_{n-1} - \tilde{V}_{n-1} | + \delta}{a}, \ n \leq 1.$$

Consequently an induction by n gives the formula

$$E | V_n - \tilde{V}_n | \leq \delta \sum_{k=1}^{n} a^{-k} \leq \frac{\delta}{a - 1}.$$

As the minimum of the complex probability metrics $E | V_n - \tilde{V}_n |$ by all joint distributions which conserve marginal distributions of r.v.`s $V_n, \tilde{V}_n$ is $L_1(\varphi_n, \psi_n)$ (Zolotarev 1976) so from Theorem 1 we obtain the inequality (16). The theorem is proved.

It is easy to obtain from (Tsitsiashvili 2004, Kalashnikov & Rachev 1988) that in conditions

$$a > 1, \ \max_{g \in Q} \int_0^{\infty} \overline{F}_q(t) dt = C < \infty \tag{17}$$

there is nonincreasing function $\psi(t)$ so that $\psi(0) = 1, \ \psi(t) \to 0, t \to \infty$ and

$$\lim_{n \to \infty} \psi_n(t) = \psi(t), \ t \geq 0.$$

Indeed from Theorem 1 and the formula (3) the sequence $\psi_n(t), \ n \geq 0,$ satisfies the inequalities $\psi_{n+1}(t) \geq \psi_n(t), \ n \geq 0,$ and so it has the limit $\psi(t)$. Choosing r.v. V_∞ so that $P(V_\infty > t) = \psi(t),$

$t \geq 0,$ and applying Theorem 4 proof to the sequence $EV_n,\ n > 0,$ it is possible to obtain the inequality

$$EV_\infty \leq \frac{C}{a-1} < \infty$$

and consequently $\psi(t) \to 0, t \to \infty.$

Theorem 5. If the conditions (17) are true then

$$L_1(\psi, \psi_n) \leq \frac{C}{(a-1)a^{n-1}},\ n > 0. \tag{18}$$

Proof. For n=1 the formula (18) is true. Prove the formula (18) using an induction by n. Suppose that (18) takes place for some $n > 0$. Introduce the following joint distribution of r.v.`s $V_n, V_\infty$ which conserves their marginal distributions $V_n = \psi_n^{-1}(\omega), V_\infty = \psi^{-1}(\omega).$ Here r.v $\omega$ is independent on r.v.`s V_n, V_∞ and has uniform distribution on the interval [0, 1] so $E\,|V_n - V_\infty| = L_1(\psi_n, \psi).$ Then for r.v.`s $Z_{n+1}, Y_{n+1}$ independent on r.v.`s V_n, V_∞ we have the equalities

$$Y_{n+1}\max(0, V_n + Z_{n+1} - 1) = V_{n+1},\ Y_{n+1}\max(0, V_\infty + Z_{n+1} - 1) \overset{(d)}{=} V_\infty.$$

So from minimal property of metrics L_1 we obtain using mathematical induction by n that

$$L_1(\psi_{n+1}, \psi) \leq E\,|V_{n+1} - V_\infty| = E\,|Y_{n+1}\max(0, V_n + Z_{n+1} - 1) - Y_{n+1}\max(0, V_\infty + Z_{n+1} - 1)| \leq$$

$$\leq EY_{n+1}E\,|V_n - V_\infty| = a^{-1}E\,|V_n - V_\infty| \ \leq a^{-n}E\,|V_1 - V_\infty| \leq \frac{EV_\infty}{a^n}.$$

The formula (18) is true. The theorem is proved.

Denote $n(\varepsilon) = \inf(n : L_1(\psi, \psi_n) < \varepsilon)$ then from Theorem 5 we have the inequality

$$n(\varepsilon) \leq \inf(n : R^{-n+1}L_1(\psi, \psi_1) < \varepsilon)$$

and so

$$n(\varepsilon) \leq 2 + \frac{\ln L_1(\psi, \psi_1) - \ln \varepsilon}{\ln R} = n_1(\varepsilon). \tag{19}$$

Remark 1. The formula (19) allows to establish that if $L_1(\psi, \psi_{n(\varepsilon)}) < \varepsilon$ then it is enough to find $\psi_n,\ 1 \leq n \leq n_1(\varepsilon).$ From Theorem 2 we obtain that to calculate $\varphi_n(t)$ it is necessary $O(n^{m+1})$ arithmetical operations for $n \to \infty$. So to find $\varphi_{n(\varepsilon)}(t)$ we need $O(|\ln \varepsilon|^{m+1})$ arithmetical operations for $\varepsilon \to 0.$

Suppose that the condition (6) of Theorem 2 is not true then it is possible to approximate $G_q(t), q \in Q,$ in metrics L_1 so that the condition becomes true. We formulate this statement in the following way.

4 SMALL DENOMINATORS PROBLEM

Suppose that the condition (6) of Theorem 2 is not true and so we deal with zero denominators in recurrent formulas (8), (9). Then it is possible to approximate $G_q(t),\ q \in Q,$ in metrics L_1 so that the condition becomes true. We formulate this statement in the following way.

Theorem 6. Assume that for some $\delta > 0$ positive numbers $\lambda_1, ..., \lambda_l$ satisfy the condition

$$|\lambda_i - \lambda_j| > 3\delta,\ 1 \leq i \neq j \leq l.$$

Suppose that $r_q = t_q / T$ where t_q, T are coprimes and T is prime, $q \in Q$. Then for any $\varepsilon > 0$ there are positive and rational numbers $\tilde{\lambda}_1, ..., \tilde{\lambda}_l$ so that

$$|\lambda_i - \tilde{\lambda}_i| < \varepsilon,\ \tilde{\lambda} \neq R^K \tilde{\lambda}_j,\ 1 \le i \ne j \le l, |K| > 0.$$

Proof. Fix $\varepsilon,\ 0 < \varepsilon < \delta.$ There are integers $N,\ s_1,...,s_l$ so that

$$\frac{1}{NT} < \frac{\varepsilon}{2},\ \left|\lambda_i - \frac{s_i}{N}\right| < \frac{\varepsilon}{2}.$$

Choose $\tilde{\lambda}_i = (s_i T + 1)/NT$ then $|\lambda_i - \tilde{\lambda}_i| < \varepsilon$ and $s_i T + 1, T$ are coprimes, $1 \le i \le l$, so rational number

$$\prod_{q \in Q} t_q^{k_q} \frac{s_i T + 1}{T^{|K|}}$$

can not be integer and consequently

$$s_i T + 1 \neq \prod_{q \in Q} t_q^{k_q} \frac{s_i T + 1}{T^{|K|}},\ 1 \le i \ne j \le l, |K| > 0, K = (k_1,...,k_m).$$

The theorem is proved.

Remark 2. Fix $\varepsilon > 0$. If $r_q > 0$ and r_q is noninteger then there is prime T and rational noninteger number $r_q^* = t_q / T$ so that $|r_q - r_q^*| < \varepsilon, q \in Q.$

Introduce Markov chain $V_0^* = 0,\ V_n^* = Y_n^* \max(0, \tilde{X}_n + V_{n-1}^*),\ \ n = 1,2,...,\ \varphi_n^*(x) = P(V_n^* > x).$

Theorem 7. Suppose that $|r_q - r_q^*| < \varepsilon, L_1(F_q, G_q) < \varepsilon, q \in Q.$ If the condition (17) is true and $1/a < d < 1,\ 0 < \varepsilon < d - 1/a$ then

$$L_1(\varphi_n, \varphi_n^*) \le \frac{\varepsilon D}{1-d}, n > 0, D = \frac{aC + ad - 1}{a - 1} < \infty. \tag{20}$$

Proof. From Theorem 7 condition we have that $E\tilde{Z}_n \le C + d - 1/a$ and from Theorem 5 proof we obtain $E\tilde{V}_\infty \le (C + d - 1/a)/(a-1).$ Assume that $P(Y_n = r_q, Y_n^* = r_q^*) = p_q,\ q \in Q,$ then

$$E|\tilde{V}_{n+1} - V_{n+1}^*| \le EY_{n+1} E|\tilde{V}_n - V_n^*| + E|Y_{n+1} - Y_{n+1}^*|(V_n^* + \tilde{Z}_{n+1}) \le \frac{E|\tilde{V}_n - V_n^*|}{a} +$$

$$+\varepsilon(C + d - 1/a + E\tilde{V}_n + E|\tilde{V}_n - V_n^*|) \le E|\tilde{V}_n - V_n^*| d + \varepsilon(C + d - 1/a + E\tilde{V}_\infty) \le E|\tilde{V}_n - V_n^*| d + \varepsilon D.$$

Using mathematical induction by *n* and minimal property of metrics L_1 we obtain the formula (20).

5 BERSTEIN THEOREM IN L_1 METRICS

Bernstein theorem allows for any $q \in Q$ to approximate d.f. $F_q(t)$ by a mixture of exponential distributions in uniform metrics. But we need analogous approximation in L_1 metrics. Suppose that d.f. $F(t), \bar{F}(t) = 1 - F(t)$ concentrated on [0, ∞) has mean

$$M = \int_0^\infty \bar{F}(t) dt < \infty \tag{21}$$

and continuous positive density $f(t)$ so that for any $T > 0$

$$\inf_{0 \le t \le T} f(t) = \frac{1}{A(t)} > 0. \tag{22}$$

Lemma 1. If d.f. *F* satisfies the conditions (21), (22) then for any $\varepsilon > 0$ it is possible to choose discrete d.f. G_n with finite number *n* of positive atoms so that $L_1(F, G_n) < 2\varepsilon.$

Proof. Fix positive ε and using the condition (21) find T_ε so that

$$\int_{T_\varepsilon}^{\infty} \overline{F}(t)dt < \varepsilon. \tag{23}$$

Using the condition (22) define integer *n* so that

$$\frac{A(T_\varepsilon)}{n} < \varepsilon \tag{24}$$

and put $\nu = \overline{F}(T_\varepsilon)$. Define $t_i, 1 \le i < n,$ from the equalities

$$F(t_i) = \frac{i(1-\nu)}{n}, 1 \le i < n,\ F(t_n) = F(T_\varepsilon) = 1-\nu.$$

Suppose that discrete d.f. G_n satisfies equalities

$$G_n(t) = \begin{cases} 0,\ 0 \le t < t_1, \\ F(t_2), t_1 \le t < t_2, \\ F(t_3), t_2 \le t < t_3, \\ \ldots \\ F(t_n), t_{n-1} \le t < T_\varepsilon, \\ 1, T_\varepsilon \le t < \infty. \end{cases} \tag{25}$$

Using the formulas (15), (23) - (25) it is easy to obtain the inequality $L_1(F, G_n) < 2\varepsilon$. The lemma is proved.

Theorem 8. If d.f. *F* satisfies the conditions (21), (22) then for any $\varepsilon > 0$ it is possible to choose d.f. $R_n(t)$ concentrated on [0, ∞) with tail

$$\overline{F}(t) = \sum_{i=1}^{r} a_i \exp(-b_i t),\ t > 0,\ -\infty < a_i < \infty,\ 0 < b_i < \infty, \tag{26}$$

so that $L_1(F, R_n) < 4\varepsilon$.

Proof. From Lemma 1 it is easy to obtain that $G_n(t)$ is probability mixture of point distributions

$$G_n(t) = \frac{1-\nu}{n}\left[2\,\mathbf{1}(t - t_1) + \sum_{i=2}^{n-1} \mathbf{1}(t - t_i)\right] + \nu\,\mathbf{1}(t - T_\varepsilon)$$

where $\mathbf{1}(t - u)$ is d.f. concentrated in real point *u*.

Fix $\varepsilon > 0$ and natural *k*. Following (Dufresne 2005, Ko & Ng 2007) let $Erl(m, \lambda)$ denote the Erlang distribution with density function

$$\frac{\lambda^m t^{m-1} \exp(-\lambda t)}{(m-1)!}, t > 0,$$

which describes the sum of *m* independent r.v.`s with exponential distribution with parameter λ. Then $Erl(m, \lambda)$ has mean m/λ and variance m/λ^2. To approximate a degenerate distribution at $k > 0$ we consider $Erl(m, m/k)$ and let *m* tend to infinity. So we may choose m_ε so that variance of d.f. $H_k = Erl(m, m/k)$ is smaller than ε^2 and consequently $L_1(\mathbf{1}(t - k), H_k(t)) \le \varepsilon.$ If Erlang d.f.`s $H_1(t), ..., H_n(t)$ satisfy the inequalities

$$L_1(\mathbf{1}(t - t_1), H_1(t)) < \varepsilon, ..., L_1(\mathbf{1}(t - t_n), H_n(t)) < \varepsilon$$

then $L_1(G_n, Q_n) < \varepsilon$ where

$$Q_n(t) = \frac{1-\nu}{n}\left[2\,H_1(t) + \sum_{i=2}^{n-1} H_i(t)\right] + \nu\,H_n(t).$$

Following (Ko & Ng 2007) let $\eta_1,...,\eta_m$ are i.i.d. r.v.`s with exponential d.f. and with parameter $\lambda$. Then random sum $\sum_{i=1}^{m}\eta_i$ has d.f. $Erl(m,\lambda)$. Suppose that independent r.v.`s $\xi_1,...,\xi_m$ with exponential distributions and with parameters $\lambda_1,...,\lambda_m$ so that

$$\sum_{i=1}^{m}\left|\frac{1}{\lambda}-\frac{1}{\lambda_i}\right|<\varepsilon,\ \lambda_i\neq\lambda_j,\ 1\le i\neq j\le m,$$

and there are i.i.d. r.v.`s $\omega_n,\, n\ge 1,$ uniformly distributed on interval [0,1] so that

$$\eta_i=-\frac{\ln\omega_i}{\lambda},\ \xi_i=-\frac{\ln\omega_i}{\lambda_i},\ 1\le i\le m.$$

Denote $S(t)=P\left(\sum_{i=1}^{m}\xi_i<t\right).$ Then we have the inequality

$$E\left|\sum_{i=1}^{m}\eta_i-\sum_{i=1}^{m}\xi_i\right|\le\sum_{i=1}^{m}E\,|\,\eta_i-\xi_i\,|\le\varepsilon.$$

If we replace in last inequality complex probability metric $E\,|\,X-Y\,|$ by its minimum $L_1(P(X<t),P(Y<t))$ among all joint distributions which conserve marginal distributions of $P(X<t),P(Y<t)$ then we obtain $L_1(Erl(m,\lambda),S)<\varepsilon.$ Using the formula (12) it is easy to represent d.f. $S(t)$ in the form (26).

So it is possible to find d.f. S concentrated on $[0,\infty)$ with the tail $\overline{S}(t)$ which has the form (26) so that $L_1(Erl(m,\lambda),S)<\varepsilon.$ If d.f.`s $S_1(t),...,S_n(t)$ which has the form (26) satisfy the inequalities

$$L_1(H_1,S_1)<\varepsilon,...,L_1(H_n,S_n)<\varepsilon.$$

Then $L_1(Q_n,R_n)<\varepsilon$ where

$$R_n(t)=\frac{1-\nu}{n}\left[2\,S_1(t)+\sum_{i=2}^{n-1}S_i(t)\right]+\nu\,S_n(t)$$

and d.f. $R_n(t)$ satisfies the condition (26) also.

Consequently from Lemma 1 we obtain the inequalities

$$L_1(F,R_n)\le L_1(F,G_n)+L_1(G_n,Q_n)+L_1(Q_n,R_n)<4\varepsilon.$$

The theorem is proved.

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