ON TOTAL TIME ON TEST TRANSFORM ORDER

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ABSTRACT

In this paper, ordering of two lifetime random variable based on convex Total Time on Test (CXTTT) transform and increasing convex Total Time on Test (ICXTTT) transform of their distributions are introduced and, their implication with stochastic ordering and hazard rate ordering are proved.

1 INTRODUCTION

Stochastic orders and inequalities are being used at an accelerated rate in much diverse area of probability and statistics. This paper introduces the stochastic ordering of two life distributions based on convex Total Time on Test (CXTTT) transform and increasing convex Total Time on Test (ICXTTT) transform.

The simplest way of comparing two distribution functions is by comparison of associated means. However, such a comparison is based on only two single number (the means), and therefore it is often not very informative. When one wishes to compare two distribution functions that have the same mean (or that are centered about the same value), one is usually interested in the comparison of the dispersion of these distributions. In many situations in applications, one has more detailed information, for the comparison of two distribution functions, that take in account various forms of possible knowledge about the two underlying distributions, see Shaked and Shanthikumar (1994).

Total Time on Test (TTT) transform plots are useful for analyzing non-negative data. The plots help in choosing a mathematical model for the data and provide information about failure rate. Also incomplete data can be analyzed and there is a theoretical basis for such an analysis, see Barlow and Campo (1975). As TTT is useful in analyzing incomplete data, we can order the distributions according to TTT of respective distributions. Kochar et al. (2002) defined TTT transform order and Shaked and Shanthikumar (2007) studied it explicitly. Nair et al. (2008) provided applications of TTT of order n in reliability analysis.

But if the mean values of the two distributions are same, we need to go for variability measures for ordering. Convex and increasing convex ordering is usually used to order two distributions according to the variability of their random variables. In this paper, we introduce the ordering of two distributions based on convex TTT and increasing convex TTT, which can be used to order two distributions according to the TTT of the convex and increasing convex functions of the respective random variables. When we consider censored data, CXTTT and ICXTTT is more suitable for ordering two distributions according to the variability.

In section 2, the notions of usual stochastic ordering and TTT are briefly recalled. In section 3, the definition of TTT ordering is given. In section 4, the concept of CXTTT ordering and ICXTTT ordering are provided and some implications between stochastic ordering and hazard rate ordering with CXTTT and ICXTTT ordering are proved. Conclusions are given at last section.

2. STOCHASTIC , HAZARD RATE AND MEAN RESIDUAL LIFE ORDER

Let X and Y be two random variables such that

$$P(X > u) \le P(Y > u), \forall u \in (-\infty, \infty).$$

Then X is said to be smaller than Y in the usual stochastic order (denoted by $X \le_{st} .Y$). It means that X is less likely than Y to take large values, where ”large” means the value greater than u, and that this is the case for all u's. (2.1) is same as

$$P(X \le u) > P(Y \le u), \forall u \in (-\infty, \infty).$$

Let X and Y has distributions F and G respectively independent of each other. Let $h_f(x) = \frac{f(x)}{1 - F(x)}$ and $h_g(x) = \frac{g(x)}{1 - G(x)}$ be the hazard rate functions of F and G where f(x) and g(x) are the probability density functions of F and G respectively. Clearly higher the hazard rate smaller the X should be stochastically.

Definition 2.1 Let X and Y are two non-negative random variables with absolutely continuous distributions F and G respectively independent of each other. X is said to be smaller than Y in hazard rate order (denoted by $X \le_{hr} .Y$) if $h_f(x) \ge h_g(x)$, $x \ge 0$.

Another important order is mean residual life order. The definition of mean residual life is given below.

Definition 2.2 If X is a non-negative random variable with a survival function $\overline{F}(x)$ and a finite mean μ, the mean residual life of X at x is defined as

$$m(x) = E(X - x \mid X > x), \forall x \in [0, \infty] \text{ and 0 otherwise.}$$

Clearly, the smaller the mean residual life function is the smaller X should be in some stochastic sense. Let $m_f(x)$ and $m_g(x)$, $x \ge 0$ be the mean residual life functions of X and Y respectively.

Definition 2.3 Let X and Y be two non-negative random variables with absolutely continuous distributions F and G respectively independent of each other. X is said to be smaller than Y in mean residual life order if $m_f(x) \le m_g(x)$, $x \ge 0$ (denoted by $X \le_{mrl} Y$).

More details of stochastic orders can be seen in Shaked and Shanthikumar (1994). Now we recall the TTT order in the following section.

3. TOTAL TIME ON TEST TRANSFORM

Let X and Y have distributions F and G respectively independent of each other. Given a sample of size n from the non-negative random variables X and Y, let $X_{(1)} \le X_{(2)} \le ... \le X_{(k)} \le ... \le X_{(n)}$ and $Y_{(1)} \le Y_{(2)} \le ... \le Y_{(k)} \le ... \le Y_{(n)}$ be the order statistics corresponding to the samples. TTT to the rth failure from distributions F and G are, respectively,

$$T(X_{(r)}) = nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + ... + (n - r + 1)(X_{(r)} - X_{(r-1)}) = \sum_{i=1}^{r} X_{(i)} + (n - r)X_{(r)}$$

$$T(Y_{(r)}) = nY_{(1)} + (n-1)(Y_{(2)} - Y_{(1)}) + ... + (n - r + 1)(Y_{(r)} - Y_{(r-1)}) = \sum_{i=1}^{r} Y_{(i)} + (n - r)Y_{(r)}.$$

Define

$$H_n^{-1}(r/n) = \frac{1}{n}T(X_{(r)}) \text{ and } K_n^{-1}(r/n) = \frac{1}{n}T(Y_{(r)})$$

$$H_n^{-1}(r/n) = \int_0^{F_n^{-1}(r/n)} (1-F_n(u))du,\quad K_n^{-1}(r/n) = \int_0^{G_n^{-1}(r/n)} (1-G_n(u))du$$

where $F_n(u) = \begin{cases} 0 & u < X_{(i)} \\ i/n & X_{(i)} \le u < X_{(i+1)} \\ 1 & X_{(n)} > u \end{cases}$ and $G_n(u) = \begin{cases} 0 & u < Y_{(i)} \\ i/n & Y_{(i)} \le u < Y_{(i+1)} \\ 1 & Y_{(n)} > u \end{cases}$

$F_n^{-1}(x) = \inf\{x : F_n(x) \ge u\}$ and $G_n^{-1}(x) = \inf\{x : G_n(x) \ge u\}$.

The fact that $F_n(u) \to F(x)$ a.s. and $G_n(u) \to G(x)$ a.s. implies, by Glivenko Cantelli Theorem, $\lim_{n\to\infty, r/n\to t} \int_0^{F_0^{-1}(r/n)} (1-F_n(u))du = \int_0^{F^{-1}(t)} (1-F(u))du$ and

$\lim_{n\to\infty, r/n\to t} \int_0^{G_0^{-1}(r/n)} (1-G_n(u))du = \int_0^{G^{-1}(t)} (1-G(u))du$ uniformly in $t \in [0,1]$.

We define TTT transform of F as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1-F(u))du \quad t \in [0,1].$$

and TTT transform of G as

$$H_G^{-1}(t) = \int_0^{G^{-1}(t)} (1-G(u))du \quad t \in [0,1].$$

We define the following order of two random variables with absolute continuous distribution functions F and G respectively. Clearly lower the empirical TTT of X is lower the that of Y only when the value of $X_{(1)} \le X_{(2)} \le ... \le X_{(k)} \le ... \le X_{(n)}$ and are lower than that of $Y_{(1)} \le Y_{(2)} \le ... \le Y_{(k)} \le ... \le Y_{(n)}$. That is, $T(X_{(n)}) \le T(Y_{(r)})$.

Now we recall the following.

Definition 3.1 Let X and Y be two non-negative random variables with absolute continuous distributions F and G respectively. X is said to be smaller than that of Y in the Total Time on Test Transform order if $H_F^{-1}(t) \le H_G^{-1}(t), \quad \forall t \in [0,1]$.

We denote the TTT order as $X \le_{TTT} Y$. More details of TTT ordering can be seen in Shaked and Shanthikumar (2007) and its application in Chacko et al. (2010).

In the following section, we introduce the Convex TTT order and Increasing convex TTT order which take an account of variability of random variables.

4 CONVEX AND INCREASING CONVEX TTT

Let

$$T^g(X_{(r)}) = ng(X_{(1)}) + (n-1)(g(X_{(2)}) - g(X_{(1)})) + ... + (n-r+1)(g(X_{(r)}) - g(X_{(r-1)})) = \sum_{i=1}^{r} g(X_{(i)}) + (n-r)g(X_{(r)})$$

where g is a convex function. Then define, for $g(x) = x^2$,

$$(H_n^{-1})^2(r/n) = \int_0^{(F_n^{-1}(r/n))^2} (1-F_n(u))du = \frac{T_n^{x^2}(X_{(r)})}{n}$$

where

$$\frac{T_n^{x^2}(X_{(r)})}{n}=\sum_{i=1}^{r}(X_{(i)})^2+(n-r)(X_{(r)})^2 \quad \frac{T_n^{x^2}(X_{(r)})}{n}=\sum_{i=1}^{r}(X_{(i)})^2+(n-r)(X_{(r)})^2$$

$$\lim_{n\to\infty,r/n\to t}\int_0^{(F_0^{-1}(r/n))^2}(1-F_n(u))du=\int_0^{(F^{-1}(t))^2}(1-F(u))du=(H_F^{-1})^2(t), say.$$

More generally, define for every convex function $g:R\to R$

as $g(H_n^{-1})(r/n)=\int_0^{g(F_n^{-1}(r/n))}(1-F_n(u))du=\frac{T_n^g(X_{(r)})}{n}$

$$\lim_{n\to\infty,r/n\to t}\int_0^{g(F_0^{-1}(r/n))}(1-F_n(u))du=\int_0^{g(F^{-1}(t))}(1-F(u))du=g(H_F^{-1})(t), say.$$

Similarly we can define

$$g(K_n^{-1})(r/n)=\int_0^{g(G_n^{-1}(r/n))}(1-G_n(u))du=\frac{T_n^g(Y_{(r)})}{n}$$

$$\lim_{n\to\infty,r/n\to t}\int_0^{g(G_0^{-1}(r/n))}(1-G_n(u))du=\int_0^{g(G^{-1}(t))}(1-G(u))du=g(K_F^{-1})(t), say.$$

Let X and Y be two random variables such that $T^g(X_{(n)})\le T^g(Y_{(r)})$ for all convex functions $g:R\to R$ and all samples of size n. Then X is smaller than Y in some stochastic sense, since $T^g(X_{(n)})$ is average of total observed convex transformed time of a test. The X values are less likely to take larger values than Y values. Therefore we define the following convex TTT ordering.

Definition 4.1 Let X and Y be two non-negative random variables with absolutely continuous distribution functions F and G respectively. If

$$(H_F^{-1})^g(t)\le(H_G^{-1})^g(t), \quad \forall t\in[0,1],$$

and g is convex function, then X is smaller than Y in convex TTT order (denoted as $X\le_{CXTTT}Y$).

Roughly speaking, convex functions are functions that take on them (relatively) larger values over region of the form $(-\infty,a)\cup(b,\infty)$ for a < b.

Now we introduce the increasing convex TTT ordering.

Definition 4.2 Let X and Y be two non-negative random variables with absolutely continuous distribution functions F and G respectively. If

$$(H_F^{-1})^g(t)\le(H_G^{-1})^g(t), \quad \forall t\in[0,1]$$

and g is increasing convex function, then X is smaller than Y in increasing convex TTT order (denoted $X\le_{ICXTTT}Y$).

Roughly speaking X is both 'smaller' and 'less variable' than Y in some stochastic sense.

Example 4.1 Let $X\sim Exp(\lambda)$ and $Y\sim Exp(\theta)$ and g(x) = x, a convex function.

$H_F^{-1}(t)=\lambda t$ and $H_G^{-1}(t)=\theta t$, $t\in[0,1]$. $\therefore \hat{\lambda}=\frac{T(X_{(r)})}{n}\le\hat{\theta}=\frac{T(Y_{(r)})}{n}$ when $T(X_{(r)})\le T(Y_{(r)})$.

Hence we can conclude that $X\le_{TTT}Y$ and $X\le_{CXTTT}Y$, if $\hat{\lambda}\le\hat{\theta}.$ Again

$$\int_0^{(F^{-1}(t))^2}(1-F(u))du=\int_0^{(F^{-1}(t))^2} e^{-x/\lambda}dx=\lambda(1-e^{-(x_*)^2/\lambda})$$

and

$$\int_0^{(G^{-1}(t))^2}(1-G(u))du=\int_0^{(G^{-1}(t))^2} e^{-x/\theta}dx=\theta(1-e^{-(x_{**})^2/\theta})$$

where
$x_* = \inf\{x : F(x)\geq t\}$ and $x_{**} = \inf\{x : G(x)\geq t\}$ Then $(H_F^{-1})^2(t)\leq(H_G^{-1})^2(t),\quad \forall t\in[0,1]$ if $\lambda<\theta$ and $x_*<x_{**}$.

Now we prove the following theorem, which gives the implication of stochastic ordering and convex TTT ordering, if the expectations of random variables are finite.

Theorem 4.1 Let X and Y be two non-negative random variables having absolutely continuous distribution functions F and G respectively. Let g be an convex function $g: R\to R$. If $F^{-1}(t)\leq G^{-1}(t)$, $EX<\infty$ and $EY<\infty$ then $X\leq_{st} Y$ implies $X\leq_{CXTTT} Y$.
Proof: Clearly, under the stated conditions, $\forall u\in(0,\infty)$ and $\forall t\in[0,1]$,

$$P(X>u)\leq P(Y>u)\Rightarrow \int_0^{(F^{-1}(t))^g} P(X>u)du\leq \int_0^{(G^{-1}(t))^g} P(Y>u)du$$

where g is a convex function. Therefore $X\leq_{CXTTT} Y$. Hence the proof.

Now we prove the following theorem, which gives the implication of hazard rate ordering and convex TTT ordering, if the expectations of random variables are finite.

Theorem 4.2 Let X and Y be two non-negative random variables having absolutely continuous distribution functions F and G respectively. Let g be and convex function $g: R\to R$ If $F^{-1}(t)\leq G^{-1}(t)$, $EX<\infty$ and $EY<\infty$ then $X\leq_{hr} Y$ implies $X\leq_{CXTTT} Y$.
Proof: Clearly, under the stated conditions,

$$h_f(u)\geq h_g(u),\quad \forall u>0\Rightarrow P(X>u)=e^{-\int_0^u h_f(x)dx}\leq P(Y>u)=e^{-\int_0^u h_g(x)dx}$$

Then by above theorem, $X\leq_{CXTTT} Y$. Hence the proof.

In a similar way, we can prove the implications of stochastic ordering and increasing convex TTT ordering, and hazard rate ordering and increasing convex TTT ordering, be replacing the function g by an increasing convex function. The results are stated below without proof.

Theorem 4.3 Let X and Y be two non-negative random variables having absolutely continuous distribution functions F and G respectively. Let g be an increasing convex function $g: R\to R$. If $F^{-1}(t)\leq G^{-1}(t)$, $EX<\infty$ and $EY<\infty$ then $X\leq_{st} Y$ implies $X\leq_{CXTTT} Y$.

Theorem 4.4 Let X and Y be two non-negative random variables having absolutely continuous distribution functions F and G respectively. Let g be an increasing convex function $g: R\to R$ If $F^{-1}(t)\leq G^{-1}(t)$, $EX<\infty$ and $EY<\infty$ then $X\leq_{hr} Y$ implies $X\leq_{ICXTTT} Y$.

5. CONCLUSIONS

The main advantage of convex and increasing convex TTT order relation is to order two random variables according to their variability and closeness to 0, even when censored data is available. It needs further study to explore the closure properties as in other ordering behaviors. The concave and increasing concave TTT ordering can be defined easily. Analogous results are straight forward. The results have theoretical and practical applications in reliability theory.

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