BAYESIAN UNCERTAINTY DECISION ANALYSIS

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ABSTRACT

Bayesian statistical decision theory would be questionable when applied directly to non-random uncertainty circumstances. In this paper, we investigate the basic elements of decision analysis oriented to observational data arising from a general uncertainty environment, so that a framework for Bayesian uncertainty decision doctrine is established. Further, we propose a copula-linked uncertainty marginals mechanism for constructing the uncertainty multivariate distributions to represent both observational data and an uncertainty parameter vector. This mechanism paves the way towards the establishment of an uncertainty posterior distribution of the parameter vector given the observational data, based on uncertain measure Axiom 5. Finally, we present an illustrative example of the development of a posterior uncertainty distribution for a parameter given a single observation, step by step. The significance of this paper is to establish for the first time a Bayesian uncertainty data inference and decision framework, which constitutes a critical step towards the establishment of uncertainty statistics and a Bayesian uncertainty decision theory.

1   INTRODUCTION

Any applied mathematical model is proposed to reflect a particular aspect of the real natural world. Decision making moving from analysis of the collected data (information) to reach a final decision is actually a process to resolve the uncertainty being faced. In the real world, there are many forms of *uncertainty* surrounding us, but thus far we may only deal successfully with uncertainty as randomness or fuzziness within information. How should we solve the problems with other kind of uncertainty in real business life? For example, recently, a "Made in Japan" crisis was triggered by a Toyota Prius brake fault event and quickly spread widely over other industries widely. At the first glance, it may seem a trivial event has been exaggerated by journalists. It is a well-known fact that Japanese manufacturing arms itself to the teeth with statistical quality control. There is no reason to ascribe the fault event to an absence of total quality management. Nevertheless, we cannot deny what happened, and infer that the event indicates that some unaddressed problem exists there. The only possible answer is the methodology used to manage the quality imperative does not match the real quality problem faced. In other words, while the existing quality control and decision making doctrine, which is based on probability theoretical foundation, addressing random uncertainty problems, is powerful, nevertheless for other forms of uncertainty problems, the existing theory and methodologies may be inadequate. The law of the real world tells us that each specific form of uncertainty must be addressed by the corresponding specific uncertainty doctrine and methodology. There is no universal law for addressing all the forms of uncertainties.

In this paper, we first review the basic elements of Liu's (2007, 2009, 2010) uncertain measure theory in Section 2, and further investigate a copula-linked uncertainty marginals

approach, to construct multivariate uncertainty distributions. The purpose is to represent both observational data and an uncertainty parameter vector. In Section 3, we note the basic elements of an observational-data oriented decision analysis under general uncertainty environments, in contrast to probabilistic Bayesian decision theory (Lee (1989), Cheng (1981), Bernardo & Smith (1994)), in order to establish a framework for a Bayesian uncertainty decision theoretic foundation. In Section 4, we propose a method to construct a posterior uncertainty distribution of parameter vector given the observational data, in terms of uncertain measure Axiom 5, see Liu (2010). In Section 5 we present an illustrative example, namely the development of a posterior uncertainty distribution for a parameter given a single observation, step by step. Section 6 concludes this paper.

2   UNCERTAIN MEASURE FOUNDATION

Uncertain measure (Liu (2007, 2009, 2010)) is an axiomatically defined set function mapping from a σ-algebra of a given space (set) to the unit interval [0,1], which provides a measuring grade system of an uncertain event (a reflection of an uncertainty phenomenon) and enables the formal definition of an uncertain variable and its uncertainty distribution.

Let Ξ be a nonempty set (space), and $\mathsf{A}(\Xi)$ the σ-algebra on Ξ. Each element, let us say, $A \subset \Xi, A \in \mathsf{A}(\Xi)$ is called an uncertain event. A number denoted as $\lambda\{A\}$, $0 \le \lambda\{A\} \le 1$, is assigned to the event $A \in \mathsf{A}(\Xi)$, which indicates the uncertain measuring grade with which event $A \in \mathsf{A}(\Xi)$ occurs. The normed set function $\lambda\{A\}$ satisfies following axioms given by Liu (2007, 2009, 2010):

Axiom 1: (Normality) $\lambda\{\Xi\} = 1$.

Axiom 2: (Monotonicity) $\lambda\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B$, $\lambda\{A\} \le \lambda\{B\}$.

Axiom 3: (Self-Duality) $\lambda\{\cdot\}$ is self-dual, i.e., for any $A \in \mathsf{A}(\Xi)$, $\lambda\{A\} + \lambda\{A^c\} = 1$.

Axiom 4: (σ-Subadditivity) $\lambda\left\{\bigcup_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} \lambda\{A_i\}$ for any countable event sequence $\{A_i\}$.

Axiom 5: (Product Measure) Let $(X_k, \mathsf{A}_{X_k}, D_k)$ be the kth uncertain space, $k = 1, 2, L, n$. Then product uncertain measure D on the product measurable space (X, A_X) is defined by

$$\lambda = \lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_n = \min_{1 \le k \le n}\{\lambda_k\} \tag{1}$$

where

$$\Xi = \Xi_1 \times \Xi_2 \times \cdots \times \Xi_n = \prod_{k=1}^{n} \Xi_k \tag{2}$$

and

$$\mathsf{A}_\Xi = \mathsf{A}_{\Xi_1} \times \mathsf{A}_{\Xi_2} \times \cdots \times \mathsf{A}_{\Xi_n} = \prod_{k=1}^{n} \mathsf{A}_{\Xi_k} \tag{3}$$

That is, for each product uncertain event $L \ O \ \mathsf{A}_X$ (i.e, $L = L_1 r L_2 r L r L_n \ O \ \mathsf{A}_{X_1} r \ \mathsf{A}_{X_2} r L r \ \mathsf{A}_{X_n} = \mathsf{A}_X$), the uncertain measure of the event L is

$$\lambda\{\Lambda\} = \begin{cases} \sup\limits_{A_1 \times \cdots \times A_n \subset \Lambda} \min\limits_{1 \le k \le n} \lambda\{\Lambda_k\} & \text{if } \sup\limits_{A_1 \times \cdots \times A_n \subset \Lambda} \min\limits_{1 \le k \le n} \lambda\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{A_1 \times \cdots \times A_n \subset \Lambda^c} \min\limits_{1 \le k \le n} \lambda\{\Lambda_k\} & \text{if } \sup\limits_{A_1 \times \cdots \times A_n \subset \Lambda^c} \min\limits_{1 \le k \le n} \lambda\{\Lambda_k\} > 0.5 \\ 0.5 & \text{otherwise} \end{cases} \tag{4}$$

Definition 2.1: (Liu (2007, 2009, 2010)) Any set function $D: A(X) \circledR [0,1]$ which satisfies Axioms 1-4 is called an uncertain measure. The triple $(X, A(X), D)$ is called the uncertain measure space.

Definition 2.2: An uncertain variable ξ is a measurable mapping, i.e., $\xi: (\Xi, A(\Xi)) \to (R, B(R))$, where $B(R)$ denotes the Borel σ-algebra on $R \equiv (-\infty, +\infty)$.

Remark 2.3: The fundamental difference between a random variable and an uncertain variable is the σ-additivity: the probability measure obeys σ-additivity (Kolmogorov (1950), Primas (1999)) and the uncertain measure (Liu (2007, 2009, 2010), Liu (2008)) obeys σ-subadditivity. The way of specifying measure inevitably has impacts on the behaviour of the measurable function over the triple, and hence on the mathematical characterization of the theories. For example, in contrast to probability theory, no "uncertainty density function" can be defined and then be entered into an integral of density to characterise an uncertainty distribution. Because an uncertain measure is permitted to be σ-subadditive, any set of uncertainty distributions derived from integration, being necessarily σ-additive, will necessarily be incomplete.

Definition 2.4: (Liu (2007, 2009, 2010)) The uncertain distribution $\Psi: R \to [0,1]$ of an uncertain variable ξ on $(X, A(X), D)$ is

$$\Psi_\xi(x) = \lambda\{\tau \in \Xi \mid \xi(\tau) \le x\} \tag{5}$$

Theorem 2.5: (Peng and Iwamura (2010)) The necessary and sufficient conditions for a function $\Psi: \Box \to [0,1]$ be an uncertainty distribution function is that Ψ is non-decreasing function and

$$0 \le \Psi(x) \le 1, \ \forall x \in \Box \tag{6}$$

The function Ψ is referred to an uncertainty distribution function.

Remark 2.6: A probability distribution $F_X(x)$ requires right-continuity and $F_X(-\infty) = 0, F_X(+\infty) = 1$ in addition to those requirements of the uncertainty distribution function, while an uncertainty distribution is not limited by any continuity and $\Psi_\xi(-\infty) = 0, \Psi_\xi(+\infty) = 1$ requirements. This relaxation enables an uncertainty distribution to model even the most complicated pattern in real world data. The following definition reveals an essential characteristic of the uncertainty distribution.

Definition 2.7: Let ξ be an uncertainty variable, which takes values from a subset, denoted as E, of the real line $\Box$, with n discontinuity points collected in an ascending order as set $D \equiv \{c_1, \cdots, c_n\}$. The uncertainty distribution, Ψ, of the variable ξ is specified as follows:

1. On the set $D \equiv \{c_0, c_1, \cdots, c_n\}$,

$$\begin{aligned} \Psi(c_i -) = \psi_{i-}, \ \Psi(c_i) = \psi_i, \ \Psi(c_i +) = \psi_{i+} \\ i = 1, 2, \cdots, n \end{aligned} \tag{7}$$

where $\psi_{i-} < \psi_i < \psi_{i+}, \ \psi_{1-} \ge 0, \ \psi_{n+} \le 1, \ i = 1, 2, \cdots, n$;

2. At the inner points of the sub-intervals (c_{i-1}, c_i), $i = 1, 2, \cdots, n$, the uncertainty distribution Ψ is continuous

$$\Psi(z) = \begin{cases} \psi_{i-1+} & \text{if } z \downarrow c_{i-1} \\ \Lambda_i(z) & \text{if } z \in (c_{i-1}, c_i) \\ \psi_{i-} & \text{if } z \uparrow c_i \end{cases} \tag{8}$$

where the function Λ_i is positive, non-decreasing, and bounded by ψ_{i-1+} and ψ_{i-}, i.e., $\psi_{i-1+} \leq \Lambda_i \leq \psi_{i-}$, $i = 1, 2, \cdots, n$. Then Ψ is an uncertainty distribution of the essential form and ξ is called an essential uncertain variable.

Remark 2.8: The aim of this paper is to develop an observational-data oriented decision making doctrine. Whenever an observation is obtained, this specific observation should not be regarded as an isolated real number (or a real-valued vector), rather, it should be regarded as a representative from a population typically specified by a hypothesized uncertainty distribution. This approach matches the standard viewpoint in the statistical community. It is also a convention that the term "population" (Cheng 1981)) is equivalent to the term distribution, or to the term random variable. In the new uncertainty theory, this statistical convention continues. We formally state this convention as a definition on observational data.

Definition 2.9: An observation is a real number, (or more broadly, a symbol, or an interval, or a real-valued vector, a statement, etc), which is a representative of a population or equivalently of an uncertainty distribution under a given scheme comprising set and σ-algebra.

Remark 2.10: The uncertainty distribution is unknown but exists objectively. A workable solution is to hypothesize a family of uncertainty distributions of a specified functional form with unknown parameter q, where the family is denoted by $\{\Psi_x^q, q \in Q\}$.

Definition 2.11: (Liu (2007, 2009, 2010)) Let multivariate uncertainty variable (x_1, x_2, L, x_d) be defined on an uncertain measure space $(X, A(X), D)$, then the multivariate function $\Psi_{x_1, x_2, L, x_d} : D \circledR [0,1]$ is called an multivariate uncertainty distribution if

$$\Psi_{x_1, x_2, L, x_d}(x_1, x_2, L, x_d) = D\{x_1 J x_1, x_2 J x_2, L, x_d J x_d\} \tag{9}$$

To present a concrete form of a multivariate uncertainty distribution, Guo et al. (2010) propose a copula-linked uncertainty marginals approach.

Definition 2.12: Let (x_1, x_2, L, x_d) be a multivariate uncertainty variable with joint uncertainty distribution $\Psi_{x_1, x_2, L, x_d}(x_1, x_2, L, x_d)$, in which all the marginal uncertainty distributions $\Psi_{x_1}(\cdot), \Psi_{x_2}(\cdot), L, \Psi_{x_d}(\cdot)$ exist and are regular (i.e., $\Psi_{x_i}^{-1}(\cdot)$ exists, $i = 1, 2, L, d$). Then the uncertainty copula is defined by

$$C\left(\Psi_{x_1}(x_1), \Psi_{x_2}(x_2), L, \Psi_{x_d}(x_d)\right) = \Psi_{x_1, x_2, L, x_d}(x_1, x_2, L, x_d) \tag{10}$$

We use a bivariate uncertainty distribution as an illustrative multivariate example.

Example 2.13: Let bivariate uncertainty variable (x_1, x_2) have marginal uncertainty distributions $\Psi_{x_1}(\cdot)$ and $\Psi_{x_2}(\cdot)$ respectively. The Farlie-Gumbel-Morgenstern (FGM) copula is defined by

$$C(u_1, u_2) = u_1 u_2 (1 + v(1 - u_1)(1 - u_2)), \ v \in [-1, 1] \tag{11}$$

Further, let the bivariate uncertainty variable (x_1, x_2) have marginal uncertainty distributions $\Psi_{x_1}(\cdot)$ and $\Psi_{x_2}(\cdot)$ respectively, where

$$\Psi_{x_i}(x_i) = \frac{1}{1 + \exp\left\{-\frac{p}{\sqrt{3}s_i}(x_i - q_i)\right\}}, \ i = 1, 2 \tag{12}$$

Then the bivariate FGM-Normal joint uncertainty distribution is

$$Y_{x_1,x_2}(x_1,x_2)= \sum_{i=1}^{2}\frac{1}{1+\exp\left\{-\frac{p}{\sqrt{3}s_i}(x_i-q_i)\right\}}+v\sum_{i=1}^{2}\frac{\exp\left\{-\frac{p}{\sqrt{3}s_i}(x_i-q_i)\right\}}{1+\exp\left\{-\frac{p}{\sqrt{3}s_i}(x_i-q_i)\right\}} \qquad (13)$$

Finally, it is necessary to prepare the uncertainty expectation and the variance of an uncertainty variable to support the development of an uncertainty decision doctrine.

Definition 2.14: (Liu (2007, 2009, 2010)) Let ξ be an uncertainty variable defined on the uncertain space $(\Xi, A(\Xi), \lambda)$, then the expectation of ξ is

$$E[\xi]= \int_{0}^{+\infty}\lambda\{\xi\geq r\}dr - \int_{-\infty}^{0}\lambda\{\xi\leq r\}dr \qquad (14)$$

provided at least one of the two integrals is finite.

Definition 2.15: (Liu (2007, 2009, 2010)) Let ξ be an uncertainty variable with finite expectation $E[x]$, then the variance of ξ is

$$V[\xi]= E\left[(\xi-E[\xi])^2\right] \qquad (15)$$

Theorem 2.16: Let ξ be an uncertainty variable on uncertain measure space $(X, A(X), D)$ and h be a monotonic non-decreasing function $h: \breve{Y} \circledR \breve{Y}^+$, then the expectation of $h(x)$ is

$$E[h(\xi)]= \int_{0}^{+\infty}h(r)\lambda\{\xi\geq r\}dr - \int_{-\infty}^{0}h(r)\lambda\{\xi\leq r\}dr \qquad (16)$$

3   ELEMENTS OF BAYESIAN DECISION THEORY

A decision theory is built upon a mathematical foundation, which provides a framework (or guidelines) for decision making according to a specified criterion, based on the observational data with a distribution of the assumed uncertainty type, e.g.,

1. The statistical decision is based on probability (measure) theory, which addresses the random uncertainty;

2. The fuzzy decision theory deals with fuzziness;

3. The uncertainty decision theory deals with a general uncertainty different from randomness or fuzziness.

Recall that the statistical decision theory is established on the axiomatic foundation of probability measure.

The basic elements of statistical decision are: (1) Sample space and distributional family; (2) Decision space; (3) Loss function and decision function.

It is necessary to point out the basic elements, namely *state, action,* and *loss* in statistical decision theory (Lee (1989), Cheng (1981), Bernardo & Smith (1994)), are still the essential elements in the Bayesian uncertainty decision theory.

Firstly, in statistical decision theory, the state, termed "state of nature" is regarded as objectively in existence, at least in some consensus sense. In contrast, in any general uncertainty environments, the state may include subjective, judgmental or even phenomenological events or factors. Note here the conceptual interpretations that *state* acquires across the decision environments, i.e., "reality" in front of the decision makers, along with possible virtual actions, and virtual loss. The differentiation between the *state of nature* in the statistical decision theory and the *state* in the uncertainty decision theory is critical. The former reflects more or less reflecting the

"truth" for the frequentist school, while the uncertainty decision theory is a mixture of subjective and objective reflections.

Secondly, the connotations of *action* in the uncertainty decision theory is virtual, in that some elements are of a precautionary nature and do not correspond to any specific state element. The nature of the mapping is from multiple states to multiple actions.. However, the inclusion of virtual action elements is extremely important, because the top decision maker does not need to deal with routine decisions of day-to-day operations but with the extreme event(s) or the most important event decision(s).

Thirdly, the *loss* mechanism in both decision theories is the same. An uncertainty decision is a selection, which minimizes the loss function $l(\theta, a)$ of an action a from action space A for given state θ in the state space Θ. However, the social loss and environmental loss extract more and more attention from the public, NGO's and the governmental agencies. In the new uncertainty decision theory, the safety factor state, the health factor state, and the environmental factor state should be automatically assigned uncertain measure grades because of their intrinsic features. In the uncertainty decision theory, an action is made in terms of observational data, denoted by x, which is described by an uncertainty distribution $\Psi(x|\theta)$. Based on observational data x (i.e., representative of population $\Psi(x|\theta)$), a decision is actually a mapping from data space D into action space A. In other words,

$$a : D \rightarrow A \tag{17}$$

which can be expressed by

$$a = d(x) \tag{18}$$

The loss $l(\theta, d(x))$ is measurable on the joint uncertainty space.

Definition 3.1: The expected value of the loss with respect to the uncertainty distribution of observational data x

$$R(q, d) = \mathrm{E}_q \left[l(q, d(x)) \right] \tag{19}$$

is called a risk function.

The uncertainty distribution of observational data x depends on state θ, because the dependence of $R(\theta, d)$ on θ enters explicitly from $l(\theta, a)$ and also through the state θ in the distribution function $\Psi(x|\theta)$ for x. Therefore the uncertainty distribution of the data determines the fundamental characteristics of observational-data oriented uncertainty decision theory, which deserves further exposure.

Let us consider the uncertainty decision problem for a given uncertainty distribution. Assume a state space $\Theta = R$, and a continuous action space $A \equiv R$, and the loss function defined by

$$l(\theta, a) = w(\theta)(\theta - a)^2 \tag{20}$$

i.e., a quadratic loss function is assumed

Definition 3.2: (Uncertainty Bayes loss) Given a continuous state space Θ, the uncertainty variable θ is defined on uncertain space $(\Theta, \mathrm{B}(\Theta), \lambda_\theta)$, where $\lambda_\theta(\cdot)$ is an uncertain measure. The uncertain distribution $\Psi_\theta(\cdot)$ is defined on $(\Theta, \mathrm{B}(\Theta))$). Then the average of loss with respect to state space for a given action $a \in A$, is the quantity

$$B(a) = \mathrm{E}\left[l(\theta, a) \right] = \int_\Theta l(y, a) d\Psi_\theta(y) \tag{21}$$

and is called the uncertainty Bayes loss for a given action a.

Definition 3.3: (Uncertainty Bayes risk) The uncertainty Bayes risk is defined by

$$B(d) = E\left[l(\theta, d(x))\right] = \int_{\Theta} l(\theta, d(x)) d\Psi(\theta) \tag{22}$$

Definition 3.4: (Uncertainty Bayes rule) A Bayes decision rule, denoted as d^B, is a rule such that the Bayes risk is minimized, i.e.,

$$B(d^B) = \min_{d \in D}\{B(d)\} \tag{23}$$

Example 3.5: Given a continuous state space $\Theta = R$, the uncertainty variable θ is defined on an uncertain space $(R, B(R), \lambda_\partial)$, where $\lambda_\theta(\cdot)$ is properly defined. The uncertainty distribution is assumed to be

$$G_\theta(y) = \frac{y-a}{2(b-a)} \vartheta_{(a,b]}(y) + \frac{y+c-2b}{2(c-b)} \vartheta_{(b,c]}(y) \tag{24}$$

Then we derive the average loss with respect to the state space for a given action $a \in A$, as the uncertainty Bayes loss:

$$B(a) = E\left[l(\theta, a)\right] = \int_{\Theta} w(y)(y-a)^2 d\Psi_\theta(y) \tag{25}$$

Set $w(\theta) = w_0$, a constant, then the uncertainty Bayes loss is

$$\begin{aligned}
B(a) &= \frac{w_0}{2(\beta-\alpha)} \int_\alpha^\beta (y-a)^2 dy + \frac{w_0}{2(\gamma-\beta)} \int_\beta^\gamma (y-a)^2 dy \\
&= \frac{w_0}{2(\beta-\alpha)}\left[(\beta-a)^3 - (\alpha-a)^3\right] + \frac{w_0}{2(\gamma-\beta)}\left[(\gamma-a)^3 - (\beta-a)^3\right] \\
&= \frac{w_0}{2}\left[(\beta-a)^2 + (\alpha-a)^2 + (\beta-a)(\alpha-a)\right] + \frac{w_0}{2}\left[(\gamma-a)^2 + (\beta-a)^2 + (\gamma-a)(\beta-a)\right] \\
&= \frac{w_0}{2}\left(3a^2 - 3(\alpha+\beta)a + \alpha^2 + \beta^2\right) + \frac{w_0}{2}\left(3a^2 - 3(\gamma+\beta)a + \gamma^2 + \beta^2\right) \\
&= w_0\left(3a^2 - \frac{3}{2}(\alpha+2\beta+\gamma)a + \frac{1}{2}(\alpha^2 + 2\beta^2 + \gamma^2)\right)
\end{aligned} \tag{26}$$

With an appropriate specification of decision function in term of data, the uncertain Bayesian decision analysis can be formulated.

4   A POSTERIOR UNCERTAINTY DISTRIBUTION

When (Q, B_Q, l_q) is an uncertain (prior) space and (X, B_X, P_q) is a probability space, we actually use random sample information to make inferences on the uncertain parameter q. The critical step in the probabilistic Bayesian inference is to develop the posterior distribution for parameter q. We strongly believe that the Bayesian uncertainty inference requires parallel manipulations.

Let (Q, B_Q) be a parameter measurable space, (X, B_X) be a sample measurable space.

Definition 4.1: An uncertain measure defined on (Q, B_Q) is called an uncertain prior measure, denoted as l_q. The space (Q, B_Q, l_q) is called an uncertain prior space, the uncertain distribution $G(y) = l_q\{q \, J \, y\}$ is called an uncertain prior distribution.

An uncertainty variable, denoted by x, is defined on a measurable space (X, B_X) with uncertainty distributional family $\{Y_q, q \, O \, Q\}$ where Q is a parameter space. Formally,

Definition 4.2: The uncertainty observations are representatives of an uncertainty variable x, which is called an uncertainty population, or alternatively, called as an uncertainty distribution $\Psi_\xi(x|\theta), \theta \in \Theta$. The uncertainty variable x is defined on $(\mathsf{X}, \mathsf{A}_\mathsf{X}, \hat\lambda_X)$. The uncertainty distribution is

$$\Psi_\xi\left(x|\theta\right) = \hat\lambda_\xi\left\{\xi \le x|\theta\right\} \tag{27}$$

Remark 4.3: The uncertainty observations are presented by observers or experts, while the prior distribution (prior uncertain measure) is offered by knowledgeable experts on the observers' behaviors. In probabilistic Bayesian statistics it is typically assumed that the prior and the likelihood are independent of each other. In Bayesian uncertainty doctrine we continue to follow this convention without any theoretical justification, although the independence between prior and likelihood is debatable.

Remark 4.4: The joint cumulative distribution of the observational data $(x_1, x_2, \cdots, x_n)$ may be specified by a hypothesized copula functional according to the features of the data, as

$$\begin{aligned}&\Psi_{\xi_1, \xi_2, \cdots, \xi_n}\left(x_1, x_2, \cdots, x_n|\theta\right)\\&= \hat\lambda_{\xi_1, \xi_2, \cdots, \xi_n}\left\{\xi_1 \le x_1, \xi_2 \le x_2, \cdots, \xi_n \le x_n|\theta\right\}\\&= C_\varpi\left(\Psi_{\xi_1}\left(x_1\right), \Psi_{\xi_2}\left(x_2\right), \cdots, \Psi_{\xi_n}\left(x_n\right)\right)\end{aligned} \tag{28}$$

where $\Psi_{\xi_1}(\cdot), \Psi_{\xi_2}(\cdot), \cdots, \Psi_{\xi_n}(\cdot)$ are given marginal uncertainty distributions and $\underline{v}$ is unknown parameter vector. In contrast, within probability theory, the multivariate joint distribution function is $F_{X_1, X_2, \cdots, X_n}\left(x_1, x_2, \cdots, x_n|\theta\right)$. Given a population $F(x|\theta)$, the *i.i.d.* random sampling observations have a joint distribution function

$$F_{X_1, X_2, \cdots, X_n}\left(x_1, x_2, \cdots, x_n|\theta\right) = \prod_{k=1}^n F_{X_k}\left(x_k|\theta\right) \tag{29}$$

Also, the joint density (i.e., the likelihood function) is

$$f_{X_1, X_2, \cdots, X_n}\left(x_1, x_2, \cdots, x_n|\theta\right) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n}F_{X_1, X_2, \cdots, X_n}\left(x_1, x_2, \cdots, x_n|\theta\right) = \prod_{k=1}^n f_{X_k}\left(x_k|\theta\right) \tag{30}$$

Now, let us continue our arguments on the posterior uncertainty distribution of q. For the convenience, let us assume that a pair of observations (x_1, x_2) is obtained from the bivariate uncertainty variable, denoted by (x_1, x_2), which is defined by a hypothesized bivariate FGM-normal uncertainty distribution

$$\Psi_{x_1, x_2}(x_1, x_2) = \prod_{i=1}^2 \frac{1}{1 + \exp\left\{-\frac{\pi}{\sqrt{3}s_i}(x_i - q_i)\right\}}\left[1 + v \prod_{i=1}^2 \frac{\exp\left\{-\frac{\pi}{\sqrt{3}s_i}(x_i - q_i)\right\}}{1 + \exp\left\{-\frac{\pi}{\sqrt{3}s_i}(x_i - q_i)\right\}}\right] \tag{31}$$

with marginals

$$\Psi_{x_i}(x_i) = \frac{1}{1 + \exp\left\{-\frac{\pi}{\sqrt{3}s_i}(x_i - q_i)\right\}}, \quad i = 1, 2 \tag{32}$$

Then the bivariate uncertainty distribution has parameter vector $q = (q_1, s_1, q_2, s_2, v)$. For simplification only, we set $q = q_1 = q_2$, and assume that both $s_0 = s_1 = s_2$, and v are known. Then what we aim to derive the posterior distribution of parameter q, i.e., $\Psi_q(y/x_1, x_2, s_0, v_0)$.

For the parameters, there is again a specification issue of the joint multivariate uncertainty prior distribution. For example, the FGM-normal bivariate uncertainty distribution $Y_{x_1,x_2}(x_1,x_2)$ has five parameters, i.e., $q = (q_1, s_1, q_2, s_2, v)$. The full specification of prior vector needs a five-dimensional copula, $C_j(v_1, v_2, v_3, v_4, v_5)$ with marginals $p_i(v_i), i = 1, 2, L, 5$, and the prior takes a form

$$Y_{v_1, v_2, L, v_5}(y_1, y_2, L, y_5) = C_j(p_1(v_1), p_2(v_2), L, p_5(v_5)) \tag{33}$$

Definition 4.5: Let $\Psi(x_1, x_2, \cdots, x_n, \underline{\theta})$ be the joint uncertainty distribution of the uncertainty observations $(x_1, x_2, \cdots, x_n)$ together with the parameter vector $\underline{q}$. Note the event

$$\Lambda = \{\underline{X} \le \underline{x}, \underline{\theta} \le \underline{y}\} \tag{34}$$

Then the joint distribution of $\underline{X}$ and $\underline{\theta}$ defined on (X, A_X, D_X) and (Q, A_Q, D_q) respectively, according to Axiom 5, Liu (2010), is the joint uncertainty measure defined by

$$\Psi_{\underline{X}, \underline{\theta}}(\underline{x}, \underline{y}) = \lambda_{\underline{X}, \underline{\theta}}\{\underline{X} \le \underline{x}, \underline{\theta} \le \underline{y}\}$$

$$= \begin{cases} \sup_{A_1 \times A_2 \subset \Lambda} \min(\lambda_{\underline{\theta}}\{A_1\}, \lambda_{\underline{X}}\{A_1\}) & \text{if if} \sup_{A_1 \times A_2 \subset \Lambda} \min(\lambda_{\underline{\theta}}\{A_1\}, \lambda_{\underline{X}}\{A_1\}) > 0.5 \\ 1 - \sup_{A_1 \times A_2 \subset \Lambda^C} \min(\lambda_{\underline{\theta}}\{A_1\}, \lambda_{\underline{X}}\{A_1\}) & \text{if if} \sup_{A_1 \times A_2 \subset \Lambda^C} \min(\lambda_{\underline{\theta}}\{A_1\}, \lambda_{\underline{X}}\{A_1\}) > 0.5 \\ 0.5 & \text{otherwise} \end{cases} \tag{35}$$

Definition 4.6: We denote $\Psi_{\underline{X}}(\underline{x}) = \Psi_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n)$ as the absolute joint uncertainty distribution.

$$\Psi_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) = \sup_{\underline{y} \in \Theta}\left(\Psi_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n, \underline{y})\right) \tag{36}$$

For example, if a pair of bivariate FGM-normal uncertainty observation (x_1, x_2) is obtained, then

$$\Psi_{X_1, X_2}(x_1, x_2) = u_1 u_2 (1 + \varpi(1 - u_1)(1 - u_2)) \tag{37}$$

where

$$u_i = Y_{x_i}(x_i) = \frac{1}{1 + \exp\left(-\frac{p}{\sqrt{3}s_i}(x_i - q_i)\right)}, \ i = 1, 2 \tag{38}$$

Finally, we define a Bayesian uncertainty posterior for $\underline{q}$.

Definition 4.7: We denote $\Psi_{\underline{\theta}}(\underline{y}|x_1, x_2, \cdots, x_n)$ as the posterior uncertainty distribution under the Maximum Uncertainty Principle, The MUP posterior uncertainty distribution is thus

$$\Psi_{\underline{\theta}}(\underline{y}|x_1, x_2, \cdots, x_n) = \frac{\Psi_{\xi_1, \xi_2, \cdots, \xi_n, \underline{\theta}}(x_1, x_2, \cdots, x_n, \underline{y})}{\Psi_{\xi_1, \xi_2, \cdots, \xi_n}(x_1, x_2, \cdots, x_n)} \tag{39}$$

5   A BAYESIAN POSTERIOR UNCERTAINTY EXAMPLE

In this section, we take the uncertainty zigzag distribution as the uncertainty prior, and Liu's (2007, 2009, 2010) normal distribution as uncertainty observation distribution, and in a step by step manner, illustrate the construction of an posterior uncertainty distribution.

The observational data is assumed to be a representative value of the population observed, which can be specified by a hypothesized uncertainty distribution. For our example, the hypothesized uncertainty distribution is Liu's (2007, 2009, 2010) uncertainty normal distribution:

$$\Psi\left(x|\theta,\sigma_0\right)=\frac{1}{1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma_0}\left(x-\theta\right)\right)} \tag{40}$$

As an illustration, it is assumed that the standard deviation is given, denoted by s_0, the only unknown is parameter q, the mean or expectation of the uncertainty distribution. Because in Bayesian treatments the parameter in the distribution function is no longer an unknown real number, the parameter is treated as an uncertainty variable, Its distribution is supposed to be uniquely specified by a given uncertainty measure D_q defined on an uncertain measurable space $(Q,A(Q))$. In practice, the unknown parameter is usually specified by an uncertainty distribution. Although the uncertainty distribution can induce an uncertain measure on Borel measurable space $(Y,B(Y))$, nevertheless, it is unique in the sense of an equivalence class. The uncertainty prior distribution is assumed to be

$$G_\theta\left(y\right)=\frac{y-a}{2(b-a)}\vartheta_{(a,b]}\left(y\right)+\frac{y+c-2b}{2(c-b)}\vartheta_{(b,c]}\left(y\right) \tag{41}$$

where

$$\vartheta_{(a,b]}\left(y\right)=\begin{cases}1 & \text{if} \quad a<y\leq b \\ 0 & \text{otherwise}\end{cases} \tag{42}$$

We further assume that a single observation $x=2.3$ is taken from hypothesized Liu's uncertainty normal distribution $\Psi_x\left(x|\theta,2.00\right)=1/\left(1+\exp\left(-\pi(x-\theta)/2\sqrt{3}\right)\right)$ and the uncertainty prior parameter $(a,b,c)=(0,2,3)$, i.e., the uncertainty prior distribution is

$$G_\theta\left(y\right)=\frac{1}{4}y\vartheta_{(0,2]}\left(y\right)+\frac{1}{2}(y-1)\vartheta_{(b,c]}\left(y\right) \tag{43}$$

The Axiom 5 based posterior uncertainty distribution of θ given uncertainty observation $x=2.3$ is

$$\Psi_\theta\left(y|x=2.3,\sigma_0=2\right)=\frac{\Psi_{\xi,\theta}\left(2.3,y\right)}{\sup_{y\in\Theta}\left(\Psi_{\xi,\theta}\left(2.3,y\right)\right)} \tag{44}$$

where

$$\Psi_{\xi,\theta}\left(2.3,y\right)=\begin{cases}\sup_{A_1\times A_2\subset\Lambda}\min\left(G_\theta\left(y\right),\Psi_\xi\left(2.3|y,2.0\right)\right) & \text{if} \quad \sup_{A_1\times A_2\subset\Lambda}\min\left(G_\theta\left(y\right),\Psi_\xi\left(2.3|y,2.0\right)\right)>0.5 \\ 1-\sup_{A_1\times A_2\subset\Lambda^C}\min\left(G_\theta\left(y\right),\Psi_\xi\left(2.3|y,2.0\right)\right) & \text{if} \quad \sup_{A_1\times A_2\subset\Lambda^C}\min\left(G_\theta\left(y\right),\Psi_\xi\left(2.3|y,\sigma_0\right)\right)>0.5 \\ 0.5 & \text{otherwise}\end{cases} \tag{45}$$

and

$$\sup_{y\in\Theta}\left(\Psi_{\xi,\theta}\left(2.3,y\right)\right)=0.60392 \tag{46}$$

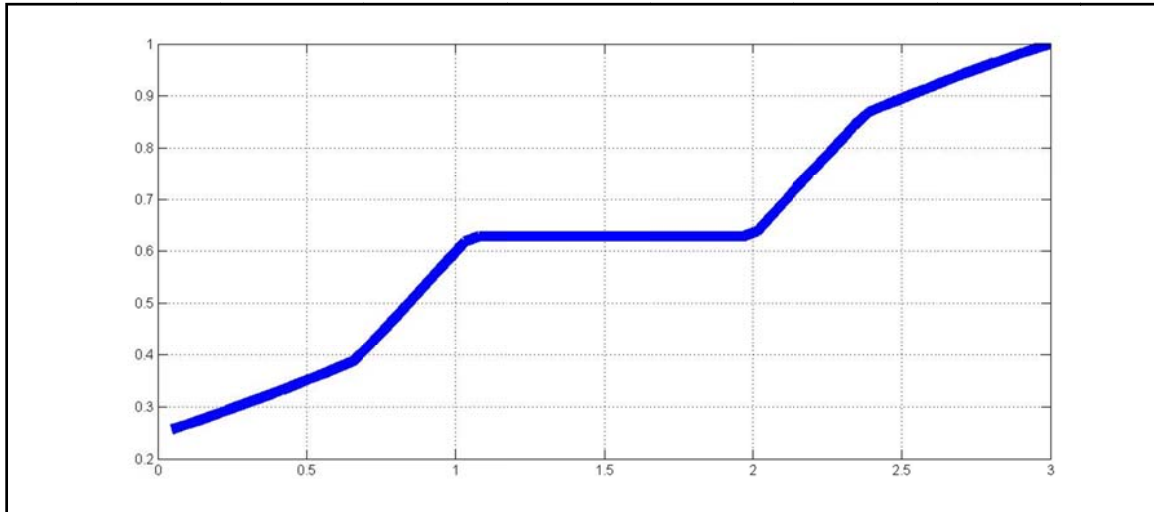The plot of the posterior uncertainty distribution $\Psi_\theta\left(y|x=2.3,\sigma_0=2\right)$ is shown in Figure 1.

Figure 1. Posterior uncertainty distribution $\Psi_\theta\left(y\,\middle|\,x=2.3,\sigma_0=2\right)$

Once we find the posterior uncertainty distribution $\Psi_\theta\left(y\,\middle|\,x=2.3,\sigma_0=2\right)$, it is very natural to calculate the uncertainty posterior mean $E\left[q\,/\,x=2.3\right]$ and variance $V\left[q\,/\,x=2.3\right]$, and carry on further Bayesian inferential analysis.

6   CONCLUSIONS

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In this paper, a Bayesian uncertainty decision theoretical framework is proposed under the uncertain measure foundation, which paves the way toward data-oriented inferential uncertainty statistics.

The contributions of this paper are listed as follows: (1) for the first time a concrete uncertainty multivariate uncertainty distribution, in terms of an uncertainty copula with uncertainty marginals, is presented in the uncertainty theory literature; (2) for the first time an uncertainty product measure data-oriented posterior uncertainty distribution is developed from axiom 5; and (3) a detailed illustrative example is given in stepwise manner.

Definitely, the treatments in this paper are debatable. Particularly, the independence between prior measure and observational data uncertainty distribution measure may be contested. Also, we have not explored the necessary and sufficient conditions for an uncertainty copula constructed multivariate uncertainty distribution to be uncertainty measure. There are many unaddressed questions ahead of us for the new Bayesian uncertainty decision theory to become fully applicable in industries and business.

7   ACKNOWLEDGEMENTS

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