HYBRID RELIABILITY MODELLING WITH IMPRECISE PARAMETER

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ABSTRACT

The real world phenomena are often facing the co-existence reality of different formality of uncertainty and thus the probabilistic reliability modeling practices are very doubtful. Under complicated uncertainty environments, hybrid variable modeling is important in reliability and risk analysis, which includes Bayesian distributional theory, random fuzzy distributional theory, as well as fuzzy random distributional theory as special distribution families. In this paper, we define a new hybrid lifetime which is specified by a random lifetime distribution with imprecise parameter with an uncertainty distribution. We furthermore define the average chance distribution as a quality index for quantifying the hybrid lifetime and accordingly the average chance reliability is derived.

1 INTRODUCTION

System reliability, as a quality index, is the capability to complete the specified functions accurately in mutually harmonious manner under the specified conditions within specified period. The quality and reliability engineering facilitates the specification of the system reliability function on the ground of probability and statistics theory. The Toyota crisis does not only tear off the brand image of quality but also shake the belief of existing quality and reliability engineering practices and the underlying probability and statistics theory, which treat the random uncertainty. Uncertainty in real world is intrinsic and diversified in formality. For example, the vagueness is another form of uncertainty, which is more and more aware of in today's industrial environments, just as Carvalho & Machado (2006) commented, "In a global market, companies must deal with a high rate of changes in business environment. … The parameters, variables and restrictions of the production system are inherently vagueness." Therefore quality and reliability engineering is no longer a blind exercise of applying the traditional techniques from existing probabilistic reliability engineering literature.

The coexistence of randomness and other forms of uncertainty in reliability concept is intrinsic and inherent and therefore modern reliability analysis inevitably engages hybrid lifetime modeling.

Accordingly, the methodology to solve the reliability of hybrid lifetime should be developed in terms of the basic concept of general uncertain measure theory.

The remaining structure of the paper is stated as follows: Section Two serves reviewing Liu's axiomatic uncertain measure and defines the concept of impreciseness in terms of uncertainty distribution; Section Three is utilized to establish the hybrid variable theory. Particularly, the hybrid variable is constituted by a random lifetime with an imprecise parameter governed by an uncertainty distribution; Section Four defines the average chance measure for hybrid variable; Section Five is used to investigate the construction of hybrid variable; while in Section Six the commonly used lifetime models for construction of hybrid lifetime models are discussed; Section Seven uses exponential lifetime with imprecise uncertainty parameter for develop the average chance reliability as an illustrative examples; and Section Eight concludes the paper.

2 UNCERTAIN MEASURE AND IMPRECISENESS

Uncertain measure (Liu (2010)) is an axiomatically defined set function mapping from a σ-algebra of a given space (set) to the unit interval [0,1], which provides a measuring grade system of an uncertain phenomenon and facilitates the formal definition of an uncertain variable.

Let Ξ be a nonempty set (space), and $\mathrm{A}(\Xi)$ the σ-algebra on Ξ. Each element, let us say, $A\subset\Xi, A\in\mathrm{A}(\Xi)$ is called an uncertain event. A number denoted as $ƛ\{A\}$, $0\le ƛ\{A\}\le 1$, is assigned to event $A\in\mathrm{A}(\Xi)$, which indicates the uncertain measuring grade with which event $A\in\mathrm{A}(\Xi)$ occurs. The normal set function $ƛ\{A\}$ satisfies following axioms given by Liu (2007, 2009, 2010):

Axiom 1: (Normality) $ƛ\{\Xi\}=1$.

Axiom 2: (Monotonicity) $ƛ\{\cdot\}$ is non-decreasing, i.e., whenever $A\subset B$, $ƛ\{A\}\le ƛ\{B\}$.

Axiom 3: (Self-Duality) $ƛ\{\cdot\}$ is self-dual, i.e., for any $A\in\mathrm{A}(\Xi)$, $ƛ\{A\}+ƛ\{A^c\}=1$.

Axiom 4: (σ-Subadditivity) $ƛ\left\{\bigcup_{i=1}^{\infty}A_i\right\}\le\sum_{i=1}^{\infty}ƛ\{A_i\}$ for any countable event sequence $\{A_i\}$.

Axiom 5: (Product Measure) Let $(\mathrm{X}_k,\mathrm{A}_{\mathrm{X}_k},\mathrm{D}_k)$ be the k^{th} uncertain space, $k=1,2,\mathrm{L},n$. Then product uncertain measure D on the product measurable space $(\mathrm{X},\mathrm{A}_{\mathrm{X}})$ is defined by

$$ƛ=ƛ_1\wedge ƛ_2\wedge\cdots\wedge ƛ_n=\min_{1\le k\le n}\{ƛ_k\} \tag{1}$$

where

$$\Xi=\Xi_1\times\Xi_2\times\cdots\times\Xi_n=\prod_{k=1}^{n}\Xi_k \tag{2}$$

and

$$\mathrm{A}_{\Xi}=\mathrm{A}_{\Xi_1}\times\mathrm{A}_{\Xi_2}\times\cdots\times\mathrm{A}_{\Xi_n}=\prod_{k=1}^{n}\mathrm{A}_{\Xi_k} \tag{3}$$

That is, for each product uncertain event $\mathrm{L\,O\,A}_{\mathrm{X}}$ (i.e, $\mathrm{L}=\mathrm{L}_1\,\mathrm{r}\,\mathrm{L}_2\,\mathrm{r}\,\mathrm{L}\,\mathrm{r}\,\mathrm{L}_n\,\mathrm{O\,A}_{\mathrm{X}_1}\,\mathrm{r}\,\mathrm{A}_{\mathrm{X}_2}\,\mathrm{r}\,\mathrm{L}\,\mathrm{r}\,\mathrm{A}_{\mathrm{X}_n}=\mathrm{A}_{\mathrm{X}}$), the uncertain measure of the event L is

$$ƛ\{\Lambda\}=\begin{cases}\sup\limits_{A_1\times\cdots\times A_n\subset\Lambda}\ \min\limits_{1\le k\le n}ƛ\{\Lambda_k\} & \text{if } \sup\limits_{A_1\times\cdots\times A_n\subset\Lambda}\ \min\limits_{1\le k\le n}ƛ\{\Lambda_k\}>0.5\\ 1-\sup\limits_{A_1\times\cdots\times A_n\subset\Lambda^c}\ \min\limits_{1\le k\le n}ƛ\{\Lambda_k\} & \text{if } \sup\limits_{A_1\times\cdots\times A_n\subset\Lambda^c}\ \min\limits_{1\le k\le n}ƛ\{\Lambda_k\}>0.5\\ 0.5 & \text{otherwise}\end{cases} \tag{4}$$

Definition 2.1: (Liu (2007, 2009, 2010)) Any set function $\mathrm{D}:\mathrm{A}(\mathrm{X})\text{®}[0,1]$ which satisfies Axioms 1-4 is called an uncertain measure. The triple $(\mathrm{X},\mathrm{A}(\mathrm{X}),\mathrm{D})$ is called the uncertain measure space.

Definition 2.2: (Liu (2007, 2009, 2010)) An uncertain variable ξ is a measurable mapping, i.e., $\xi:(\Xi,\mathrm{A}(\Xi))\to(\mathrm{R},\mathrm{B}(\mathrm{R}))$, where $\mathrm{B}(\mathrm{R})$ denotes the Borel σ-algebra on $\mathrm{R}\equiv(-\infty,+\infty)$.

Remark 2.3: The fundamental difference between a random variable and an uncertain variable is the σ-additivity: the probability measure obeys σ-additivity (Kolmogorov (1950), Primas (1999)) and the uncertain measure (Kaufmann (1975), Liu (2007, 2009, 2010)) obeys σ-subadditivity. The way of specifying measure inevitably has impacts on the behaviour of the measurable function over the triple, and hence on the mathematical characterization of the theories. For example, in contrast to probability theory, no "uncertainty density function" can be defined and then be entered into an integral of density to characterise an uncertainty distribution. Because an uncertain measure is permitted to be σ-subadditive, any set of uncertainty distributions derived from integration, being necessarily σ-additive, will necessarily be incomplete.

Definition 2.4: (Liu (2007, 2009, 2010)) The uncertain distribution $\Psi : \mathrm{R} \to [0,1]$ of an uncertain variable ξ on $(\mathrm{X}, \mathrm{A}(\mathrm{X}), \mathrm{D})$ is

$$\Psi_{\xi}(x) = \lambda\left\{\tau \in \Xi \middle| \xi(\tau) \le x\right\} \tag{5}$$

Theorem 2.5: (Peng and Iwamura (2010)) The necessary and sufficient conditions for a function $\Psi : \mathbb{R} \to [0,1]$ be an uncertainty distribution function is that Ψ is non-decreasing function and

$$0 \le \Psi(x) \le 1, \ \forall x \in \mathbb{R} \tag{6}$$

The function Ψ is referred to an uncertainty distribution function.

Remark 2.6: A probability distribution $F_X(x)$ requires right-continuity and $F_X(-\infty) = 0, F_X(+\infty) = 1$ in addition to those requirements of the uncertainty distribution function, while an uncertainty distribution is not limited by any continuity and $\Psi_{\xi}(-\infty) = 0, \Psi_{\xi}(+\infty) = 1$ requirements. This relaxation enables an uncertainty distribution to model even the most complicated pattern in real world data. The following definition reveals an essential characteristic of the uncertainty distribution.

Definition 2.7: Let ξ be an uncertainty variable, which takes values from a subset, denoted as E, of the real line $\mathbb{R}$, with n discontinuity points collected in an ascending order as set $\mathrm{D} \equiv \{c_1, \cdots, c_n\}$. The uncertainty distribution, Ψ, of the variable ξ is specified as follows:

1. On the set $\mathrm{D} \equiv \{c_0, c_1, \cdots, c_n\}$,

$$\begin{aligned} &\Psi(c_i -) = \psi_{i-}, \ \Psi(c_i) = \psi_i, \ \Psi(c_i +) = \psi_{i+} \\ &i = 1, 2, \cdots, n \end{aligned} \tag{7}$$

where $\psi_{i-} < \psi_i < \psi_{i+}$, $\psi_{1-} \ge 0$, $\psi_{n+} \le 1$, $i = 1, 2, \cdots, n$;

2. At the inner points of the sub-intervals (c_{i-1}, c_i), $i = 1, 2, \cdots, n$, the uncertainty distribution Ψ is continuous

$$\Psi(z) = \begin{cases} \psi_{i-1+} & \text{if } z \downarrow c_{i-1} \\ \Lambda_i(z) & \text{if } z \in (c_{i-1}, c_i) \\ \psi_{i-} & \text{if } z \uparrow c_i \end{cases} \tag{8}$$

where the function Λ_i is positive, non-decreasing, and bounded by ψ_{i-1+} and ψ_{i-}, i.e., $\psi_{i-1+} \le \Lambda_i \le \psi_{i-}$, $i = 1, 2, \cdots, n$. Then Ψ is an uncertainty distribution of the essential form and ξ is called an essential uncertain variable.

Remark 2.8: Whenever an "observation" is obtained, this specific observation should not be regarded as an isolated real number (or a real-valued vector), rather, it should be regarded as a representative from a "population" typically specified by a hypothesized uncertainty distribution. This approach matches the standard viewpoint in the statistical community, see wikipedia (2010). It is also a convention that the term "population" (Bernardo & Smith (1994), Lee (1989)) is equivalent to the term distribution, or to the term random variable. In the new uncertainty theory, this statistical convention should be retained. We formally state this convention as a definition on observational data.

Definition 2.9: An observation is a real number, (or more broadly, a symbol, or an interval, or a real-valued vector, a statement, etc), which is a representative of a population or equivalently of an uncertainty distribution under a given scheme comprising set and σ-algebra.

Remark 2.10: The uncertainty distribution is unknown but exists objectively. A workable solution is to hypothesize a family of uncertainty distributions of a specified functional form with unknown parameter θ, where the family is denoted by $\{\Psi_\xi^\theta, \theta \in \Theta\}$.

Definition 2.11: (Liu (2007, 2009, 2010)) Let multivariate uncertainty variable $(\xi_1, \xi_2, \cdots, \xi_d)$ be defined on an uncertain measure space $(\Xi, \mathrm{A}(\Xi), \Delta)$, then the multivariate function $\Psi_{\xi_1,\xi_2,\cdots,\xi_d} : D \to [0,1]$ is called an multivariate uncertainty distribution if

$$\Psi_{\xi_1,\xi_2,\cdots,\xi_d}(x_1, x_2, \cdots, x_d) = \Delta\{\xi_1 \le x_1, \xi_2 \le x_2, \cdots, \xi_d \le x_d\} \tag{9}$$

To present a concrete form of a multivariate uncertainty distribution, Guo et al. (2010) propose a copula-linked uncertainty marginals approach.

Definition 2.12: Let $(\xi_1, \xi_2, \cdots, \xi_d)$ be a multivariate uncertainty variable with joint uncertainty distribution $\Psi_{\xi_1,\xi_2,\cdots,\xi_d}(x_1, x_2, \cdots, x_d)$, in which all the marginal uncertainty distributions $\Psi_{\xi_1}(\cdot), \Psi_{\xi_2}(\cdot), \cdots, \Psi_{\xi_d}(\cdot)$ exist and are regular (i.e., $\Psi_{\xi_i}^{-1}(\cdot)$ exists, $i = 1, 2, \cdots, d$). Then the uncertainty copula is defined by

$$C\left(\Psi_{\xi_1}(x_1), \Psi_{\xi_2}(x_2), \cdots, \Psi_{\xi_d}(x_d)\right) = \Psi_{\xi_1,\xi_2,\cdots,\xi_d}(x_1, x_2, \cdots, x_d) \tag{10}$$

We use a bivariate uncertainty distribution as an illustrative multivariate example.

Example 2.13: Let bivariate uncertainty variable (ξ_1, ξ_2) have marginal uncertainty distributions $\Psi_{\xi_1}(\cdot)$ and $\Psi_{\xi_2}(\cdot)$ respectively. The Farlie-Gumbel-Morgenstern (FGM) copula is defined by

$$C(u_1, u_2) = u_1 u_2 \left(1 + \nu (1 - u_1)(1 - u_2)\right), \ \nu \in [-1, 1] \tag{11}$$

Further, let the bivariate uncertainty variable (ξ_1, ξ_2) have marginal uncertainty distributions $\Psi_{\xi_1}(\cdot)$ and $\Psi_{\xi_2}(\cdot)$ respectively, where

$$\Psi_{\xi_i}(x_i) = \frac{1}{1 + \exp\left[-\frac{\pi}{\sqrt{3}\sigma_i}(x_i - \theta_i)\right]}, \ i = 1, 2 \tag{12}$$

Then the bivariate FGM-Normal joint uncertainty distribution is

$$\Psi_{\xi_1,\xi_2}(x_1, x_2) = \prod_{i=1}^{2} \frac{1}{1 + \exp\left[-\frac{\pi}{\sqrt{3}\sigma_i}(x_i - \theta_i)\right]} \left[1 + \nu \prod_{i=1}^{2} \frac{\exp\left[-\frac{\pi}{\sqrt{3}\sigma_i}(x_i - \theta_i)\right]}{1 + \exp\left[-\frac{\pi}{\sqrt{3}\sigma_i}(x_i - \theta_i)\right]}\right] \tag{13}$$

Finally, it is critical to define impreciseness with mathematical rigor. To achieve this goal, we review the discussions on randomness concept in statistics first for comparison purpose. Randomness in classical (i.e., probabilistic) statistics is referred to a term with an intrinsic property "governed by or involving equal chances for each of the actual or hypothetical members of a population; (also) produced or obtained by such a process, and therefore unpredictable in detail".

Randomness is "closely connected, therefore, with the concepts of chance, probability, and information entropy, randomness implies a lack of predictability. More formally, in statistics, a random process is a repeating process whose outcomes follow no describable deterministic pattern, but follow a probability distribution, such that the relative probability of the occurrence of each outcome can be approximated or calculated", see wikipedia (2010). In other words, randomness is an intrinsic property of a variable or an observation being characterized by a probability measure. Just as Kolmogrov (1950) emphasized probability measure specification is the prerequisite to randomness.

Remark 2.14: Parallel to revelation of the connotation of randomness. Impreciseness in uncertainty statistics is referred to a term with an intrinsic property governed by an uncertain measure or an uncertainty distribution for each of the actual or hypothetical members of an uncertainty population; (also) produced or obtained by such a process, and therefore unpredictable in detail. An uncertainty process is a repeating process whose outcomes follow no describable deterministic pattern, but follow an uncertainty distribution, such that the uncertain measure of the occurrence of each outcome can be only approximated or calculated.

Definition 2.15: Impreciseness is an intrinsic property of a variable or an expert's knowledge being specified by an uncertainty measure.

Remark 2.16: Impreciseness exists in engineering, business and research practices. Just as Utikin & Gurov (2000) as well as Walley (1991) argued strongly that “it very often happens that probabilities cannot be determined exactly, either due to measurement imperfections, or due to more fundamental reasons, such as insufficient available information, ... , or "is of a linguistic nature, i.e. the information is conveyed by statements in natural language”, …, a part of “the reliability assessments may be supplied by experts” or reliability “assessments may be made by the user of the system during the experimental service”. Thus it is an unarguable fact that impreciseness exists intrinsically in expert’s knowledge on the real world.

Definition 2.17: Let ξ be a uncertainty quantity of impreciseness on an uncertainty measure space $(\Xi, \mathcal{A}(\Xi), \lambda)$. The uncertainty distribution of ξ is $\Psi_{\xi}(x) = \lambda\{\tau \in \Xi \mid \xi(\tau) \le x\}$.

Remark 2.18: An imprecise variable ξ is an uncertainty variable and thus is a measurable mapping, i.e., $\xi : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$. An observation of an imprecise variable is a real number, (or more broadly, a symbol, or an interval, or a real-valued vector, a statement, etc), which is a representative of the population or equivalently of an uncertainty distribution $\Psi_{\xi}(\cdot)$ under a given scheme comprising set and σ-algebra. The single value of a variable with impreciseness should not be understood as an isolated real number rather an interval or a set.

3 HYBRID VARIABLE THEORY

Since Zadeh (1965, 1978) proposed fuzzy set theory, fuzzy random fuzzy set, a special case of hybrid variable, soon proposed by Kaufmann (1975). Liu (2007) defined that a random fuzzy variable, another special case of hybrid variable, is a mapping from the credibility space $(\Theta, 2^{\Theta}, \mathrm{Cr})$ to a set of random variables. Let us start with a general hybrid variable definition.

Definition 3.1: (Liu (2007)) A hybrid variable is a real-valued measurable mapping, i.e., $\eta : (\Xi, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$.

Remark 3.2: It is obvious that the order of the formation of a hybrid variable does matter. For example, Random fuzzy variable (Liu (2007)) and fuzzy random variable (Kaufmann (1975)) are two types of hybrid variable, even with the same component uncertain variables. Therefore, it is necessary to define them separately when specifying the hybrid variable with different uncertain variables.

Definition 3.3: A random-uncertain hybrid variable is a measurable mapping η from product space $(X,A(X),D)r\ (W,F(W),Pr)$ into $(R,B(R),n)$, which is called as hybrid variable of Type I; An uncertain-random hybrid variable is a measurable mapping η from product space $(W,F(W),Pr)r\ (X,A(X),D)$ into $(R,B(R),n)$, which is called hybrid variable of Type II.

In the remaining of the paper, we only deal with hybrid variable of Type I, i.e., random-uncertain hybrid variable. Therefore, for convenience we simply use the term hybrid variable. For reliability engineers and managers armed with introductory probability and statistics, this definition will be difficult to understand. For a more intuitive understanding, we would like to present a definition similar to that of stochastic process in probability theory and expect readers who are familiar with the basic concept of stochastic processes can understand our comparative definition.

Definition 3.4: A hybrid variable (of Type I), denoted by $\xi = \left\{X_{\beta(\tau)}, \tau \in \Xi\right\}$, is a collection of random variables X_β defined on the common probability space $(\Omega, F, \Pr)$ and indexed by an uncertain variable $\beta(\tau)$ defined on the uncertainty space $(X,A(X),D)$.

Similar to the interpretation of a stochastic process $X = \left\{X_t, t \in \mathbf{R}^+\right\}$, a hybrid variable is also a bivariate mapping from $(\Omega \times \Xi, F \times A)$ to the space (R, B). As to the index set, in stochastic process theory, index set used is referred to as time typically, which is a positive (scalar variable), while in the random fuzzy variable theory, the "index" is an uncertain variable β. Using uncertain parameter as index is not starting in hybrid variable definition. In stochastic process theory we already know that the stochastic process $X = \left\{X_{\tau(\omega)}, \omega \in \Omega\right\}$ uses stopping time $t(w)$, $w\,O\,W$, which is an random variable as its index.

4 AVERAGE MEASURE FOR A HYBRID VARIABLE

Hybrid variable can be quantified in terms of chance measure concept, see Liu (2007).

Definition 4.1: Let ξ be a random-uncertain hybrid variable and B a Borel set of real numbers. Then the chance measure of random fuzzy event $\{\xi \in B\}$ is a function mapping from $(0,1]$ to $[0,1]$,

$$\mathrm{Ch}\,\{x\,O\,B\}(a) = \sup_{D\{A\}i\ a}\ \inf_{qOA} \Pr\{q : x(q)\,O\,B\} \tag{14}$$

However, we notice the potential mathematical complexity associated with the chance measure formulation, see Liu (2008). Therefore, it is necessary to explore a convenient way to deal with the chance measure specification. Recall that in probability theory, the distribution of a random variable ξ on probability space $(W, A, \Pr)$, $F_\xi(\cdot)$ links to the probability measure of event " $\{w : x(w)\,J\ x\}\,O\,A$

$$F_x(x) = \Pr\{w : x(w)\,J\ x\}. \tag{15}$$

In random-uncertain hybrid variable theory, we may say that that average chance measure plays an equivalent role similar to probability measure, denoted as $\Pr$, in probability theory.

Definition 4.2: Let x be a random-uncertain hybrid variable, then the *average* chance measure, denoted as $\mathrm{ch}\,\{\}$, of a random-uncertain event $\{t\ O\,X : x(t)\,J\ x\}$, is

$$\mathrm{ch}\{\xi \le x\} = \int_0^1 \Delta\{\tau \in \Xi \mid \Pr\{\xi(\tau) \le x\} \ge \alpha\} d\alpha \tag{16}$$

Then function $\Psi(\cdot)$ is called as average chance distribution if and only if

$$\Psi(x) = \mathrm{ch}\{\xi \le x\} \tag{17}$$

Now, we are required to establish a theoretical framework in terms of average chance measure concepts. Once the average chance measure for the basic event form $\{\xi \le x\}$ is given, then the average chance measure for any event A should be established in terms of the basic event $\{\xi \le x\}$. In this way, we may define average chance measure for an arbitrary event A. The triple space $(\Omega\times\Xi, \mathcal{F}\times\mathcal{A}, \mathrm{ch})$ is called the average chance space.

Proposition 4.3: Let $\mathrm{ch}(\cdot)$ be an average chance measure on a product measure space $(\Omega\times\Xi, \mathcal{F}(\Omega)\times\mathcal{A}(\Xi))$. Then

(i) $\mathrm{ch}\{\emptyset\} = 0$ and $\mathrm{ch}\{\Omega\times\Xi\} = 1$;

(ii) (Normality) $\forall A \in \mathcal{F}\times\mathcal{A}$, $0 \le \mathrm{ch}\{A\} \le 1$;

(iii) (Self-Duality) For $\forall A \in \mathcal{F}\times\mathcal{A}$, then $\mathrm{ch}\{A^c\} = 1 - \mathrm{ch}\{A\}$

(iv) (Weak monotone increasing) For $\forall A \subset B$, $A, B \in \mathcal{F}\times\mathcal{A}$, $\mathrm{ch}\{A\} \le \mathrm{ch}\{B\}$;

(v) (Semi-Continuity) For $\forall A_n, A \in \mathcal{F}\times\mathcal{A}$, $n = 1, 2, \cdots$, if $A_n \to A$, then

$$\lim_{A_n \to A} \mathrm{ch}\{A_n\} = \mathrm{ch}\{A\} \tag{18}$$

if and only if one of the following conditions holds:

(a) $\Delta\{A_n\} \le 0.5$ & $A_n \uparrow A$,

(b) $\lim_{n\to\infty} \Delta\{A_n\} < 0.5$ & $A_n \uparrow A$,

(c) $\Delta\{A_n\} \ge 0.5$ & $A_n \downarrow A$, and

(d) $\lim_{n\to\infty} \Delta\{A_n\} > 0.5$ & $A_n \downarrow A$.

(vi) (Sub-Additivity) For $\forall A \subset B$, $A, B \in \mathcal{F}\times\mathcal{A}$,

$$\mathrm{ch}\{A \cup B\} \le \mathrm{ch}\{A\} + \mathrm{ch}\{B\} \tag{19}$$

Proposition 4.4: Let $\Psi_\xi(\cdot)$ be average chance distribution of (random-uncertain) hybrid variable ξ on the chance measure space $(\Omega\times\Xi, \mathcal{F}\times\mathcal{A}, \mathrm{ch})$. Then

(i) $\Psi_\xi(-\infty) = 0$ and $\Psi_\xi(+\infty) = 1$;

(ii) For $\forall x \in \Re = (-\infty, +\infty)$, $0 \le F_\xi(x) \le 1$;

(iii) Nonnegative real-valued function $\psi_\xi(\cdot)$ is called average chance density for a (random-uncertain) hybrid variable ξ if for $\psi_\xi(x) \ge 0$, $x \in \Re$ and

$$\Psi_\xi(x) = \int_{-\infty}^{x} \psi_\xi(u) du \tag{20}$$

5 CONSTRUCTION OF HYBRID VARIABLE

Liu (2007) mentioned an exponentially distributed random fuzzy variable ξ has a density function

$$\phi(x)=\begin{cases}\frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right) & \text{if } x\geq 0\\ 0 & \text{otherwise}\end{cases} \tag{21}$$

if the value of β is assumed to be a fuzzy variable, then ξ is a random fuzzy variable. Similarly, let parameter β be an uncertain variable following a distribution function $\Lambda_{\beta}(\cdot)$, and the probability density is defined by Equation (11), then the random-uncertain hybrid variable ξ is said to be exponentially distributed. This example hints a constructive definition for specifying hybrid variable, i.e., random-uncertain variable or equivalently, the average chance distribution.

Definition 5.1: Let $\{F(x;b(t)), t \text{ O X}\}$ be a family of probability distributions on the probability space $(\text{W}, \text{A}, \Pr)$ with a common uncertain parameter β on the uncertain measure space $(\text{X}, \text{A}(\text{X}), \text{D})$, then the average distribution derived from $(F(x,\text{b}), \text{D})$ defines a (random-uncertain) hybrid variable x.

Theorem 5.2: Let ξ be a random-uncertain hybrid variable. If the expectation $E_P\left[\xi(\tau_0)\right]$ exists for any given $\tau_0 \in \Xi$, then $E_P\left[\xi(\cdot)\right]$ is an uncertain variable.

6 RANDOM LIFETIME WITH IMPRECISE PARAMETER

Analyzing hybrid lifetimes, or survival times, or failure times, is the focus of lifetime modeling and analysis under randomness and general uncertainty co-existence environments. Different from the statistical lifetime modeling and analysis, where the random lifetimes are concerned, also different from the uncertainty lifetime modeling and analysis, where the uncertainty lifetimes are concerned, hybrid lifetime modeling analysis provides a general guideline with a rigorous theoretical foundation.

A (random-uncertain) hybrid lifetime, denoted by x, which is a special case of hybrid (of Type I), takes only a positive real values. In other words, hybrid lifetime is a bivariate mapping from $(\Omega\times\Xi, \text{F}\times\Xi)$ to the space $\left(\text{R}^{+}, \text{B}\left(\text{R}^{+}\right)\right)$.

6.1 Basic construction of continuous hybrid lifetimes

It is well-known fact the probability distribution contains the full information on system lifetime and there are many related concepts, particularly, hazard function reveals an aspect of lifetime distribution, which links to the physical structure of a system.

Theorem 6.1: Let x be a continuous hybrid lifetime having probability distribution function $F(t;\text{b}(\text{t}))$, where the imprecise parameter b is defined on the uncertain measure space $(\text{X}, \text{A}(\text{X}), \text{D})$. Then function $\text{P}(t;\text{b}) = \text{L}(F(t;\text{b}))$ can uniquely define the hybrid lifetime x if the operator or function L is invertible.

Table 1 lists four commonly used operators or functions.

Table 1. Examples of operators or functions

Name	Form of $\Pi(t;\beta)$	$\Lambda(\cdot)$
Survival function	$\bar{F}(t;\beta)=1-F(t;\beta)$	$F(t;\beta)=1-\bar{F}(t;\beta)$
Density function	$f(t;\beta)=dF(t;\beta)/dt$	$F(t;\beta)=\int_0^t f(u;\beta)du$
Hazard function	$h(t;\beta)=f(t)/(1-F(t;\beta))$	$F(t;\beta)=1-\exp\left(-\int_0^t h(u;\beta)du\right)$
Moment generating function	$m(\theta;\beta)=\int_0^{+\infty} e^{\theta t}dF(t;\beta)$	$F(t;\beta)=\int_0^t\left(\frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} m(s;\beta)e^{su}ds\right)du$

6.2 Continuous hybrid lifetime models

In statistical lifetime modeling and analysis, the elementary lifetime models are exponential, Weibull, Log-normal, gamma, Cox-Lewis, bathtub, and etc. These are essential for the construction of hybrid lifetimes. Table 2 lists these models.

In Table 2, $I(\beta,\lambda t)$ denotes the incomplete gamma function of the first-type and $\Phi(\cdot)$ represents the cumulative distribution of a standard normal variable.

Table 2. Commonly used distributional lifetime models

Name	Probability density & hazard function	
Exponential	density	$\beta\exp(-\beta t)$
	hazard	β
Weibull	density	$(\beta/\eta)(t/\eta)^{\beta-1}\exp\left(-(t/\eta)^{\beta}\right)$
	hazard	$(\beta/\eta)(t/\eta)^{\beta-1}$
Extreme - value	density	$(1/u)\exp((t-b)/u)\exp(-\exp((t-b)/u))$
	hazard	$(1/u)\exp((t-b)/u)$
Log-Normal	density	$\left(1/(\sqrt{2\pi}\sigma t)\right)\exp\left(-(\ln t-\mu)^2/2\sigma^2\right)$
	hazard	$\left(\left(1/(\sqrt{2\pi}\sigma t)\right)\exp\left(-(\ln t-\mu)^2/2\sigma^2\right)\right)/\left(1-\Phi((\ln t-\mu)/\sigma\right.$
Gamma	density	$\left(\lambda(\lambda t)^{\beta-1}/\Gamma(\beta)\right)e^{-\lambda t}$
	hazard	$\left(\lambda(\lambda t)^{\beta-1}/\Gamma(\beta)\right)e^{-\lambda t}/\left(1-I(\beta,\lambda t)\right)$
Bathtub	density	$(\beta/\eta)(t/\eta)^{\beta-1}\exp\left((t/\eta)^{\beta-1}\right)\exp\left(-\exp\left((t/\eta)^{\beta-1}\right)\right)$
	hazard	$(\beta/\eta)(t/\eta)^{\beta-1}\exp\left((t/\eta)^{\beta-1}\right)$

6.3 Proportional hazard models

Covariate models play very important roles in lifetime analysis. Cox (1972) initiated proportional hazards (abbreviated as PH) model as following:

$$h(t;\beta,\gamma)=h_0(t;\beta)V(\gamma^T y) \tag{22}$$

where $h_0(t;\beta)$ is called the baseline hazard function having a fuzzy parameter β defined on the credibility measure space $(\mathrm{X},\mathrm{A},\Delta)$, while $V:\mathbf{R}\rightarrow\mathbf{R}^{+}$ with

$$\gamma^T y = \gamma_0 + \gamma_1 y_1 + \cdots + \gamma_p y_p \tag{23}$$

where $y = (1, y_1, \cdots, y_p)^T$ is covariate vector and $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_p)^T$ is covariate effect parameter vector. A typically function of $V: \mathbf{R} \to \mathbf{R}^+$ used is the exponential function $V(x) = e^x$. It is easy to show that the accumulated hazard if covariate y is not time-dependent is

$$H(t;\beta,\gamma) = H_0(t;\beta)e^{\gamma^T y}. \tag{24}$$

And therefore the average chance distribution with covariate y is

$$\Phi(t,y) = \int_0^1 \mathrm{D}\left\{(t_1,t_2): H_0(t;\beta(t_1))e^{\gamma^T y(t_2)} \geq -\ln(1-\alpha)\right\}d\alpha \tag{25}$$

where covariate y is assumed to be uncertain distributed but parameter γ is assumed to be determined. Other options are also possible to be formulated.

7 EXPONENTIAL RANDOM VARIABLE WITH IMPRECISE PARAMETER

The purpose to have this section is double-folded: (a) exponential hybrid lifetime is an important member for system lifetime analysis; (b) the arguments for deriving the average chance distribution are demonstration in line with hybrid variable reliability analysis. Bearing this agenda in mind, the following step-by-step developments will be very beneficial.

Let us use exponentially distributed hybrid lifetime which has probability density

$$f(t;\beta) = \begin{cases} 0 & t \leq 0 \\ \beta e^{-\beta t} & t > 0 \end{cases} \tag{26}$$

where the imprecise parameter β has a five-piece-wise linear uncertainty distribution function (Liu (2007))

$$\Lambda(x) = \mathcal{M}\{\xi \leq x\} = \begin{cases} 0 \quad \text{if } x \leq a \\ \dfrac{x-a}{2(b-a)} \quad \text{if } a < x \leq b \\ 0.5 \quad \text{if } b < x \leq c \\ \dfrac{x+d-2c}{2(d-c)} \quad \text{if c} < x \leq d \\ 1 \quad \text{if } x > d \end{cases}. \tag{27}$$

Note that

$$\Pr\{\xi(\theta) \leq t\} = 1 - e^{\beta(\theta)t} \tag{28}$$

Therefore the event $\{t : \Pr\{x(t) \leq t\} \geq \alpha\}$ is an uncertain event and is equivalent to the uncertain event $\{\theta : \beta(\theta) \geq -\ln(1-\alpha)/t\}$. As a critical step toward the derivation of the average chance distribution, it is necessary to calculate the uncertain measure for the uncertain event $\{\theta : \beta(\theta) \geq -\ln(1-\alpha)/t\}$, i.e., obtain the expression for

$$\mathrm{D}\{\theta : \beta(\theta) \geq -\ln(1-\alpha)/t\} \tag{29}$$

Accordingly the range for integration with respect to α can be determined as shown in Table 3. Recall that the expression of $x = -\ln(1-\alpha)/t$ appears in Equation (29), which facilitates the link between intermediate variable α and average chance measure.

Table 3. Range analysis for α

Range for x	α and credibility measure expression	
$-\infty < x \le a$	Range for α	$0 \le \alpha \le 1 - e^{-at}$
	$\mathrm{Cr}\{\theta : \beta(\theta) \ge -\ln(1-\alpha)/t\}$	1
$a < x \le b$	Range for α	$1 - e^{-at} < \alpha \le 1 - e^{-bt}$
	$\mathrm{Cr}\{\theta : \beta(\theta) \ge -\ln(1-\alpha)/t\}$	$1 - (x-a)/(2(b-a))$
$b < x \le c$	Range for α	$1 - e^{-bt} < \alpha \le 1 - e^{-ct}$
	$\mathrm{Cr}\{\theta : \beta(\theta) \ge -\ln(1-\alpha)/t\}$	0.5
$c < x \le d$	Range for α	$1 - e^{-ct} < \alpha \le 1 - e^{-dt}$
	$\mathrm{Cr}\{\theta : \beta(\theta) \ge -\ln(1-\alpha)/t\}$	$(d-x)/(2(d-c))$
$d < x < +\infty$	Range for α	$1 - e^{-dt} < \alpha \le 1$
	$\mathrm{Cr}\{\theta : \beta(\theta) \ge -\ln(1-\alpha)/t\}$	0

The average chance distribution for the exponentially distributed hybrid lifetime is then derived by splitting the integration into five terms according to the range of α and the corresponding mathematical expression for the uncertain measure $\mathrm{Cr}\{\theta : \beta(\theta) \ge -\ln(1-\alpha)/t\}$, which is detailed in Table 3. Then the exponential random fuzzy lifetime has an average chance distribution function:

$$\begin{aligned}\Psi_\xi(\mathrm{t}) &= \int_0^1 \mathrm{Cr}\{\theta : \beta(\theta) \ge -\ln(1-\alpha)/t\}\mathrm{d}\alpha \\ &= 1 + \frac{e^{-bt} - e^{-at}}{2(b-a)t} + \frac{e^{-dt} - e^{-ct}}{2(d-c)t}\end{aligned} \tag{30}$$

and the average chance density is

$$\begin{aligned}\psi_\xi(\mathrm{t}) &= \frac{e^{-at} - e^{-bt}}{2(b-a)t^2} + \frac{be^{-bt} - ae^{-at}}{2(b-a)t} \\ &+ \frac{e^{-ct} - e^{-dt}}{2(d-c)t^2} + \frac{ce^{-ct} - de^{-dt}}{2(d-c)t}\end{aligned} \tag{31}$$

Similar to the probabilistic reliability theory, we define a reliability function or survival function for a random fuzzy lifetime and accordingly name it as the average chance reliability function, which is defined accordingly as

$$R_\xi(t) = 1 - \Psi_\xi(t) \tag{32}$$

Then, for exponential random fuzzy lifetime, its average chance reliability function is

$$R_\xi(t) = \frac{e^{-at} - e^{-bt}}{2(b-a)t} + \frac{e^{-ct} - e^{-dt}}{2(d-c)t} \tag{33}$$

In standard statistical lifetime modelling and analysis reliability function reveals the system functioning behaviour. The average chance reliability function should play similar roles in hybrid lifetime modelling and analysis. In order to gain an intuitive perceptions on the average chance reliability function, let us assume that the trapezoidal identification function defined by (0.1, 0.15, 0.25, 0.30), i.e., the parameters for specifying the identification function are $a = 0.1$, $b = 0.15$,

$c = 0.25, d = 0.30$. For comparison purpose, we define an exponentially distributed random lifetime with fixed valued parameter, 0.20, which is obtained by

$$m_b = E(b) = 0.20 \tag{34}$$

Then the reliability function for the exponentially distributed random lifetime with parameter $m_b = 0.20$ is

$$R(t;0.20) = \exp(-0.2t) \tag{35}$$

The corresponding average chance reliability function, $R_\xi(t;\beta)$:

$$R_x(t;b) = \frac{10(e^{-at} - e^{-bt})}{t} + \frac{10(e^{-ct} - e^{-dt})}{t} \tag{36}$$

Figure 1 gives a graphic comparison between $R_x(t;b)$ and $R(t;0.20)$.

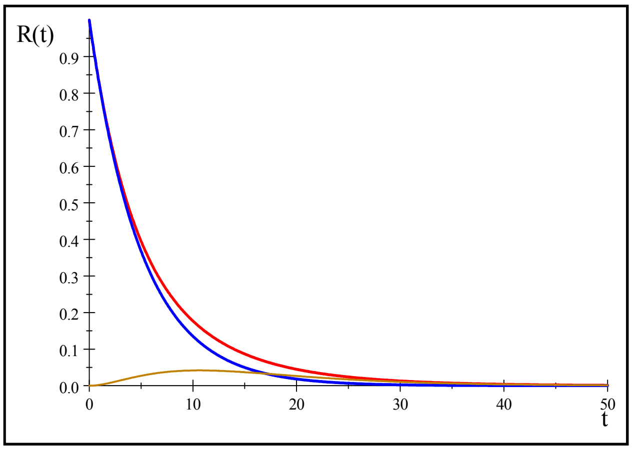

Figure 1. Exponential hybrid lifetime average chance reliability $R_x(t;b)$ (Red), corresponding exponential lifetime reliability $R(t;0.20)$ (Blue), and the difference function $d(R_x(t;b), R(t;0.20))$ (Sienna)

Intuitively, we can see that given two systems: the first one is an exponentially distributed hybrid system with trapezoidal uncertain distributed parameter $b = (0.10, 0.15, 0.25, 0.30)$ and the second one is an exponentially distributed random system with parameter $m_b = 0.20$, the first one enjoys a higher reliability than that of the second one. Definitely, a rigorous mathematical proof should be pursued before stating this impression as a general statement.

However, the purpose for us to develop hybrid lifetime analysis theory is a serious effort to facilitate a foundation for analyzing reliability data collected from system performance.

8 CONCLUDING REMARKS

In this paper, we develop a framework for modeling hybrid lifetimes (of Type I) and the average chance distribution as well as the average chance reliability. The models are constructive. We use exponentially distributed hybrid lifetime with an imprecise parameter having a five-piece-like uncertainty function as an example to illustrate the model developments on hybrid lifetimes. It should mention that for two-parameter with impreciseness, the bivariate copula-linked uncertainty marginals approach can facilitate a bivariate uncertainty distribution for imprecise parameters and further the derivation of the average chance distribution. Guo et al. (2007) demonstrated hybrid variable theory in repairable modeling, although in random fuzzy context. However, many research work need to be done ahead, for example, the parameter estimation, the asymptotic distribution for the estimated parameters, the small sample theory, etc.

9 ACKNOWLEDGEMENTS

This research is supported by VC's Incentive for NRF Rated Researchers, University of Cape Town, and South African National Research Foundation Grant IFR2010042200062, and IFR2009090800013 as well.

The research is also partially supported by Tsinghua University during Professor Guo and Mr YH Cui’s visit to the Uncertainty Theory Laboratory, Department of Mathematical Science, Tsinghua University from July 18 to October 14, 2010.

The authors would like to thank Professor B.D. Liu and his PhD students: X.W. Chen, W. Dai, K. Yao, and Z. X. Peng for their invaluable debates and inputs.

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