PROBABILITY-STATISTICAL ANALYSIS OF QUEUING NETWORKS WITH INFINITE NUMBER OF SERVERS

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ABSTRACT

In this paper queuing networks with infinite number of servers and exponentially distributed service times are considered. Limit distributions for numbers of customers in aggregated nodes are calculated. Using these calculations statistical estimates of limit distributions parameters are constructed. Mean times of customer sojourn time in opened network and its analog for closed network are estimated also. Obtained results are spread onto estimates of populations and life cycle parameters.

Keywords: ruin probability, transportation problem, asymptotic formula, enumeration problem

INTRODUCTION

In (Tsitsiashvili & Osipova, 2009) a problem of estimates of product limit distributions parameters is formulated and is solved for opened and closed queuing networks with single server nodes. In this paper opened and closed queuing networks with infinite number of servers in each node are considered. A problem of an aggregation of nodes in these networks is analyzed and limit product distributions are represented via some analogs of nodes load coefficients. Such approach allows to obtain formulas for efficiency indexes of networks and to construct algorithms of their estimates by observations of numbers of customers in different nodes. We speak about a customer mean sojourn time in opened network and about mean time between customer successive appearances in some node for closed network. Obtained results allow analyzing a dynamics of a population with individuals which have many states (Hun & Gulevich, 2008) and life cycle of technical systems with many states (Yung-Wen Liu & Kailash C. Kapur, 2008).

1. OPENED EXPONENTIAL NETWORKS

Consider opened queuing network with the states set $S=\{0,1,...,m\}$ and irresolvable matrix of transition probabilities $\Theta=\|\theta_{ij}\|_{i,j=0}^{m}$ and Poisson input flow with intensity $\lambda>0$. In each node $i\in I$, $I=\{1,...,m\}$, there is infinite number of servers with service time which has exponential distribution with parameter $\mu_i>0$. The node 0 is dummy, it is a source of arriving customers and is a runoff of customers which depart from network. For fixed $\lambda>0$ the system of linear algebraic equations (Basharin & Tolmachev, 1983)

$$(\lambda,\lambda_1,...,\lambda_m)=(\lambda,\lambda_1,...,\lambda_m)\Theta \quad (1)$$

has single solution $\lambda_i>0$, $i\in I$. Markov process $(n_1(t),...,n_m(t))$ which describes numbers of customers in nodes of opened network is ergodic (Foss, 1991) and its limit distribution has the following form (Serfozo, 1999, Example 1.29) (see (Basharin & Tolmachev, 1983) also)

$$p(n_1,...,n_m) = \prod_{i=1}^{m} \exp(-\rho_i) \frac{\rho_i^{n_i}}{n_i!}, \; \rho_i = \frac{\lambda_i}{\mu_i}, \; i \in I, \tag{2}$$

Theorem 1. Almost surely we have

$$\lim_{T\to\infty} n_i^T = \rho_i, \; n_i^T = \frac{\int_0^T n_i(t)dt}{T}, \; i \in I. \tag{3}$$

Theorem 1 statement is obtained from the law of large numbers for discrete Markov processes (Serfozo, 1999, Theorem 1.2).

Construct a model of aggregated network uniting nodes from not intersected subsets $I_1,...,I_r$ of set I and conserve initial order of customers service and routing. Denote $(N_1(t),...,N_r(t))$ random process (non Markov) describing numbers of customers in united nodes, $N_k(t) = \sum_{i\in I_k} n_i(t)$, $k = 1,...r$.

Theorem 2. The process $(N_1(t),...,N_r(t))$ has limit distribution

$$P(N_1,...,N_r) = \prod_{k=1}^{r} \exp(-R_k) \frac{R_k^{N_k}}{N_k!}, \; R_k = \sum_{i\in I_k} \rho_i \tag{4}$$

and almost surely

$$\lim_{T\to\infty} N_j^T = R_j, \; N_j^T = \frac{\int_0^T N_j(t)dt}{T}, \; j = 1,...,r. \tag{5}$$

Proof. Formula (4) is a corollary of formula (2) and well known fact that (Feller, 1984, Chapter XI) a sum of independent random variables with Poisson distributions and parameters a, b has Poisson distribution with parameter $a+b$. Formula (5) is a corollary of formulas (3), (4).

Suppose that f is mean sojourn time of customer in opened network which equals with mean time of customer service in network nodes. The quantity f may be interpreted as a mean life time of individual represented by a customer of input flow. Then we have (Borovkov, 1972, § 31, Theorem 6)

$$f = \frac{R}{\lambda}, \; R = \sum_{i\in I} \rho_i. \tag{6}$$

So if we estimate the input intensity λ then theorem 2 allows to estimate the parameter R and consequently the parameter f.

2. DYNAMICS OF POPULATION OF INDIVIDUALS WITH MANY STATEDS

In (Hun & Gulevich, 2008) a model of a dynamics of a population with n individuals is considered. A state of k-th individual (a stage of illness) is described by Markov chain $x_k(t)$, $t = 0,1,...$, with the states set $I = \{1,...,m\}$ and with the transition probabilities

$$\theta_{i,i}, \; \theta_{i,1} > 0, \; i = 1,...,m, \; \theta_{i,i+1} > 0, \; i = 1,...,m-1.$$

A transition of an individual into state 1 means that it dies and immediately new individual burns with same number. If the process $x_k(t)$ is interpreted as a displacement of particle then it is in i-th state random time τ_i with geometric distribution and with parameter $1-\theta_{i,i}$. For ergodic process $x_k(t)$ it is possible easily to calculate stationary distribution $\pi(i)$, $i \in I$.

In (Hun & Gulevich, 2008) if Markov chains $x_k(t)$, $k = 1,...,n,$ are independent then numbers of individuals which are in state i-th satisfy the law of large numbers (Serfozo, 1999, Theorem 1.2) and almost surely

$$\lim_{T\to\infty}\frac{n_i^T}{n}=\pi_i,\ i\in I.$$

These formulas allowed to analyze optimal strategies of a prevention of non infectious diseases.

In this paper a model of a population dynamics in continuous time is constructed. In constructed model non realistic assumption of constant number of individuals in a population is cancelled.

We consider population model as opened queuing network with infinite number of servers in each node. Then a vector of numbers of individuals in different stages of diseases is described by Markov process $(n_1(t),...,n_m(t))$. In this case sojourn time of individual in i-th state has exponential distribution with the parameter μ_i. A presence of infinite number of servers in each node guarantees independent individuals displacements along network nodes.

Analogously to (Hun & Gulevich, 2008) assume that the route matrix Θ of opened queuing network contains following non zero elements:

$$\theta_{i,i+1},\ \theta_{i,0}=1-\theta_{i,i+1},\ i=1,...,m-1,\ \theta_{m,0}=\theta_{0,1}=1.$$

Then it is easy to obtain from (1) that

$$\lambda_{i+1}=\lambda_i\theta_{i,i+1},\ i=1,...,m,$$

And consequently from (6) we have that mean individual life time is

$$f=\sum_{i=1}^{m}\prod_{k=0}^{i-1}\frac{\theta(k,k+1)}{\mu_i}.$$

Remark 1. Suppose that customer-individual passing through i-th node may receive additional service-prevention in parallel node i' with intensity $\mu_{i'}<\mu_i$. That increases mean sojourn time of customer in this node. If customer arrives in node i then with probability a_i it is served on server of this node and with probability $b_i=1-a_i$ it is served in node i'. As result equality $\rho_i=\lambda_i/\mu_i$ is replaced by equality

$$\rho_i=\frac{a_i\lambda_i}{\mu_i},\ \rho_{i'}=\frac{b_i\lambda_i}{\mu_{i'}}.$$

Using these equalities it is possible to estimate influence of prevention on main indexes of this network.

3. CLOSED EXPONENTIAL NETWORKS

Consider closed queuing network with the states set S, the route matrix Θ, n customers circulating along this network and n servers in each node with service intensity μ_i on a server of i-th node, $i\in S$. Suppose that for fixed $\lambda_0>0$ the system of linear algebraic equations (1) has single solution $\lambda_1>0,...,\lambda_m>0$. Then Markov process $(n_0(t),...,n_m(t))$ characterizing numbers of customers in nodes of closed network is ergodic (Ivchenko & Kashtanov & Kovalenko, 1982, Theorem 2.4) and has limit multinomial distribution (Serfozo, 1999, Example 1.29)

$$P(n_0,...,n_m)=n!\prod_{i=0}^{m}\frac{d_i^{m_i}}{n_i!},\ d_i=\frac{\rho_i}{\rho_0+...+\rho_m},\ \rho_0=\frac{\lambda_0}{\mu_0},\ i\in S,\ \sum_{i=0}^{m}n_i=n. \tag{7}$$

Theorem 3. Almost surely

$$\lim_{T\to\infty}\frac{n_i^T}{n}=d_i,\ i\in S. \tag{8}$$

Proof. From binomial theorem we have that the process

$$(n_0(t),n_1(t),...,n_{m-2}(t),n_{m-1}+n_m(t))$$

has limit distribution

$$P(n_0,n_1,...n_{m-2},n_{m-1}+n_m)=n!\frac{(d_{m-1}+d_m)^{n_{m-1}+n_m}}{(n_{m-1}+n_m)!}\prod_{i=0}^{m-2}\frac{d_i^{m_i}}{n_i!}. \qquad (9)$$

Using mathematical induction it is easy to prove that the process $\left(n_0(t), \sum_{k\in I} n_k(t)\right)$ has limit distribution

$$P(n_0,n_1,...,n_m)=\frac{n!d_0^{n_0}(d_1+...+d_m)^{n-n_0}}{n_0!(n-n_0)!}.$$

Consequently the process $n_0(t)$ has binomial limit distribution with mean nd_0. Analogously it is possible to prove that random process $n_i(t)$ has binomial limit distribution with mean nd_i, $i=1,...,m$. Then from the law of large numbers for ergodic Markov processes (Serfozo, 1999, Theorem 1.2) we obtain the formula (8).

Construct a model of aggregated network uniting nodes in not intersected subsets $I_1,...,I_r$ of the set I, conserving initial characteristics of customers service and routing. Denote $(N_1(t),...,N_r(t))$ random process (non Markov) which describes numbers of customers in united nodes, $N_k(t)=\sum_{i\in I_k} n_i(t)$, $k=1,...,r$.

Theorem 4. The process $(N_1(t),...,N_r(t))$ has multinomial limit distribution

$$P(N_1,...N_r)=n!\prod_{k=1}^{r}\frac{D_k^{N_k}}{N_k!},\ D_k=\sum_{i\in I_k} d_i,\ k=1,...,r, \qquad (10)$$

and almost surely

$$\lim_{T\to\infty}\frac{N_k^T}{n}=D_k,\ k=1,...,r. \qquad (11)$$

Proof. The formula (10) may be obtained from the formula (9) by mathematical induction. The formula (11) is a corollary of the formula (10) and the theorem 3.

Denote f_i mean time between successive appearances of a customer in i-th node of closed network. To calculate f_0 consider opened network with route matrix Θ and with input intensity λ_0. It is obvious that

$$f_0=\frac{1}{\mu_0}+f=\frac{1}{\mu_0}+\frac{R}{\lambda_0}=\frac{\rho_0+...+\rho_m}{\lambda_0}=\frac{\rho_0}{d_0\lambda_0}=\frac{1}{\mu_0 d_0}. \qquad (12)$$

Consequently if we know statistical estimate of service intensity μ_0 then using the theorem 3 it is possible to estimate d_0 and consequently f_0. Analogously it is possible to obtain the equality

$$f_i=\frac{1}{\mu_i d_i}, \qquad (13)$$

and using it to estimate mean f_i by estimate of service intensity μ_i, $i\in I$.

CONCLUSION

Theorems 2, 4 allow to aggregate nodes of initial network and consequently to simplify a procedure of statistical estimates of its parameters. It means that we may delete a solution of the system (1) and to simplify a procedure of the network observation. Besides of product distribution parameters suggested approach allows to estimate mean sojourn times of customers in the network. Obtained results may be generalized onto queuing networks with non exponential service time distributions using (Harrison & Lemoine, 1981) and may be applied in technical gerontology and demography to estimate mean time of endowment for persons of fixed age.

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