ASYMPTOTIC FORMULA FOR DISCONNECTION PROBABILITY OF GRAPH ON TWO DIMENSIONAL MANIFOLD

G. Tsitsiashvili

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IAM, FEB RAS, Vladivostok, Russia

e-mails: guram@iam.dvo.ru

1. INTRODUCTION

A problem of a calculation of a graph disconnection probability is considered in a lot of papers, see for example [1]-[4]. In [1] upper and low bounds of the graph disconnection probability (a reliability polynomial) are constructed using maximal systems of disjoint cross sections. For a graph with sufficiently small number of arcs in [2] accelerated algorithms are constructed. These algorithms showed good results in a comparison with Maple 11. In [3] this problem is solved using Monte-Carlo method and some specific combinatory indexes and formulas.

But when a number of arcs increases this problem becomes much more complicated. So it is necessary to construct convenient asymptotic formulas for connectivity or disconnection probability of graph with high reliable arcs. In this paper such problem is solved for planar graphs or graphs arranged on two dimensional manifolds. Such graphs appear in honeycombed structures which are widely used in different applications.

2. PRELIMINARIES

Consider unoriented and connected graph G with finite sets of nodes U and of arcs W. Denote $\mathcal{L}(u,v)$ the set of all cross sections in G which divide nodes $u,v \in U, u \neq v$,

$$\mathcal{L} = \bigcup_{u \neq v} \mathcal{L}(u,v).$$

Put $d(L)$ a number of arcs in cross section L and define

$$D(u,v) = \min\left(d(L) : L \in \mathcal{L}(u,v)\right), \quad D = \min_{u \in v} D(u,v), \quad \mathcal{L}_* = \{L \in \mathcal{L} : d(L) = D\},$$

C is a number of cross sections from the set $\mathcal{L}_*$

Theorem 1. Suppose that graph arcs $w \in W$ fail independently with the probability h then the probability P of the graph G disconnection satisfies the formula

$$P \sim Ch^D \ , \ h \to 0. \tag{1}$$

So to calculate asymptotic of graph disconnection probability it is necessary to find the constants C, D. Suppose that G is two dimensional integer rectangle with the size $M \times N$. If $M, N > 1$ then the set $\mathcal{L}_*$ consists of four angle cross sections with two arcs [5] and so

$$P \sim 4h^2, \ \ h \to 0.$$

In general case for $M > 0, N > 0$ we obtain the formula

$$P \sim \left(4 + I(M = 1)N + I(N = 1)M\right)h^2, \ h \to 0.$$

But to make asymptotic analysis of disconnection probability in honeycombed structures it is necessary to pass from integer rectangle to more general graphs.

3. MAIN RESULTS

This generalization is based on a concept of a graph G arranged on connected and two dimensional smooth manifold without edge $\mathcal{T}$ [6,chapter 1].
Suppose that between two nodes of the graph G there is not more than two arcs and there are not arcs beginning and ending at the same node (loops). Arcs do not intersect and may have only common nodes. Each node and each arc belong to some cycle with more than two arcs and more than two nodes.

Call faces (or cells) areas $S_i, i=0,...,m$, of the manifold $\mathcal{T}$ limited by its cycles minimal by the set theory inclusions. So faces may have common nodes, common arcs but have not common internal points. Put two faces adjacent if there is their common arc. Each arc belongs to two faces (is adjacent to two faces). Denote by δS_i the face S_i boundary.

Suppose that faces $S_i,...,S_m$ are bounded and call them internal. Then the face

$$S_0 = \mathcal{T} \setminus \bigcup_{i=1}^{m} S_i$$

may be called external. The face S_0 may be unbounded if for example the manifold $\mathcal{T}$ is a plane. It may be bounded also if for example $\mathcal{T}$ is a sphere or a torus.

(A). Suppose that each two internal faces $S_i, S_j, 1 \le i \le m$, may have no more than single common arc.

Examples of graphs satisfied Condition (A) are connected aggregations of quadrates from rectangular lattice or connected aggregations of hexagons from hexagonal lattice.

Denote $A_{i,j}$ the set of arcs adjacent to faces $S_i, S_j, 0 \le i \ne j \le m$, and put $n_{i,j}$ a number of arcs in the set $A_{i,j}$. Designate $M_{i,j} = C^2_{n_{i,j}}$, if $n_{i,j} > 1$ and $M_{i,j} = 0$ if $n_{i,j} \le 1$. Define $N = \Sigma_{1 \le i \le m} M_{i,0}$, $M = \Sigma_{0 \le i < j \le m} M_{i,j}$.

Theorem 2. Suppose that Condition (A) and the inequality $N > 0$ are true then $C = N, D = 2$.

An example of a graph satisfied Theorem 2 conditions is integer rectangle.

Theorem 3. If $M > 0$ then the equalities $C = M, D = 2$ are true.

Remark that Condition (A) is absent in Theorem 3. Denote U_3 the set of the graph G nodes which are connected with three arcs and put K_3 the number of elements in U_3.

Theorem 4. If $M = 0$, $K_3 > 0$, then $C = K_3, D = 3$.

Examples of graphs which satisfy Theorem 3 conditions are the dodecahedron [3,Chapter 4, Figure 4.2] and integer tube obtained by a gluing of a pair of opposite sides in an integer rectangle with a size $M \times N$, $M > 1$, $N > 1$.

Theorem 5. If $M = 0$, $K_3 = 0$, $K_4 > 0$ then $C = K_4, D = 4$.

An example of a graph satisfies Theorem 5 conditions is a graph arranged on two dimensional torus and obtained by a gluing of two pairs of opposite sides in an integer rectangle with a size $M \times N$, $M > 1$, $N > 1$.

4. PROOF OF MAIN RESULTS

Theorem 1.

Proof. From the Burtin-Pittel formula [3] the probability $P(u,v)$ of the nodes u,v disconnection in G satisfies the relation $P(u,v)\sim C(u,v)h^{D(u,v)}$, $h\to 0$ where $C(u,v)$ is a number of sections $L\in \mathcal{L}(u,v): d(L)=D(u,v)$. Assume that V_L is random event when all arcs of cross section L fail then

$$P=P\left(\bigcup_{L\in\mathcal{L}} V_L\right)=P\left(\left(\bigcup_{L\in\mathcal{L}*} V_L\right)\cup\left(\bigcup_{L\in\mathcal{L}\setminus\mathcal{L}*} V_L\right)\right)\sim P\left(\bigcup_{L\in\mathcal{L}*} V_L\right),\ h\to 0,$$

as

$$P(V_L)=o\left(h^D\right),\ L\in\mathcal{L}\setminus\mathcal{L}*,\ P\left(\bigcup_{L\in\mathcal{L}*} V_L\right)\sim Ch^D.$$

Theorem 2.

Proof. 1. Prove at first that $D>1$. If $D=1$ then there are nodes $u,v\in U$ dividing by single arc $w\in W$. Suppose that Γ is a way in the graph G connecting nodes u,v and the arc w belongs to the face S_i.Without a restriction of a generality assume that the nodes $u,v\in S_i$. Then it is possible to construct a way Γ' in which the arc w is replaced by a way along the face S_i boundary which bypasses the arc w (see fig. 1). Consequently the arc w does not divide the nodes u,v and so minimal number of arcs which create the graph G cross section is larger than one.

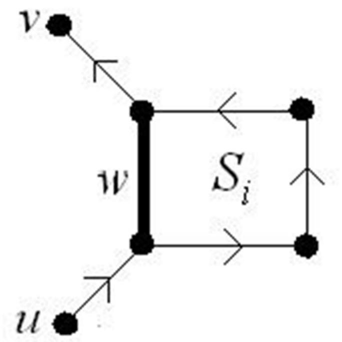

Fig. 1. Item 1 illustration

2. Suppose that w_1,w_2 is a pair of graph arcs. Prove that if these arcs do not belong to common internal face then the set $\{w_1,w_2\}\notin\mathcal{L}$. Indeed the arc w_1 (the arc w_2) may be bypassed by its internal face boundary (see fig. 1). And a way around the arc w_1 (around the arc w_2) does not contain the arcs w_1,w_2. So the set of arcs $\{w_1,w_2\}\notin\mathcal{L}$.

3. Assume that the arcs w_1,w_2 belong to internal face S_i but do not belong simultaneously to external face S_0.Prove that the set $\{w_1,w_2\}\notin\mathcal{L}$.

a) At first consider the case when there are internal faces S_j,S_k, $j\neq i,k\neq i$ so that $w_1\in S_j$, $w_2\in S_k$. As any two internal faces have not more than single common arc then $j\neq k$.

So the arc w_1 (the arc w_2) may be bypassed along δS_j (along δS_k) without the arcs w_1, w_2. Consequently the set $\{w_1, w_2\} \notin \mathcal{L}$.

b) Suppose now that the arc $w_1 = (u_1, v_1) \in S_0$ and the arc $w_2 = (u_2, v_2) \notin S_0$. Then the arc w_1 may be bypassed by the way along δS_0. The arc w_2 may be bypassed along δS_i from the node v_2 to the node v_1. Then this way may be prolonged along δS_0 to the node u_1 and then it may return along δS_i from the node u_1 to the node u_2 (see fig. 2). Consequently the set $\{w_1, w_2\} \notin \mathcal{L}$.

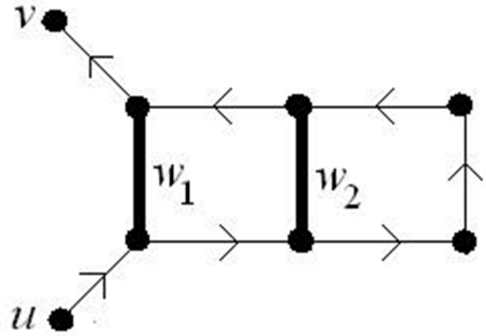

Fig. 2. Item 3, subitem b) illustration.

4. Suppose that $w_1 \in S_i \cap S_0$, $w_2 \in S_i \cap S_0$. Prove that $\{w_1, w_2\} \in \mathcal{L}$. Take a cycle around the face S_i and suppose that u, v are first nodes of this cycle touching w_1, w_2 appropriately. Extract from the face S_i the set $\delta' S_i$ of arcs which do not belong to S_0 Exclude trivial case when the arcs w_1, w_2 have common node. Contrast each arc w from $\delta' S_i$ the set of internal faces $\{S_k, k \in J_w\}$ defined by the symbols set $J_w, i \in J_i$. This set satisfies recurrent conditions: if $k \in J_w, t \neq k$ and $S_k \cap S_t \neq 0$ then $t \in J_w$. Denote $R_w = \bigcup_{k \in J_w} S_k$ and obtain that if $u \in R_{w'}$, $v \in R_{w''}$ then $R_{w'} \cap R_{w''} = \varnothing$ (see fig. 3). In opposite case $w_1 \notin S_i \cap S_0$ (see fig. 4). This contradiction proves that the set $\{w_1, w_2\} \in \mathcal{L}$.

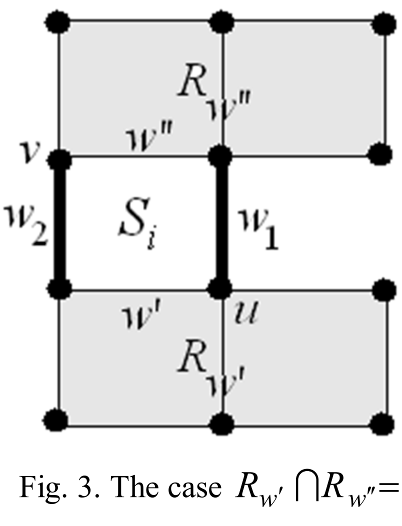

Fig. 3. The case $R_{w'} \cap R_{w''} = \varnothing$.

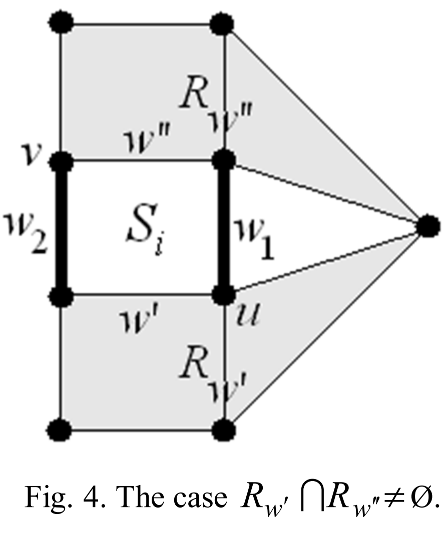

Fig. 4. The case $R_{w'} \cap R_{w''} \neq \varnothing$.

The illustration of the figure 4 may be added by following formal inference. Assume that the arcs w', w_1 have common node u and the arcs w'', w_2 have common node v and $w' \notin S_i$, $w'' \notin S_i$. Suppose that the node $z \in R_{w'} \cap R_{w''}$.

From the recurrent definition of the set $J_{w'}$ it is easy to prove that between the nodes u, z there is acyclic way $\Gamma(u,z)$ consisting of the faces $S_t, t \in J_{w'}$ arcs excluding the face S_i arcs. Analogously between the nodes z, v there is acyclic way $\Gamma(z,v)$ consisting of the faces $S_t, t \in J_{w''}$ arcs without the face S_i arcs. Consider a polygon bounded by the way $\Gamma(u,z) \cup \Gamma(z,v)$ and by the way $\Gamma(u,v)$which connects the nodes u, v and passes along $\delta S_i, w_1 \in \Gamma(u,v)$. It is obvious that this polygon consists of internal faces and does not contain the face S_i. So the arc w_1 is adjacent to some internal face which does not coincide with S_i. This statement contradicts with initial suggestion that $w_1 \in S_i$. Consequently $R_{w'} \cap R_{w''} \neq \varnothing$.

Theorem 3.
Proof. This statement may be proved analogously to Theorem 2 statement if to pass from the faces S_0, S_i to the faces S_j, S_k, see item 3, subitem a) in Theorem 2 proof.

Theorem 4.
Proof. As $M = 0$ and each face may have no more than single common arc with the face S_0 then $D > 0$. From Theorem 2the graph G which satisfies Theorem 5 conditions has not cross sections with two arcs. Prove that there are graph cross sections with three arcs and all these cross sections $\{w_1, w_2, w_3\}$contain arcs connected with a node from U_3 . Indeed if arcs w_1, w_2, w_3 are connected with a node from U_3 then the set $\{w_1, w_2, w_3\} \in \mathcal{L}$. Prove that if arcs w_1, w_2, w_3 are not connected with a node from U_3 then the set $\{w_1, w_2, w_3\} \notin \mathcal{L}$. Then Theorem 4 will be proved.

As $M = 0$ so each two faces of the graph G may have no more than single common arc and each node connects with more than two arcs. Assume that arcs w_1, w_2, w_3 belong to the faces S_1, S_2, S_3 appropriately and some of these faces may coincide.

1. Suppose that there is not a pair of arcs from w_1, w_2, w_3 in the same face. Then each arc w_i may be bypassed by a way along δS_i (see fig. 1). So any way $\Gamma(u,v)$ from u to v which contains arcs from w_1, w_2, w_3 may be replaced by a way $\Gamma'(u,v)$ which does not contain arcs from w_1, w_2, w_3. For this aim it is enough to replace arc from this set by a way which bypasses this arc. Consequently the set $\{w_1, w_2, w_3\} \notin \mathcal{L}$.

2. Suppose that all three arcs from w_1, w_2, w_3 belong to common face S. Then each arc w_i may be bypassed by a way along δS_i without arcs from w_1, w_2, w_3 (see fig. 1). Consequently the set $\{w_1, w_2, w_3\} \notin \mathcal{L}$.

3. Suppose that faces S_1, S_2, S_3, S are different and $w_1 \in S_2$, $w_2 \in S_3$, $w_3 \in S$. Then the arc w_1 may be bypassed by a way along δS_1, the arc w_3 may be bypassed by a way along δS (see fig. 1) and the arc w_2 - by a way from δS_3 to the arc w_3, then around the arc w_3 along δS

and then along δS_3 to the arc w_2 second node (see fig. 2). And the way around w_i does not contain arcs from w_1, w_2, w_3. Consequently the set $\{w_1, w_2, w_3\} \notin \mathcal{L}$.

4. Suppose that the faces S_1, S_2, S_3, S are different and $w_1 \in S$, $w_2 \in S$, $w_3 \notin S$. Then each arc w_i may be bypassed by a way along δS_i without arcs from the set w_1, w_2, w_3 (see fig. 1). Consequently the set $\{w_1, w_2, w_3\} \notin \mathcal{L}$.

Theorem 5.

Proof. From Theorem 4 we have that the graph G satisfying Theorem 5 has not cross sections with three arcs. Prove that in G there are cross sections with four arcs w_1, w_2, w_3, w_4 and all these cross sections consist of arcs connected with some node from U_4. Indeed if the arcs w_1, w_2, w_3, w_4 are connected with some node from U_4 t$\{w_1, w_2, w_3\} \notin \mathcal{L}$hen the set $w_1, w_2, w_3, w_4 \in \mathcal{L}$ .Prove that all other sets are not cross sections in G.

From $M = 0$ we have that each two faces in G may have no more than single common arc. Denote faces S_1, S_2, S_3, S_4 which contain arcs w_1, w_2, w_3, w_4 appropriately.

1. Suppose that there is not a pair of arcs from the set $\{w_1, w_2, w_3, w_4\}$ which belong to the same face. Then each arc w_i may be bypassed by a way along δS_i without arcs from the set $\{w_1, w_2, w_3, w_4\}$. Consequently each way $\Gamma(u,v)$ from u to v which contains some arcs from the set $\{w_1, w_2, w_3, w_4\}$may be replaced by a way $\Gamma'(u,v)$ without arcs from the set $\{w_1, w_2, w_3, w_4\}$ (see fig. 1). Consequently the set $\{w_1, w_2, w_3, w_4\} \notin \mathcal{L}$.

2. Suppose that all arcs from $\{w_1, w_2, w_3, w_4\}$ belong to the same face S. Then each arc w_i may be bypassed by a way along δS_i adjacent with w_i (see fig. 1) without arcs from the set $\{w_1, w_2, w_3, w_4\}$. Consequently the set $\{w_1, w_2, w_3, w_4\} \notin \mathcal{L}$.

3. Suppose that the faces S_1, S_2, S_3, S_4, S are different and $w_1 \in S_2$, $w_3 \in S_4$, $w_4 \notin S$. Then the arc w_1 may be bypassed by a way along S_1 the arc w_2 - by parts of δS_2 and along δS_1 around w_1 (see fig. 2), the arc w_4 - along S and the arc w_3 - by a way along parts of S_4 and along S around the arc w_4 (see fig. 2). Consequently the set $\{w_1, w_2, w_3, w_4\} \notin \mathcal{L}$.

4. Suppose that faces S_1, S_2, S_3, S_4, S are different and

$$w_1 \in S,\ w_2 \in S, w_3 \in S, w_1 \notin S_4,\ w_2 \notin S_4,\ w_3 \notin S_4.$$

Then each arc from w_i, $i = 1,2,3$, may be bypassed by a way along δS_i without arcs from $\{w_1, w_2, w_3, w_4\}$. Analogously the arc w_4 may be bypassed along S_4 (see fig. 1). Consequently the set $\{w_1, w_2, w_3, w_4\} \notin \mathcal{L}$.

5. Suppose that the faces S_1, S_2, S_3, S_4, S are different and

$$w_1 \in S,\ w_2 \in S, w_3 \in S, w_4 \in S_1.$$

Then the arc w_2 may be bypassed by a way along δS_2 the arc w_3 - along δS_3 the arc w_4 - along δS_4 (see fig. 1). The arc w_1 may be bypassed along parts of δS_1 and around the arc w_4 along

δS_4 (see fig. 2). Each way bypassing the arc w_i does not contain arcs from the set $\{w_1, w_2, w_3, w_4\}$. Consequently $\{w_1, w_2, w_3, w_4\} \notin \mathcal{L}$.

5. CONCLUSION

So we obtain asymptotic formulas for disconnection probability of a wide variety of graphs with high reliable arcs. This problem is sufficiently complicated especially for the coefficient C because it is necessary to solve some N-P problem. But a consideration of planar graphs or graphs arranged on two dimensional smooth manifold without edge simplifies this calculations significantly. It takes place because though a problem to arrange a graph on a manifold is sufficiently complicated but in different applications this problem is solved by a designer without any calculations.

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