DISTRIBUTIONS OF NUMBERS OF CONNECTIVITY COMPONENTS IN RECURSIVELY DEFINED GRAPHS WITH UNRELIABLE ARCS

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ABSTRACT

In this paper a problem of accuracy and approximate calculations of connectivity characteristics in recursively defined random graphs is considered. This problem is solved using low and upper bounds for numbers of connectivity components in graphs and limit theorems of probability theory: law of large numbers and central limit theorem.

1. INTRODUCTION

In this paper a problem of an accuracy and approximate calculations of connectivity characteristics in recursively defined random graphs is considered. This problem is analyzed in papers [1] – [6] and many other ones. But when a graph becomes more complicated a complexity of this solution increases significantly. So it is necessary to introduce additional characteristics of connectivity like numbers of connectivity components. It allows to widen a set of analyzed random graphs essentially.

Analogously to [6, Figure 4] connectivity probability and mean number of connectivity components for parallel aggregation of chains with identical arcs is calculated accurately. But accuracy formulas do not allow to consider manifold practically interesting random graphs. So first step to obtain estimates of the connectivity is to analyze completely connective random graph (where each pair of nodes is connected by single arc). Sufficient conditions of tendency of connectivity probability of this graphs to one are obtained.

Then we transit from accuracy formulas to upper and low bounds. Analogously to [7] upper and low bounds of numbers of connectivity components are constructed for recursively defined graphs which are obtained by a gluing of defined graphs in few nodes. The gluing in single node creates graphs of treelike structure with a bridge or radial-circle generating graphs. But it is not enough and a step to the gluing in a few nodes is made.

In this case upper and low bounds of numbers of connectivity components are obtained by numbers of failed arcs and some deterministic summands. Applying to these bounds law of large numbers and central limit theorem it is possible to remove deterministic summands and to obtain variants of limit theorems. These results are used to parallel aggregations of chains with equal lengths.

2. CONNECTIVITY PROBABILITY IN PARALLEL AGGREGATIONS OF CHAINS OF IDENTICAL ARCS

Consider parallel aggregation of m chains with lengths $n_1 > 0,...,n_m > 0$. Each chain consists of independently working arcs with the failure probability $q = 1 - p$, $0 < p < 1$. Our problem is to calculate the probability Q of the event C, that this aggregation is not connective.

Define the event A that there is a chain with more than one failed arc and the event B, that there is single failed arc in each chain. It is obvious that the events A, B are inconsistent and $A \subset C$, $B \subset C$, $C \cap \overline{A} = B$, consequently, $C = A \cup B$, $Q = P(C) = P(A) + P(B)$, where

$$P(A) = 1 - \prod_{i=1}^{m} \left(p^{n_i} + n_i p^{n_i - 1} q \right), \; P(B) = q^m \prod_{i=1}^{m} n_i p^{n_i - 1} \quad . \tag{1}$$

If the chain i consists of arcs which work with the probability p_i and fail with the probability $q_i = 1 - p_i$, then the formulas (1) transforms as follows

$$P(A) = 1 - \prod_{i=1}^{m} \left(p_i^{n_i} + n_i p_i^{n_i - 1} q_i \right), \; P(B) = q^m \prod_{i=1}^{m} n_i p_i^{n_i - 1} q_i \quad . \tag{2}$$

If the graph $G = G_1 \to G_2$ is constructed by a gluing of the graphs G_1, G_2 in a single node then the connectivity probability of the graph G equals a product of the connectivity probabilities of the graphs G_1, G_2.

3. NETWORKS WITH LARGE NUMBERS OF NODES AND ARCS

E.A. Nurminsky (oral information) using numerical experiments formulated a hypothesis that if a number of graph nodes and a number of arcs is large also then this graph connectivity probability is close with one. In this section a model of a graph which satisfies this hypothesis is constructed and its sufficient conditions are formulated.

Consider non oriented connective graph G_n with the arcs set W_n and the nodes set $U_n = \{u_1, ..., u_n\}$. Suppose that each pair of nodes may be connected no more than by a single arc. Denote $\varphi_n(i, j)$ a number of nodes $u_k \in U_n$ so that the arcs $w_{ik} = (u_i, u_k) \in W_n$, $w_{kj} = (u_k, u_j) \in W_n$ and put

$$\varphi_n = \min_{1 \le i < j \le n} \varphi_n(i, j), \; \psi_n = \min_{1 \le i < j \le n} \psi_n(i, j),$$

$$\psi_n(i, j) = \min\left(p(w_{ik}) p(w_{kj}) : u_k \in U_n, k \ne i, k \ne j \right).$$

Theorem 1. Suppose that the graph G_n arcs work independently with probabilities $p(w)$, $w \in W_n$ and

$$-\varphi_n \psi_n + 2 \ln n \to -\infty, \; n \to \infty.$$

Then the connectivity probability of the graph G_n satisfies the relation

$$\lim_{n \to \infty} P(G_n) = 1 \quad . \tag{3}$$

Proof. Denote $P_n(u_i, u_j)$ the probability that the nodes $u_i, u_j \in U_n$ are connected in the graph G_n. It is obvious that $\overline{P}_n(u_i, u_j) = 1 - P_n(u_i, u_j)$ does not exceed failure probability of all ways (w_{ik}, w_{kj}), which pass through some nodes $u_k \in U_n$, $k \ne i$. Consequently from the inequality $x \ge 1 - \exp(-x)$, $x > 0$, we obtain that

$$\overline{P}_n(u_i, u_j) \le (1 - \psi_n)^{\varphi_n} \le \exp(-\varphi_n \psi_n).$$

As the number of the graph G_n arcs does not exceed $n(n-1)/2$ then

$$0 \le 1 - P(G_n) \le \sum_{1 \le i < j \le n} \overline{P}_n(u_i, u_j) \le \frac{n(n-1)}{2} \exp(-\varphi_n \psi_n) \le \frac{1}{2} \exp(-\varphi_n \psi_n + 2\ln n).$$

From this inequality we obtain the limit relation (3).

Corollary1. If the graphs G_n are completely connective, $n \ge 1$, and for some $c > 0$ the inequalities

$$p\left(w \ge \sqrt{\frac{(2+c)\ln n}{n}} \right), \ w \in W_n,$$

are true then the formula (3) takes place.

Corollary2. If in the graphs G_n , $n \ge 1$, the conditions $\varphi_n - \ln n \to \infty$, $n \to \infty$, are true and for some $p > 0$ the inequalities $p(w) \ge p$, $w \in W_n$ take place then the formula (3) is true also.

4. MEAN NUMBER OF CONNECTIVITY COMPONENTS IN PARALLEL AGGREGATION OF CHAINS

Calculate now the mean number S of connectivity components in parallel aggregation of m chains with $n_1 = ... = n_m = n$ arcs.

Theorem 2. The following formula is true:

$$S = mnq + \left(1 - p^n\right)^m + (1 - m) + mp^n \ . \tag{4}$$

Proof. Define auxiliary expressions $P_n(k_1, ..., k_m)$- the probability of k_i failures in the chain $i, k_i = 0, ..., n$, $i = 1, ..., m$, $p_n(k) = C_n^k q^k p^{n-k}$, $k_i = 0, ..., n$,

$$S_1 = \sum_{k_1 > 0, ..., k_m > 0} (k_1 + ... + k_m - m + 2) P_n(k_1, ..., k_m) =$$
$$= \sum_{k_1 > 0, ..., k_m > 0} (k_1 + ... + k_m - m + 2) p_n(k_1) \cdot ... \cdot p_n(k_m) =$$
$$= mnq\left(1 - p^n\right)^{m-1} + (2 - m)\left(1 - p^n\right)^m,$$

$$S_2 = \sum_{r=1}^{m} C_m^r p^{nr} \left[(m - r) nq \left(1 - p^n\right)^{m-r-1} + (1 - m + r)\left(1 - p^n\right)^{m-r} \right], \ S = S_1 + S_2.$$

Here S_1 - is mean number of connectivity components if there are failures in all chains, S_2 is mean number of connectivity components if there is positive number of chains without failures. Denote

$$S_2' = \sum_{r=1}^{m} C_m^r p^{nr} (m - r) nq \left(1 - p^n\right)^{m-r-1} = (t = m - r) =$$
$$= \left(1 - p^n\right)^{-1} nq \sum_{t=1}^{m-1} C_m^t p^{n(m-t)} t \left(1 - p^n\right)^t =$$
$$= \left(1 - p^n\right)^{-1} nq \sum_{t=0}^{m-1} C_m^t p^{n(m-t)} t \left(1 - p^n\right)^t = \left(1 - p^n\right)^{-1} nq \left[m\left(1 - p^n\right) - m\left(1 - p^n\right)^{m-1} \right] =$$
$$= mnq\left[1 - \left(1 - p^n\right)^{m-1} \right],$$

$$S_2'' = \sum_{r=1}^{m} C_m^r p^{nr} (1 - m + r)\left(1 - p^n\right)^{m-r} = \left(\sum_{r=0}^{m} - \sum_{r=0}^{0} \right) C_m^r p^{nr} (1 - m + r)\left(1 - p^n\right)^{m-r} =$$

$$=(1-m)\left(1-\left(1-p^n\right)^m\right)+\sum_{r=0}^{m}C_m^r p^{nr} r\left(1-p^n\right)^{m-r}=(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n .$$

Consequently

$$S_2=S_2'+S_2''=mnq\left[1-\left(1-p^n\right)^{m-1}\right]+(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n ,$$

and so

$$S=mnq\left(1-p^n\right)^{m-1}+(2-m)\left(1-p^n\right)^m+mnq\left[1-\left(1-p^n\right)^{m-1}\right]+$$
$$+(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n=mnq+(2-m)\left(1-p^n\right)^m+(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n=$$
$$=mnq+\left(1-p^n\right)^m+(1-m)+mp^n .$$

5. DISNRIBUTION OF NUMBER OF CONNECTIVITY COMPONENTS IN RECURSIVELY DEFINED GRAPHS

Recursively defined class of graphs. Consider recursively defined class $\mathcal{A}$ of graphs with identical arcs. Suppose that $A-$ is enumerable set of arcs called a system of generating arcs. Each graph $g\in A$ characterizes by numbers $n(g)=1$, $m_i(g)=0$, $i\ge 1$. The class $\mathcal{A}$ is defined by rolls: $A\subset\mathcal{A}$, if, $g_1\subset\mathcal{A}$, $g_2\subset\mathcal{A}$ and the sets of these graphs arcs do not intersect, then the aggregation $g_1\overset{(i)}{\cdot}g_2$ constructed by a gluing of the graphs g_1,g_2 in $i\ge 1$ nodes belongs to the class $\mathcal{A}$.also and

$$n\left(g_1\overset{(i)}{\cdot}g_2\right)=n(g_1)+n(g_2),\quad m_j\left(g_1\overset{(i)}{\cdot}g_2\right)=m_j(g_1)+m_j(g_2)+\delta_{ij},\quad 1\le i,\quad 1\le j .$$

Here δ_{ij} - is Kroneker symbol, $n(g)$- is a number of arcs and $m_i(g)$ - is a number of "$\overset{(i)}{\cdot}$" connections in a graph $g\in A$.

Example 1. The class of parallel-sequential graphs is an example of recursively defined class $\mathcal{A}$ which is widely used in reliability theory [1].

Inequalities for numbers of connectivity components in random realizations of graphs. Assume that arcs of a graph $g\in A$ work independently with the probability $p, 0<p<1$ and fail with the probability $q=1-p$. For each realization of the graph $g'\in A$ arcs it is possible to define random number $l(g')$ of failed arcs and random number $k(g')$ of connectivity components. Assume that edges of failed arcs belong to this graph realization. Designate $m(g)=\sum_{i\ge1}(i-1)m_i(g)$.

Lemma 1. For each random realization g' of the graph $g\in A$ the following inequalities take place:

$$l(g')-2m(g)+1\le k(g')\le l(g')+1 . \tag{5}$$

Proof. Using recursive definition of the class $\mathcal{A}$ it is easy to prove that almost surely the following formulas are true: for realizations g_1',g_2' of graphs $g_1\in A$, $g_2\in A$ and for a realization g' of a graph $g\in A$

$$k(g_1')+k(g_2')-2i+1\le k\left(g_1'\overset{(i)}{\cdot}g_2'\right)\le k(g_1')+k(g_2')-1,$$
$$k(g')=l(g')+1,\ l\left(g_1'\overset{(i)}{\cdot}g_2'\right)=l(g_1')+l(g_2'),\ i\ge 1. \tag{6}$$

Indeed for $g\in A$ the inequalities (5) are true as $k(g')=l(g')+1$. Assume that these inequalities take place for random realizations g_1',g_2' of the graphs $g_1\in A$, $g_2\in A$. Then

$$k\left(g_1'\overset{(i)}{\cdot}g_2'\right)\le k(g_1')+k(g_2')-1\le l(g_1')+1+l(g_2')+1-1=l\left(g_1'\overset{(i)}{\cdot}g_2'\right)+1,$$

$$k\left(g_1'\overset{(i)}{\cdot}g_2'\right)\ge k(g_1')+k(g_2')-2i+1\ge l(g_1')-2m(g_1)+1+l(g_2')-2m(g_2)+1-2i+1=$$

$$=l\left(g_1'\overset{(i)}{\cdot}g_2'\right)-m\left(g_1\overset{(i)}{\cdot}g_2\right)+1.$$

Limit theorems for numbers of connectivity components in recursively defined graphs. Remark that random quantity $l(g')$ may be represented as a sum $\sum_{i=1}^{n(g)}\eta_i$ of independent random variables η_i, $P(\eta_i=1)=q$, $P(\eta_i=0)=p$, $i=1,...,l(g)$.

Theorem 3. Suppose that $m(g)/n(g)\to 0$, $n(g)\to\infty$. Then almost surely

$$\frac{k(g')}{qn(g)}\to 1\ ,n(g)\to\infty\ . \tag{7}$$

Proof. Rewrite the inequalities (4) as follows

$$\frac{l(g')}{qn(g)}+\frac{1-2m(g)}{qn(g)}\le\frac{k(g')}{qn(g)}\le\frac{l(g')}{qn(g)}+\frac{1}{qn(g)}. \tag{8}$$

Theorem 3 statement is a corollary of the inequalities (7) and enforced law of large numbers [8, chapter IV, §3].

Theorem 4. Suppose that $m(g)/\sqrt{n(g)}\to 0$, $n(g)\to\infty$, then random variable $\left(k(g')-qn(g)\right)/\sqrt{pqn(g)}$ distribution tends to normal distribution with zero mean and single variation.

Proof. Rewrite the inequality (5) as follows

$$\frac{l(g')-qn(g)}{\sqrt{pqn(g)}}+\frac{1-2m(g)}{\sqrt{pqn(g)}}\le\frac{k(g')-qn(g)}{\sqrt{pqn(g)}}\le\frac{l(g')-qn(g)}{\sqrt{pqn(g)}}+\frac{1}{\sqrt{pqn(g)}}\ , \tag{9}$$

Then Theorem 4 is a corollary of the inequalities (9) and integral Muavre-Laplas theorem [8, chapter I, §6].

Limit theorems for numbers of connectivity components in parallel aggregation of chains. Consider important partial case of the graph g_m which is an aggregation of m parallel chains with the length n. Using Theorems 3, 4 it is possible to prove that for $n\to\infty$ random sequence

$k(g'_m)/qnm$ almost surely tends to 1, $n\to\infty$ for m which may depend on n arbitrarily. More over if $m/n\to 0$, $n\to\infty$, then distribution of random variable $\left(k(g')-qnm\right)/\sqrt{pqnm}$ for $n\to\infty$ tends to normal distribution with zero mean and single variation. But there is a question connected with a behavior of this sequence when $m\to\infty$ if for example there is $N<\infty$ so that $1<n<N<\infty$. This problem may be solved as follows.

It is easy to prove the recurrent formula

$$k(g'_{m+1}) = k(g'_m) + \left(\gamma_{m+1} - 1 + \chi(\gamma_{m+1}=0)\right) - \chi(\gamma_{m+1}=0)\,\chi(\gamma_1>0,\ldots,\gamma_m>0),\ m\ge 1, \tag{10}$$

and the initial condition

$$k(g'_1) = \gamma_1 + 1. \tag{11}$$

Here γ_m is a number of failed arcs in m - th chain, $\chi(.)$ is an indicator function of an event ".". Using the formulas (10), (11) it is easy to obtain that

$$k(g'_m) = 1 + \gamma_1 + \sum_{i=2}^{m}\left(\gamma_i - 1 + \chi(\gamma_i=0)\right) - \sum_{i=2}^{m}\chi(\gamma_i=0)\chi(\gamma_1>0,\ldots,\gamma_{i-1}>0),\ m\ge 1.$$

Consequently we have that

$$k(g'_m) = 2 + \sum_{i=1}^{m}\left(\gamma_i - 1 + \chi(\gamma_i=0)\right) - \sum_{i=1}^{m}\chi(\gamma_i=0)\chi(\gamma_1>0,\ldots,\gamma_{i-1}>0),\ m\ge 1. \tag{12}$$

and

$$1 + \sum_{i=1}^{m}\left(\gamma_i - 1 + \chi(\gamma_i=0)\right) \le k(g'_m) \le 2 + \sum_{i=1}^{m}\left(\gamma_i - 1 + \chi(\gamma_i=0)\right). \tag{13}$$

Calculate now a mean and a variation of the random variable $\left(\gamma_i - 1 + \chi(\gamma_i=0)\right)$:

$$M\left(\gamma_i - 1 + \chi(\gamma_i=0)\right) = nq - 1 + p^n, \tag{14}$$

$$M\left(\gamma_i - 1 + \chi(\gamma_i=0)\right)^2 = M\left[\gamma_i^2 + 1 + \chi(\gamma_i=0) - 2\gamma_i + 2\gamma_i\chi(\gamma_i=0) - 2\chi(\gamma_i=0)\right] =$$

$$= npq + n^2q^2 + 1 + p^n - 2nq - 2p^n = npq + n^2q^2 + 1 - 2nq - p^n,$$

$$D\left(\gamma_i - 1 + \chi(\gamma_i=0)\right) = npq + n^2q^2 + 1 - 2nq - p^n - \left(nq - 1 + p^n\right)^2 =$$

$$= npq + n^2q^2 + 1 - 2nq - p^n - n^2q^2 - 1 - p^{2n} + 2nq + 2p^n - 2nqp^n = npq - p^{2n} + p^n - 2nqp^n. \tag{15}$$

From the formulas (13) - (15) and enforced law of large numbers we obtain that almost surely $k(g'_m)/m \to nq - 1 + p^n$, $m\to\infty$. And from central limit theorem the distribution of random variable

$$\frac{k(g'_m) - m\left(nq - 1 + p^n\right)}{\sqrt{m\left(npq - p^{2n} + p^n - 2nqp^n\right)}}$$

tends to normal distribution with zero mean and single variation.

5. CONCLUSION

In this paper a static model of a graph with unreliable arcs is considered and a connectivity probability and a distribution of connectivity components are considered. But all obtained results may be spread onto a graph in which an arc w has failure intensity λ_w и and renewal intensity μ_w

so that $\lambda_w / (\lambda_w + \mu_w) = p$. In this case limit connectivity probability of the graph G and limit distribution of connectivity components are analogous to the same characteristics of static model.

REFERENCES

1. Ushakov I.A. et al. 1985. Reliability of technical systems: Handbook : *Moscow: Radio and Communication.* (In Russian).
2. Rodionova O.K., Rodionov A.S., Choo H. 2004. Network Probabilistic Connectivity Exact Calculation with Use of Chains. *ICCSA-2004. Springer LNCS.* Vol. 3046. P. 315-324.
3. Tanguy C. 2007. What is the probability of connecting two points? *J. Phys. A: Math. Theor.* V. 40. P. 14099-14116.
4. Popkov V.K. 2006. Mathematical models of connectivity. *Novosibirsk: Institute for Computing Mathematics and Mathematical Geophysics,* Siberian Branch of RAS. (In Russian).
5. Riabinin I.A. 2007. Reliability and safety of structural complicated systems. *Sankt-Petersberg: Edition of Sankt-Petersberg university,* (In Russian).
6. Tkachev O.A. 2011. Analytical estimation of reliability of networks consisting of identical elements. *Reliability: Theory and Applications.* Vol. 2. № 2. Pp. 9-20.
7. Polesskiy V.P. 1990. Estimates of connectivity probability of random graph. *Problems of information transmission.* T. 26. № 1. C. 90-98. (In Russian).
8. Shiriaev A.N. 1989. Probability. *M.: Science,* (In Russian).