DETERMINATION OF MEAN TIME TO FAILURE OF A NETWORK CONSISTING OF IDENTICAL NON-REPAIRABLE ELEMENTS

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ABSTRACT

It is suggested the analytical model permitting to get expression for determination of mean time to failure of a network consisting of identical non-repairable elements that fail independently of one another and have exponential distribution of time to failure. To determine values of obtained expressions it is necessary to determine probability of network failure in failure of defined quantity of its elements. This factor may be determined exactly with analysis of all possible combinations of failed elements or approximately with Monte-Carlo method.

1. INTRODUCTION

In assessment of reliability of networks consisting of non-repairable elements, determination of mean time to failure is of great interest.

In suggested work it is considered Markov process, states of which are characterized with quantity of failed elements and the state of network. The analysis of this process permits obtaining of analytical expression for mean time to failure of networks consisting of identical non-repairable elements. It is supposed that network elements fail independently one on another and have exponential distribution of time to failure.

In considered examples we will use connectivity of a network as criterion of its operability, however, obtained expressions are true for other criterion of its operability.

2. RELIABILITY MODEL OF NETWORK CONSISTING OF IDENTICAL NON-REPAIPABLE ELEMENTS

We supposed that network nods are absolutely reliable, and edges are identical in reliability, fail independently of one another and have exponential distribution of time to failure.

We will use probability of network failure due to ***i*** elements failure as main network parameter permitting us to determine values of network reliability factor under consideration. We will denote this factor as - $\mathbf{Z_i}$.

Value of $\mathbf{Z_i}$ is equal to ratio quantity of non-operable states of the network in case failure of ***i*** elements ($\mathbf{Y_i}$) to the total quantity of possible combinations of ***i*** elements of ***n***, where ***n*** is the quantity of network elements.

$$Z_i = \frac{Y_i}{\binom{n}{i}} \tag{1}$$

Let us determine values $\mathbf{Y_i}$ for network presented in Figure 1. Since considered network is biconnected, therefore one edge moving off cannot break its connectivity. Hence $\mathbf{Y_1=0}$. For definition of $\mathbf{Y_2}$ and $\mathbf{Y_3}$ we consider all possible states of the network when 2 and 3 edges are moved off respectively. It is possible to define that 2 cutsetss of capacity 2, and 14 cutsetss of capacity 3 are within the considered network, hence $Y_2=2$, $Y_3=14$. Any combination of *i* edges at ***i* > 3** is a cutsets, hence $Y_i = \binom{n}{i}$ for ***i*>3**.

Values Z_i for this network: $Z_1=0$, $Z_2=2/21$, $Z_3=14/35$, $Z_4=Z_5=Z_6=Z_7=1$. More detailed analysis of this network is described in article Tkachev (2011).

For small ***n*** the values of Y_i can be determined by means of enumeration of all possible network states. For great values of ***n*** it is necessary to use Monte-Carlo method.

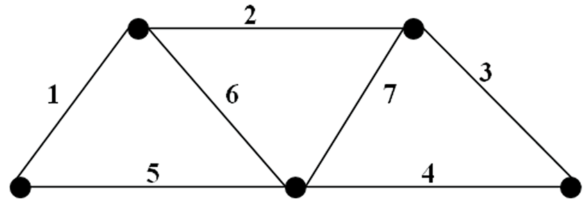

Figure1: Example of a network

Let us denote with Z^*_i the probability of network failure in failure of ***i*** element, if the network was operable for ***i-1*** failed elements. Correlation of values Z^*_i и Z_i was established in article Tkachev (1983).

Let us consider Markov chain describing change of network states at the moments of its elements failure.

If for state of ***i-1*** failed elements the network is operable, then for failure of ***i***-element it transforms with probability Z^*_i to non-operable state, or with probability $1- Z^*_i$ it will remain in operable state.

If in presence of ***i-1*** failed elements the network was non- operable, than for the failure of ***i***-element it will remain with probability of ***1*** in non-operable state.

Transition diagram of considered process is shown in Figure 2. States ***i'*** correspond to network operable states, and states ***i''*** correspond to non-operable states of network for failure of ***i*** elements.

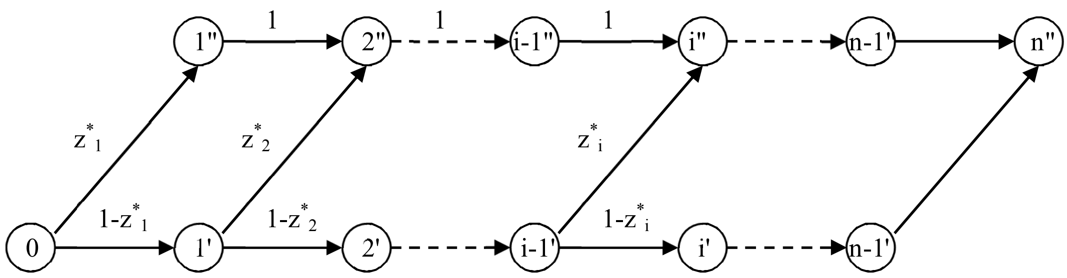

Figure 2: Network state transition diagram

Let us denote $P_{i'}$ – probability of operable state, and $P_{i''}$ –probability of non-operable state of network in the case of failure of ***i*** elements. From definition of Z_i it follows

$$P_{i'} = 1 - Z_i \tag{2}$$

$$P_{i''} = Z_i \tag{3}$$

In accordance with diagram (Figure 2), the network state transition can be written as

$$P_{i'} = P_{(i-1)'}(1 - Z^*{}_i) \tag{4}$$

wherefrom

$$Z^*_i = \frac{P_{(i-1)'} - P_{(i)'}}{P_{(i-1)'}} \quad (5)$$

Substituting in (5) values $P_{i'}$ и $P_{i''}$ from (2) and (3) we obtain

$$Z^*_i = \frac{Z_i - Z_{i-1}}{1 - Z_{i-1}} \quad (6)$$

Besides, from expression (4) it follows

$$P_{i'} = \prod_{j=1}^{i} 1 - Z^*_j = 1 - Z_i \quad (7)$$

For determination of mean time to failure we will consider continuous Markov process, describing system behavior in time. Process states are preset with quantity of failed elements and states of network. State transition diagram of this process is shown in Figure 3.

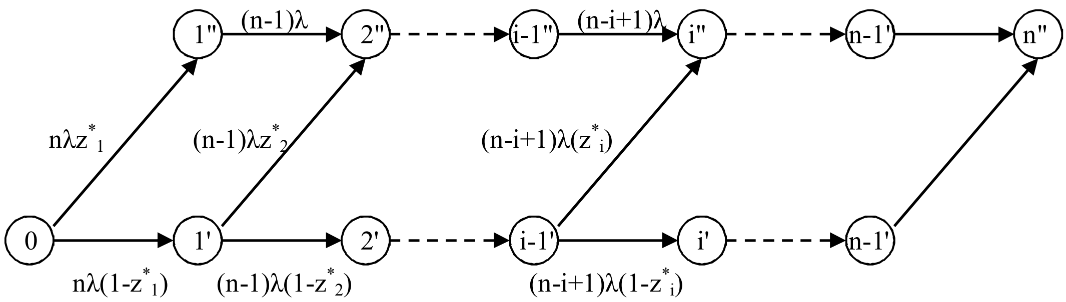

Figure 3: The Markov process of network state transition.

Let us denote elements failure rate as **λ**. Let in some moment of time to be ***i*** failed elements and the network to be operable. In infinitely small time interval Δt it can occur one of the following events:

- network will remain in operable state.

Probability of this event **(1-(n-i)λ Δt);**

- One more element will fail and the network will transit to non-operable state.

Probability of this event **(1-(n-i)λ *Z*_{i+1}* Δt);**

- One more element will fail but the network will remain in operable state.

Probability of this event **(1-(n-i)λ *(1-Z*_{i+1})* Δt).**

To simplify analytic calculations we will substitute a set of non-operable states with one absorbing state Figure 4.

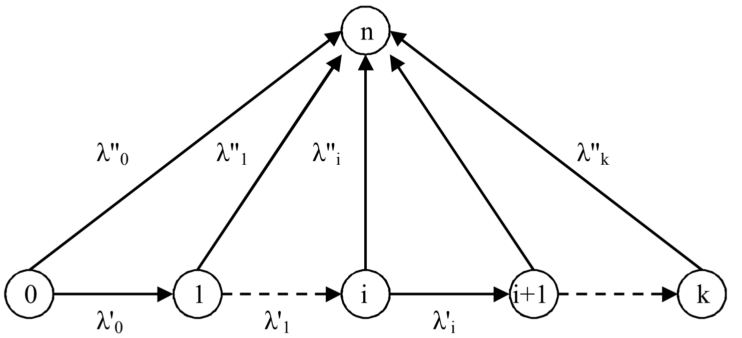

Figure 4: The Markov process of network state transition with absorbing state

Probabilities of process being in different states in arbitrary moment of time $\mathbf{P_i(t)}$ can be found by means of solving the following differential equation system:

$$\begin{aligned}
\frac{\partial P_0(t)}{\partial t} &= -\lambda_0^{'} P_0(t) - \lambda_0^{''} P_0(t) = -\lambda_0 P_0(t) \\
\frac{\partial P_i(t)}{\partial t} &= \lambda_{i-1}^{'} P_{i-1}(t) - \lambda_i^{'} P_i(t) - \lambda_i^{''} P_i(t) = \lambda_{i-1}^{'} P_{i-1}(t) - \lambda_i P_i(t) \\
\frac{\partial P_n(t)}{\partial t} &= \sum_{i=0}^{k} \lambda_i^{''} P_i(t)
\end{aligned} \quad (8)$$

Here

$\lambda_i=(n-i)\lambda$

$\lambda'_i=(1- Z^*_{i+1})\ \lambda_i$

$\lambda''_i=Z^*_{i+1}\ \lambda_i$

k – is the maximum quantity of elements, after the failure of which the network can be operable.

Let us solve differential equation system (8) using Laplace transformation under initial conditions $P_0(0)=1$, $P_i(0)=0\ \forall i \neq 0$.

Let us designate

$$F(s) = \int_0^\infty P(t) e^{-st} \partial t$$

Then differential equation system (8) is deduced into algebraic equation system relatively to F(s).

$$\begin{aligned}
sF_0(s) &= 1 - \lambda_0 F_0(s) \\
sF_i(s) &= \lambda_{i-1}^{'} F_{i-1}(s) - \lambda_i F_i(s) \\
sF_n(s) &= \sum_{i=0}^{k} \lambda_i^{''} F_i(s)
\end{aligned} \quad (9)$$

Wherefrom we have

$$\begin{aligned}
F_0(s) &= \frac{1}{s+\lambda_0} \\
F_i(s) &= \frac{\lambda_{i-1}^{'}}{s+\lambda_i} F_{i-1}(s) \\
F_n(s) &= \frac{1}{s} \sum_{i=0}^{k} \lambda_i^{''} F_i(s)
\end{aligned} \quad (10)$$

This solution permits obtaining of expression for mean time to failure. According to definition Kozlov&Ushakov (1975).

$$T = \sum_{i=0}^{k} F_i(s) \Big| s = 0 \quad (11)$$

Expanding (10) we obtain

$$F_0(0) = \frac{1}{\lambda_0} = \frac{1}{n\lambda} \quad (12)$$

$$F_i(0) = \frac{\lambda'_{i-1}}{\lambda_i} F_{i-1}(0) = \frac{\prod_{j=1}^{i}(1 - Z_j^*)}{(n-i)\lambda} \quad (13)$$

Using (7) the expression (13) may be simplified

$$F_i(0) = \frac{\prod_{j=1}^{i}(1 - Z_j^*)}{(n-i)\lambda} = \frac{1 - Z_i}{(n-i)\lambda} \quad (14)$$

Then from (11) it follows

$$T = \frac{1}{\lambda}\sum_{i=0}^{k}\frac{(1 - Z_i)}{n - i} \quad (15)$$

Expression (15) permits determine the value of mean time to failure for known values $\boldsymbol{Z_i}$ and $\boldsymbol{\lambda}$.

Let us determine value $\boldsymbol{T}$ for network, shown in Figure 1, for λ=0,01 (1/hour)

$$T = 100(\frac{1}{7} + \frac{1}{6} + (\frac{19}{21})\frac{1}{5} + (\frac{21}{35})\frac{1}{4}) = 100 * \frac{269}{420} \approx 64{,}047(hour)$$

3. COMPARISON WITH KNOWN RESULTS

To check obtained expression let us determine mean time to failure of a system, for which there are known analytical assessments Figure 5.

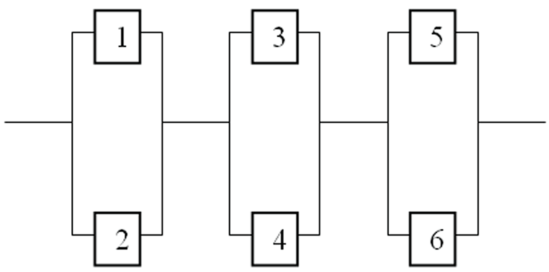

Figure 5: Series-parallel system

In Gnedenko&Belyayev&Solovyov (1965) there can be found following expressions. Probability of failure-safe operation of system consisting of 2 parallel identical elements.

$$P_{(t)} = 1 - (1 - e^{-\lambda t})^2 = 2e^{-\lambda t} - e^{-2\lambda t}$$

Probability of failure-safe operation of system consisting of 3 parallel identical, series-connected subsystems.

$$P_{c(t)} = (P_{(t)})^3$$

$$P_{c(t)} = (2e^{-\lambda t} - e^{-2\lambda t})^3 = 8e^{-3\lambda t} - 12e^{-4\lambda t} + 6e^{-5\lambda t} - e^{-6\lambda t}$$

Mean time to failure is equal to

$$T = \int_0^\infty P_{c(t)} \partial t = \frac{1}{\lambda}(\frac{8}{3} - \frac{12}{4} + \frac{6}{5} - \frac{1}{6}) = \frac{1}{\lambda}(\frac{42}{60}) = 0{,}7\frac{1}{\lambda}$$

For determination of mean time to failure with use of expression (15), it is necessary to determine values Z_i. For considered system they are equal to: Z_1=0,0; Z_2=3/15; Z_3=12/20; Z_4 = Z_5 = Z_6=1.

Substituting these values into expression (15), we obtain:

$$T = \frac{1}{\lambda}(\frac{1}{6} + \frac{1}{5} + (\frac{12}{15})\frac{1}{4} + (\frac{8}{20})\frac{1}{3}) = \frac{1}{\lambda}(\frac{42}{60}) = 0{,}7\frac{1}{\lambda}$$

This example also shows that suggested method can be used for determination of mean time to failure of complicated series-parallel and parallel-series systems consisting of identical non-repairable elements.

4. EXAMPLE OF NETWORKS RELIABILITY ANALYSIS

Let us denote the number of nodes by ***m*** and the number of edges by ***n***. Let us consider network with parameters m=20, n=24, Figure 6. Values **Zi** of this network can be determined by means of enumeration of all possible network states. Calculation results are shown in table 5.

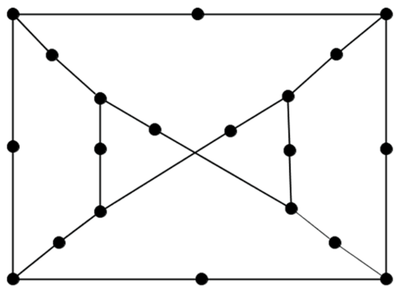

Figure 6: Network with parameters m=20, n=24.

Table 5. Values $\mathbf{Y_i}$ and $\mathbf{Z_i}$ for network on Figure 6.

i	1	2	3	4	5	i>=6
Y_i	0	12	328	4082	29960	
Z_i	0	0,043478	0,162055	0,384252	0,704875	1

Suggested method can be used for other criteria of a network operability. For example: the network is operable, if number of connected nodes **>= m-k**.

Obtained results can also be used in this case, but it is necessary to make some corresponding corrections in determination algorithm of values $\mathbf{Z_i}$**.** In table 6 there are listed values $\mathbf{Z_i}$ for different values of **k** for network shown in Figure 6. Determination of values $\mathbf{Z_i}$ for k>0 was carried out with Monte-Carlo method. Number of tests amounted 10^6.

Table 6. Values of $\mathbf{Z_i}$ for different values of **k** for network on Figure 6.

i / k	0	2	4	6	8	10
1	0,0000000	0,00000	0,00000	0,00000	0,00000	0,00000
2	0,0434783	0,00000	0,00000	0,00000	0,00000	0,00000
3	0,1620553	0,01581	0,00000	0,00000	0,00000	0,00000
4	0,3841521	0,07980	0,01498	0,00435	0,00296	0,00000
5	0,7048748	0,25247	0,08813	0,03329	0,01644	0,00000
6	1,0000000	0,56194	0,27850	0,13070	0,05976	0,00324
7	1,0000000	0,86648	0,56942	0,33252	0,17912	0,04445
8	1,0000000	1,00000	0,82642	0,58123	0,36764	0,16783
9	1,0000000	1,00000	0,96544	0,79849	0,57894	0,34029
10	1,0000000	1,00000	1,00000	0,93211	0,76173	0,52593
11	1,0000000	1,00000	1,00000	0,98807	0,89262	0,69336
12	1,0000000	1,00000	1,00000	1,00000	0,96470	0,82566
13	1,0000000	1,00000	1,00000	1,00000	0,99360	0,91743
14	1,0000000	1,00000	1,00000	1,00000	1,00000	0,97046
15	1,0000000	1,00000	1,00000	1,00000	1,00000	0,99403
α	0,214850615	0,29501	0,354027	0,411813	0,476755	0,571528

$$\alpha = \sum_{i=0}^{k} \frac{(1-Z_i)}{n-i}$$

$$T = \alpha * \lambda$$

It should be noticed that obtained results can be used in the case when nods failed, though edges are absolutely reliable. Node failure can be modeled by removal of all edges coming from this node.

5. CONCLUSION

It was obtained the analytical expression for determination of mean time to failure of networks consisting of identical non-repairable elements that fail independently of one another and have exponential distribution of time to failure. And as the nods, so the edges can be assumed as absolutely reliable.

Suggested method can also be used for reliability assessment of complicated series-parallel systems.

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