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CFBLTQ: A CLOSED FEED BACK LOOP TYPE QUEUING SYSTEM; MODELING AND ANALYSIS

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ABSTRACT

This paper presents an innovative approach to solve probability distributions of a closed feed back loop type queuing system with general service time distribution. This model is applied to a multi-processors system where some of its nodes are performed a repair procedure during a node's malfunction condition. Our model is appropriate for a multiprocessor system that employs a common bus or for a multi-node system in computer network. A meticulous analysis of the system's model has been conducted and numerical results have been obtained to scrutinize the proposed model.

1 INTRODUCTION

The queuing system is widely classified into an open-type system and a closed-type system model which are described by Baskett et al. (1975). The open-type system model customers arrive from outside and depart to the outside of the system while in the closed system, the customers operate internally where no customers arrive from outside or depart to outside of the system. Numerous research works have been extensively dedicated to investigate the open system model (classical model) which is widely used in computer systems and computer networks. However, the closed system model has not been paid much attention in spite of its paramount importance to computer systems. Some research works have devoted to find an optimal solution to the closed queuing behaviour at a group of systems that form networks (See Benson & Gregory 1961, Jeffrey & Buzen 1973, Kakubava 2010, Denning & Buzen1978, Lavenberg & Reiser1980, Reiser & Lavenberg 1980), or at a particular system such as cyclic systems (See Koenigsberg, 1958, Gordon et al. 1967, Lipsky 1985, Lavenberg 1989). The reliability and flexibility of closed queuing systems with repairable elements has been investigated by Chinho et al. (1994) & Kakubava (2010), both of these works are considered two types of service operations where in Chinho (1994) examined the flexibility and the capacity of the repair station, while Kakubava (2010) scrutinized the closed queuing system for replacements and renewals.

This paper has made very punctilious efforts to formulate the closed system's behaviour for maintenance operation at a maintenance repair center in the course of malfunctions which might develop in the system during repairing procedures. A failed element is removed from the system while keeping the system operates normally. Our proposed a closed queuing system consists of only one type of service operation with a closed feed back loop type queuing system *CFBLTQ* model. We study the reliability and the limitation of the system in case of failed elements increased, also we will investigate of how far faults tolerance that the system can offer.

2 MODEL OF A CLOSED LOOP TYPE QUEUING SYSTEM

The proposed system consists of multi elements or processors which are autonomously operated. When any of the system's elements malfunction is reported, this element is required for repair operation (or service) at the service repair center (server). The repaired element is subsequently put to work in the system again.

This kind of element's failure and repair procedure is called Closed Feed Back Loop Type *CFBLTQ* queuing system, an example of this model is illustrated in Figure 1. The *CFBLTQ* model has fault tolerance, with respect to a single or a multi-element failure. To formulate the system's model, we define λ as elements' failure rate per unit time and μ as the rate of completions' repairing per unit time when the system is busy. Meanwhile, the exponential distributions for both arrival time and service time have been considered. The system has also self configuration feature that can tolerant temporary failures while distributing tasks which have been assigned to the failure element to other active elements. The system can tolerant up to m of N elements $(m \in N)$ while the system operation will be in a normal operation condition if the number of faulty elements are less than or equal to m. The fault tolerant is defined by probability of working elements in the system which the probability of a minimum number of elements m in the system while keeping the system operates normally. The following sub-section will present a mathematical procedure of how to obtain the probability of working elements.

As mentioned above, the proposed system's model is applicable for numerous systems' applications in computer systems and we will focus on a maintenance operation at a maintenance repair station.

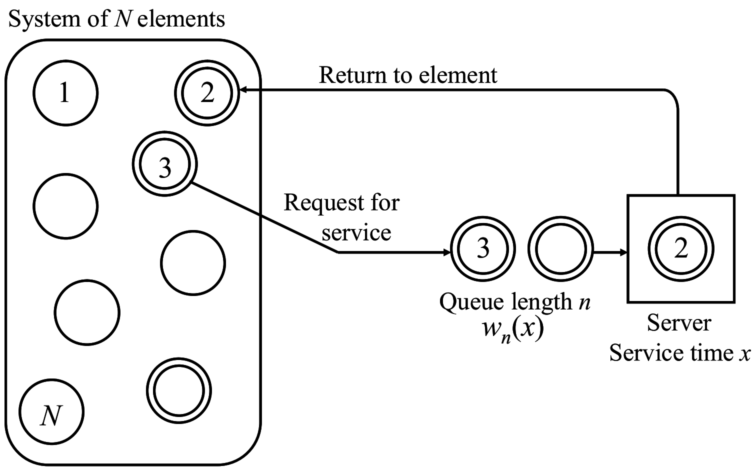

Figure 1. An example model of a closed feed back loop type queuing system model.

3 AN ANALYSIS OF THE CLOSED QUEUING SYSTEM

A systematic approach is given in this section for proper analysis of *CFBLTQ* model, an example of this model is shown in Figure 1. The *CFBLTQ* model is described below and the properties of the model is given by

(1) We consider the system is in a steady state, and the queue is first in and first service (FIFS) model's discipline.

(2) Let the number of elements in the system be N (N >1). The request arrivals for service due to elements' malfunctions follow an exponential distribution with elements' failure rate of λ.

(3) Let the service time distribution be a general distribution. Suppose the probability that a service is started between arbitrary time t and time $t+\Delta$ is equal to $\mu(x)\Delta+o(\Delta^2)$ on service time x, and the density function $f(x)$ of service time distribution is given by

$$f(x) = \mu(x) \cdot e^{-\int_0^x \mu(y)dy} \tag{1}$$

The full derivation of above equation is given in the Appendix of this paper.

(4) Let the probability density function of the service time x with n queue length be $w_n(x)$. The state probabilities p_n are given by

$$p_n = \int_0^\infty w_{n-1}(x)\,dx \qquad (n = 1,2,\cdots,N) \tag{2}$$

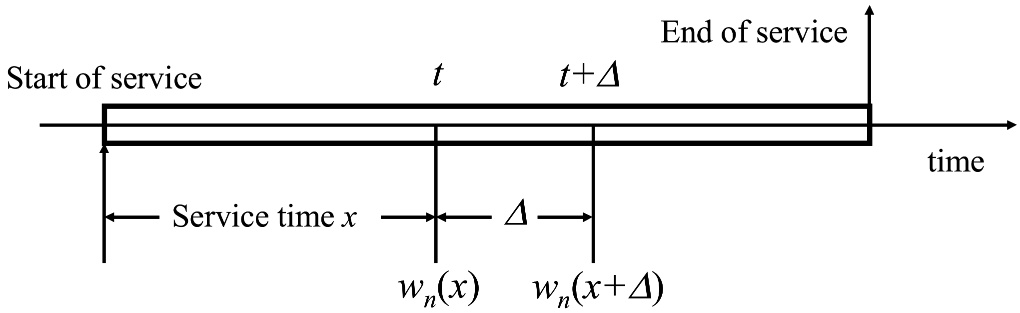

Figure 2. The relationship among n states $w_n(x)$ at arbitrary time t and $t+\Delta$, when the service continues.

From above notations and since the service is continued, as shown in Figure 2, the relationship between $w_n(x)$ in arbitrary time t and $w_n(x+\Delta)$ in time $(t+\Delta)$ is given as follows

$$w_0(x+\Delta) = \{1-(N-1-n)\cdot\lambda\Delta\}\cdot\{1-\mu(x)\Delta\}\cdot w_0(x) + o(\Delta^2) \tag{3}$$

$$\begin{aligned} w_n(x+\Delta) &= \{1-(N-1-n)\cdot\lambda\Delta\}\cdot\{1-\mu(x)\Delta\}\cdot w_n(x) \\ &\quad + (N-n)\cdot\lambda\Delta\cdot w_{n-1}(x) + o(\Delta^2) \qquad (n = 1, 2, \cdots, N-1) \end{aligned} \tag{4}$$

For $\Delta \to 0$, we can differentiate the above equations with respect to x to obtain

$$\frac{d}{dx} w_0(x) + \{(N-1)\cdot\lambda + \mu(x)\}\cdot w_0(x) = 0 \tag{5}$$

$$\frac{d}{dx} w_n(x) + \{(N-1-n)\cdot\lambda + \mu(x)\}\cdot w_n(x) = (N-n)\cdot\lambda\cdot w_{n-1}(x) \quad (n = 1,2,\cdots,N-1) \tag{6}$$

In order to solve the above differential equations, the differential Equations $w_n(x)$ of (5) and (6) become

$$w_0(x) = C_0 \cdot e^{-(N-1)\cdot\lambda x - \int_0^x \mu(x)dx} \tag{7}$$

$$\begin{aligned} w_n(x) = \{&C_n + (-1)^1 \frac{(N-n)}{1!}\cdot C_{n-1}\cdot e^{-\lambda x} + (-1)^2 \frac{(N-n+1)(N-n)}{2!}\cdot C_{n-2}\cdot e^{-2\cdot\lambda x} \\ &+ \cdots + (-1)^n \frac{(N-1)\cdots(N-n)}{n!}\cdot C_0\cdot e^{-n\cdot\lambda x}\}\cdot e^{-(N-n-1)\cdot\lambda x - \int_0^x \mu(y)dy} \\ &(n = 1,2,\cdots,N-1) \end{aligned} \tag{8}$$

where C_n (n= 0, 1, 2, . . ., N-1) is a constant value given by the boundary conditions at the start point or at the end point of service. Moreover, from (2), the state probabilities are

$$p_1 = \int_0^\infty w_0(x)dx = C_0 \cdot \frac{1-f^*\{(N-1)\lambda\}}{(N-1)\lambda} \tag{9}$$

$$p_n = \int_0^\infty w_{n-1}(x)dx = C_{n-1} \cdot \frac{1 - f^*\{(N-n)\lambda\}}{(N-n)\lambda} + (-1)^1 \frac{(N-n+1)!}{1!} \cdot C_{n-2} \cdot \frac{1 - f^*\{(N-n+1)\lambda\}}{(N-n+1)\lambda} + \cdots$$

$$+ (-1)^{n-1} \frac{(N-1)\cdots(N-n+1)}{(n-1)!} \cdot C_0 \cdot \frac{1 - f^*\{(N-1)\lambda\}}{(N-1)\lambda}$$

$$+ (-1)^{n-1} \frac{(N-1)\cdots(N-n+1)}{(n-1)!} \cdot C_0 \cdot \frac{1 - f^*\{(N-1)\lambda\}}{(N-1)\lambda} \qquad (n = 2, 3, \cdots, N) \qquad (10)$$

where $f^*(i\lambda)$ is the Laplace transform of the function $f(x)$ that was given in (1) and is given by

$$f^*(i\lambda) = \int_0^\infty f(x) \cdot e^{-i \cdot \lambda x} dx \qquad (i = 0, 1, 2, \cdots, N-1) \qquad (11)$$

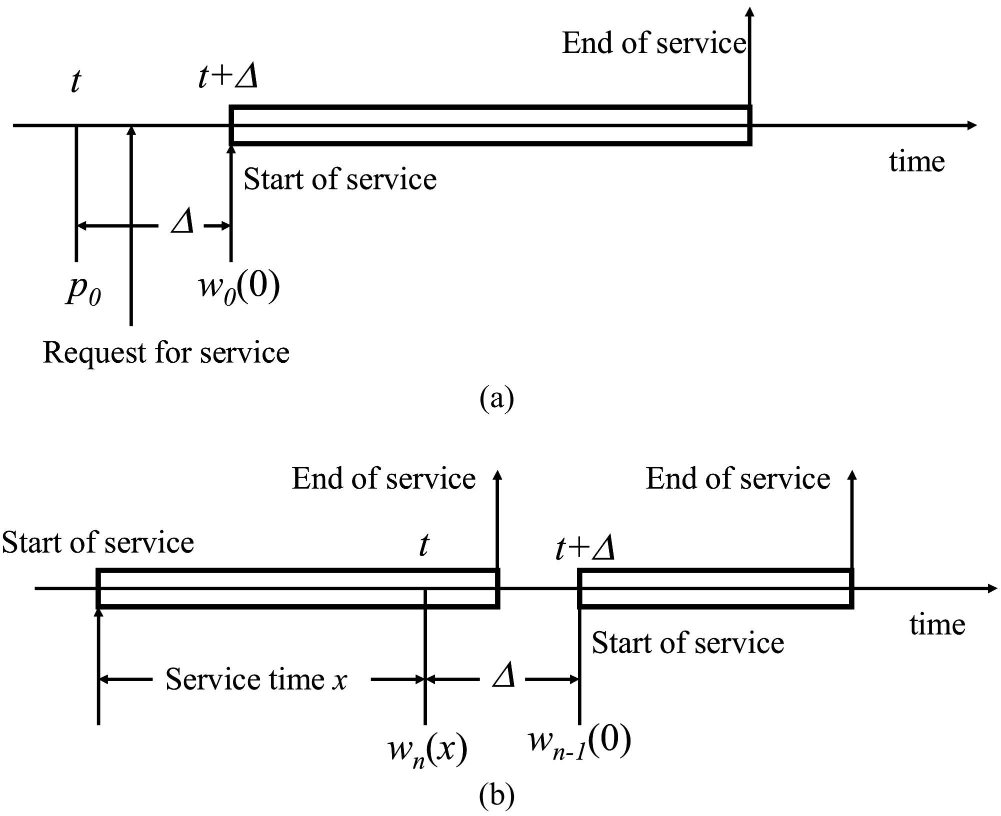

Figure 3. The states exchange between arbitrary time t and $t+\Delta$ at the start point and/or at the end point of the service. (a) when n=0 at arbitrary time t (b) when n>0 at arbitrary time t

On the other hand, the boundary conditions at the start point or at the end point of the service, as shown in Fig.3, are given by the following formulas

$$p_0 = \{1 - N \cdot \lambda\Delta\} \cdot p_0 + \int_0^\infty \mu(x)\Delta \cdot w_0(x)\, dx + o(\Delta^2) \qquad (12)$$

$$w_0(0)\Delta = \int_0^\infty \mu(x)\Delta \cdot w_1(x)\, dx + N \cdot \lambda\Delta \cdot p_0 + o(\Delta^2) \qquad (13)$$

$$w_{n-1}(0)\Delta = \int_0^\infty \mu(x)\Delta \cdot w_n(x)\, dx + o(\Delta^2) \qquad (n = 2, 3, \cdots, N-1) \qquad (14)$$

If $\Delta \to 0$, the above equations become as follows

$$N \cdot \lambda \cdot p_0 = \int_0^\infty \mu(x) \cdot w_0(x)\, dx \tag{15}$$

$$w_0(0) = \int_0^\infty \mu(x) \cdot w_1(x)\, dx + N \cdot \lambda \cdot p_0 \tag{16}$$

$$w_{n-1}(0) = \int_0^\infty \mu(x) \cdot w_n(x)\, dx \qquad (n = 2,3,\cdots,N-1) \tag{17}$$

By substituting the values of w_0 and w_n in Equations (7) and (8), respectively, into (15), (16) and (17) to obtain the constant values C_0, $C_1$and C_n which are given as follows

$$C_0 = \frac{N \cdot \lambda \cdot p_0}{f^*\{(N-1)\lambda\}} \tag{18}$$

$$C_1 = C_0 \cdot \frac{1 + (N-2) \cdot f^*\{(N-1)\lambda\}}{f^*\{(N-2)\lambda\}} \tag{19}$$

$$\begin{aligned} C_n = \frac{1}{f^*\{(N-n-1)\lambda\}} \cdot [\{1 + \frac{N-n}{1} \cdot f^*((N-n)\lambda)\} \cdot C_{n-1} \\ + (-1)^1 \frac{(N-n+1)}{1!}\{1 + \frac{N-n}{2} \cdot f^*((N-n+1)\lambda)\} \cdot C_{n-2} + \cdots \\ + (-1)^{n-1} \frac{(N-1)\cdots(N-n+1)}{(n-1)!} \cdot \{1 + \frac{N-n}{n} \cdot f^*((N-1)\lambda)\} \cdot C_0] \\ (n = 2, 3, \cdots, N-1) \end{aligned} \tag{20}$$

Insert the above constant values into the Equations (9) into (10), then we have the state probabilities p_n (n=0, 1, 2, . . ., N-1). The empty state probability p_0 is given by

$$\sum_{n=0}^{N} p_n = 1 \tag{21}$$

To calculate the average waiting time, we need to define another parameter which is average service time T_S, this parameter is given by

$$T_S = \int_0^\infty x \cdot f(x)\, dx \tag{22}$$

If the state probabilities p_n (n=0, 1, 2, . . ., N) are solved by above (9) to (10) and (19) to (22) then the average queue length L_q and the average waiting time W_q are given respectively by

$$L_q = \sum_{n=1}^{N} (n-1)\, p_n = \sum_{n=0}^{N-1} n\, p_{n+1} \tag{23}$$

$$\begin{aligned} W_q = \sum_{n=0}^{N-1} \int_0^\infty w_n(x) \left\{ nT_S + \int_0^\infty y \frac{f(x+y)}{1-F(x)} dy \right\} dx = L_q \cdot T_S + \sum_{n=0}^{N-1} \{C_n \cdot \frac{f^*((N-n-1)\lambda)}{((N-n-1)\lambda)^2} \\ + \sum_{i=1}^{n} (-1)^i C_{n-i} \cdot \frac{(N-n+i-1)\cdots(N-n)}{i!} \cdot \frac{f^*((N-n+i-1)\lambda)}{((N-n+i-1)\lambda)^2}\} \end{aligned} \tag{24}$$

Suppose P_w is working probability which it means that the probability of a minimum number of elements m in the system while keeping the system operates normally and it's given by

$$P_w = p_0 + p_1 + \cdots + p_{N-m} \qquad (N \geq m > 0) \tag{25}$$

The above (25) means that the system is able to operate normally with minimum elements in operations. Therefore, our model has fault tolerance and has the ability to respond properly to an unexpected failure, and we can realize that the system's operation works normally even the system has some faulty elements.

4 NUMERICAL RESULTS AND DISCUSSIONS

In previous section, we assumed in our previous calculations that the service time was general service distribution, however, if we consider that the service time is Erlang type k distributed then Equation (11) will be

$$f^*(i\lambda) = \int_0^\infty f(x)e^{-i\lambda x}dx = \int_0^\infty \frac{(k\mu)^k}{(k-1)!}x^{k-1}e^{-k\mu x}e^{-i\lambda x}dx = \left(\frac{k\mu}{i\lambda + k\mu}\right)^k \qquad (26)$$

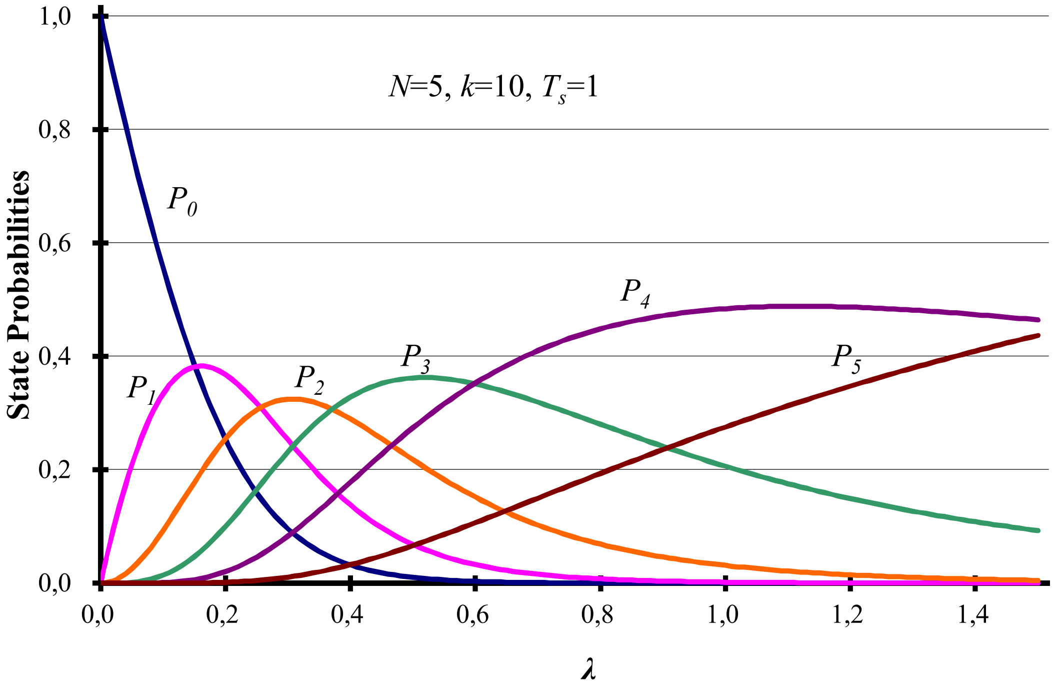

Figure 4. The state probabilities p_n versus the arrival rate λ on condition of N=5, k=10, T_S=1 (unit time)

Equation (33) is Laplace transformation of $f(x)$, where μ is any arbitrary positive service rate. We analyze the initial system performance by evaluating (9) to (21) and (22) for N=5, T_s=1 with different k phase and is plotted in Figure 4. As shown in figures, the state probability of all elements in operations p_0, where no defective elements in the system or n=0 is reported, is sharply decreased while the state probability p_n for (n=N) is increased as arrival rate λ expanded. However, for each state probabilities p_n ($0< n <N$) has a maximum value at a specific rate of λ. This maximum value becomes higher as k increased. As we mentioned above, the number of faulty elements should be less than or equal to m ($m \in N$), for keeping the system operates normally.

Figure 5 shows the probability of the system operates normally P_w versus arrival rate of the defective elements with various k phases. However, if the system has not fault tolerance capability then any defective element in the system will cause a system's termination. For example, if the system has 5 elements then the probability $P_w = p_0$ =0.269 at λ=0.2 with k=1. However, in case of the system has fault tolerance capability, suppose that the system has 3 of 5 elements are out of services, then the probability P_w=0.798 at λ=0.2 with k=1. Therefore, our model has fault tolerance and has the ability to respond reasonably to an unexpected failure, and we can realize that system's operation works normally even the system has some faulty elements.

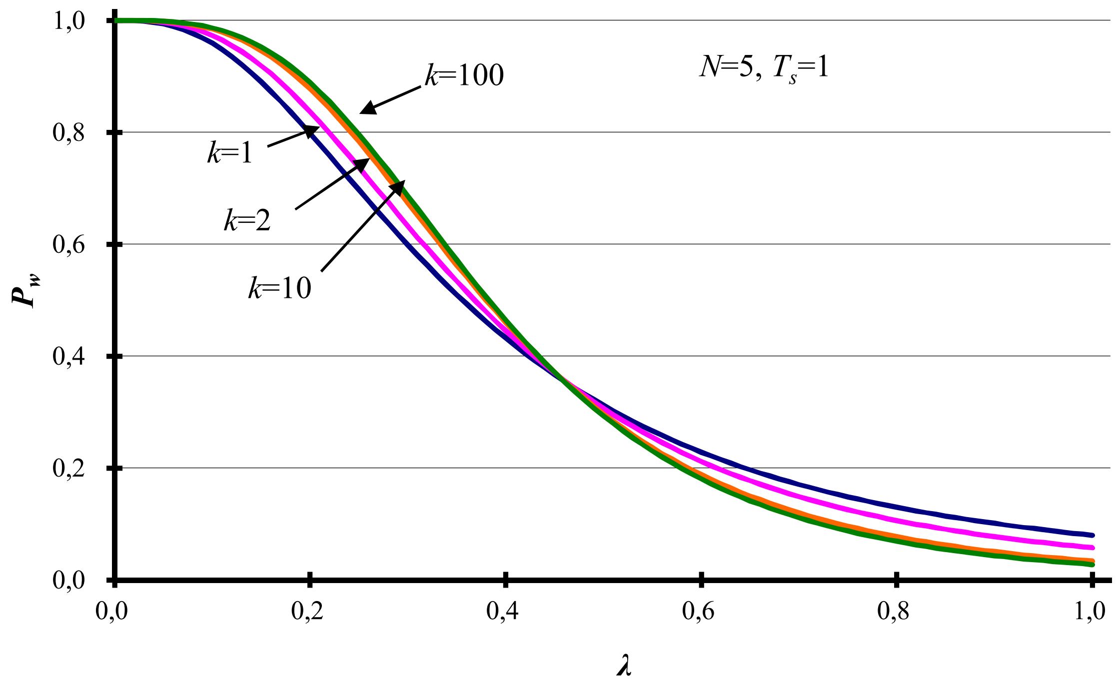

Figure 5. The probability P_w of the 3 out of 5 system versus the arrival rate λ on conditions of N=5, T_S=1 (unit time).

5 CONCLUSIONS

A study of a closed system's behavior is very important because is widely used in recent computer systems and in factories. In this paper, we presented an analytical method for a closed feed back loop type queuing *CFBLTQ* model, which is appropriated for failure and repair processes in the maintenance station. Numerical examples were given to gain a better understanding of the system's behavior. The system model has fault tolerance capabilities by considering the system has a self configuration feature that can tolerant temporary failures while operation's tasks which have been assigned to failure elements are distributed to other active elements.

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APPENDIX

In this Appendix, we will derive (1) which is the density function *f(x)* of service time distribution (See Yoshioka 1988, Yoshioka 2004). Suppose the probability that service is completed on arbitrary service time x to $x+\Delta$ is the conditional probability and is defined by

$$\frac{F(x+\Delta)-F(x)}{1-F(x)} = \frac{f(x)}{1-F(x)}\cdot\Delta + o(\Delta^2) = \mu(x)\cdot\Delta + o(\Delta^2) \tag{A1}$$

where $F(x)$ and $f(x)$ are the probability distribution function and the probability density function, respectively. From (A1), we can obtain the following function

$$\frac{f(x)}{1-F(x)} = \mu(x) \tag{A2}$$

By taking limit integration for both sides of (A2) with respect to t, where $(0 \le t \le x)$

$$\int_0^x \frac{f(t)}{1-F(t)}dt = \log|1-F(x)| = \int_0^x \mu(t)dt \tag{A3}$$

The function $F(x)$ in (A3) can be expressed as

$$F(x) = 1 - e^{-\int_0^x \mu(t)dt} \tag{A4}$$

Finally, the probability density function $f(x)$ can be obtained, as well

$$f(x) = \frac{d}{dx}F(x) = \mu(x)\cdot e^{-\int_0^x \mu(t)dt} \tag{A5}$$