COOPERATIVE EFFECTS IN COMPLETE GRAPH WITH LOW RELIABLE ARCS

G. Tsitsiashvili

•

IAM, FEB RAS, Vladivostok, Russia
e-mail: guram@iam.dvo.ru

ABSTRACT

An analysis of the limit $\lim_{n\to\infty} P_n = A$ of connectivity probability (CP) P_n of complete graph with n nodes and independent arcs which have working probability n^{-a} is made. It is proved that for $0 < a < 1$ we have the equality $A = 1$ and for $1 < a$ the equality $A = 0$.

1. INTRODUCTION

An analysis of the limit $\lim_{n\to\infty} P_n = A$ of connectivity probability (CP) P_n of complete graph with n nodes and independent arcs which have working probability n^{-a} is made. In the complete graph each pair of nodes is connected by single arc. It is proved that for $0 < a < 1$ we have the equality $A = 1$ and for $1 < a$ the equality $A = 0$.

Analogously with [1] this zero-one low may be interpreted as a transition from chaos to order in a structure with all possible connections between nodes. The parameter a may be called order parameter with critical meaning $a = 1$. Such model may be applied to an analysis of connection structure in the internet for example in social networks. Another field of applications may be a modeling of self organizing systems.

A calculation of CP for graphs with unreliable arcs is considered in a lot of monographs [2] – [5] which become classical. This list may be added by articles on upper and low bounds of CP [6] – [11], on transfer matrices [12], [13], on an application of groups of disjoint events [14] for accuracy CP calculation of two nodes in a graph, on accelerated algorithms [15] of accuracy CP calculation and on an application of Monte-Carlo simulations with some combinatory formulas for CP estimates [16]. There is a large number of other articles and monographs devoted to this very important problem of applied mathematics and applied probability.

But a consideration of complete graph with large number of nodes demand to construct special upper and low bounds of the probability P_n and its asymptotic analysis for $n \to \infty$. These approaches look like proofs of limit theorems in combinatory probability theory [17].

A formulation of considered problem appears from oral communication of E.A. Nurminsky. It is based on numerical experiment showed a fact of a transition from "chaos to order" in a graph with large number of connections between nodes. Our researches are accompanied by a lot of sufficiently long Monte-Carlo simulations which helped to define main properties of random graphs with large number of arcs and nodes.

2. FORMULATION OF MAIN RESULTS

Consider complete graph G_n with nodes $1,...,n$ and with independently working arcs. Denote p, $0 < p < 1$, working probability of an arc and put $P_n(i, j)$ CP of the nodes i, j in the graph G_n,

$\overline{P}_n(i,j)=1-P_n(i,j)$. It is obvious that $P_n(i,j)=P_n(k,s)$ for all pairs of nodes $i \neq j$, $k \neq s$. Denote P_n the CP of random realization of the whole graph, $\overline{P}_n = 1-P_n$.

Theorem 1. Suppose that $p = p_n = n^{-a}$, $a \geq 0$, then

$$\lim_{n\to\infty} P_n = 1,\ 0 < a < 1\ . \tag{1}$$

$$\lim_{n\to\infty} P_n = 0,\ a > 1\ . \tag{2}$$

Denote $Q_n(b)$ the probability that in random realization of the graph G_n there is more than $[n^b]$ connectivity components, $0 < b \leq 1$. Here $[r]$ is integer part of real number *r*.

Theorem 2. Assume that $p = p_n = n^{-a}$, $a > 0$, $0 < b \leq 1$. Then

$$\lim_{n\to\infty} Q_n(b) = 1,\ 1+b < a \leq 2,\ . \tag{3}$$

$$\lim_{n\to\infty} Q_n(1) = 1,\ a > 2\ . \tag{4}$$

Remark 1. The condition $p = n^{-a}$ may be replaced in Theorems 1, 2 by more general condition $p = \min(1, cn^{-a})$, where c is arbitrary positive number.

Remark 2. The condition of the graphs G_n, $n > 1$, completeness may be replaced by the suggestion that there is C, $1/2 < C \leq 1$, so that each node of the graph G_n is connected with more than $[Cn]-1$ other nodes.

3. PROOFS OF MAIN RESULTS

Theorem 1. Suppose that $0 < a < 1$ then it is possible to use obvious statement that the probability of the graph G_n random realization disconnection equals with the probability that there are k, $0 < k \leq [n/2]$, nodes which are not connected with rest $n-k$ nodes of the graph G_n. Consequently the inequality

$$\overline{P}_n \leq T = \sum_{0<k\leq[n/2]} C_n^k q^{k(n-k)}\ , \tag{5}$$

is true with $q = 1-p$. Remark that the functions C_n^k, $q^{k(n-k)}$ of the discrete argument k for $0 < k \leq [n/2]$ do not decrease.

Choose $\gamma > 0$ and integer K from the conditions

$$0 < \gamma < 1-a\ ,\ 0 < 1-K\gamma < \gamma\ . \tag{6}$$

and so

$$K\gamma > a\ . \tag{7}$$

Choose N so that for $n > N$ the inequality $n^{K\gamma} < n/2$ is true and put then $n > N$. Represent the sum T as follows

$$T = \sum_{i=1}^{K} T_i + T_0,\ T_i = \sum_{[n^{(i-1)\gamma}]\leq k<[n^{i\gamma}]} C_n^k q^{k(n-k)},\ 1 \leq i \leq K\ ,\ T_0 = \sum_{[n^{K\gamma}]\leq k\leq[n/2]} C_n^k q^{k(n-k)}\ . \tag{8}$$

As the functions C_n^k, $k(n-k)$, $0 < k \leq [n/2]$, do not decrease then

$$T_i \le n^{i\gamma} q^{\left[n^{(i-1)\gamma}\right]\left(n-n^{(i-1)\gamma}\right)} C_n^{\left[n^{i\gamma}\right]}, \quad T_0 \le \left[\frac{n}{2}\right] q^{\left[n^{K\gamma}\right]\left(n-n^{K\gamma}\right)} C_n^{[n/2]}.$$

From the formula $q = 1 - n^{-a}$ and monotone increasing of the sequence $\left(1-n^{-1}\right)^n$, $n \ge 1$, to the limit $\exp(-1)$ it is easy to obtain the inequalities

$$T_i \le n^{i\gamma} \exp\left(-n^{-a}\left[n^{(i-1)\gamma}\right]\left(n - n^{(i-1)\gamma}\right)\right) C_n^{\left[n^{i\gamma}\right]}, \tag{9}$$

$$T_0 \le \frac{n}{2} \exp\left(-n^{-a}\left[n^{K\gamma}\right]\left(n - n^{K\gamma}\right)\right) C_n^{[n/2]}. \tag{10}$$

From the Sterling formula [18, chapter II, paragraph 9, formula (9.15)] we obtain for $0 < \delta < 1$:

$$C_n^{\left[n^\delta\right]} = \frac{n!}{\left[n^\delta\right]!\left(n - \left[n^\delta\right]\right)!} \le$$
$$\le \frac{\exp(1/12n)\, n^n \exp(-n)\sqrt{2\pi n}}{\left[n^\delta\right]^{\left[n^\delta\right]} \exp\left(-\left[n^\delta\right]\right)\sqrt{2\pi\left[n^\delta\right]}\left(n - \left[n^\delta\right]\right)^{n-\left[n^\delta\right]} \exp\left(-\left(n - \left[n^\delta\right]\right)\right)\sqrt{2\pi\left(n - \left[n^\delta\right]\right)}}, \tag{11}$$

Denote $R_\delta = \left(1 - 2^{-\delta}\right)^{2^\delta}$ then for $n > 1$

$$\left[n^\delta\right]^{\left[n^\delta\right]} \ge \left(n^\delta - 1\right)^{n^\delta - 1} \ge n^{\delta\left(n^\delta - 1\right)} R_\delta, \tag{12}$$

$$\left(n - \left[n^\delta\right]\right)^{n-\left[n^\delta\right]} \ge \left(n - n^\delta\right)^{n-n^\delta} = n^{n-n^\delta}\left(1 - n^{\delta-1}\right)^{n-n^\delta} \ge n^{n-n^\delta} R_{1-\delta}^{n^\delta}. \tag{13}$$

From the formulas (11) - (13) for $0 < \delta < 1$ we obtain

$$C_n^{\left[n^\delta\right]} \le \frac{\exp(1/12n)\, n^{n^\delta(1-\delta)+\delta}}{\sqrt{2\pi\left(n^\delta - 1\right)\left(1 - n^{\delta-1}\right)} R_\delta R_{1-\delta}^{n^\delta}}. \tag{14}$$

Analogously we have

$$C_n^{[n/2]} = \frac{n!}{([n/2])!\left(n - [n/2]\right)!} \le \frac{\exp(1/12n)\, n^n \sqrt{2\pi n}}{[n/2]^{[n/2]}\sqrt{2\pi[n/2]}\left(n - [n/2]\right)^{n-[n/2]}\sqrt{2\pi\left(n - [n/2]\right)}}. \tag{15}$$

As $(n-1)/2 \le [n/2] \le n/2$ so

$$[n/2]^{[n/2]} \ge \left((n-1)/2\right)^{(n-1)/2} \ge \frac{n^{n/2-1/2}}{2^{n/2-1/2}}(2/3)^{3/2}, \tag{16}$$

$$\left(n - [n/2]\right)^{n-[n/2]} \ge \left(n - n/2\right)^{n-n/2} \ge [n/2]^{[n/2]} \ge \frac{n^{n/2-1/2}}{2^{n/2-1/2}}(2/3)^{3/2}, \; n > 2. \tag{17}$$

From the formulas (15) - (17) we obtain

$$C_n^{[n/2]} \le \exp\left(\frac{1}{12n}\right) \cdot 27 \cdot 2^{n-7/2}\sqrt{\frac{n}{\pi(1 - 2/n)}}. \tag{18}$$

Consequently from the formulas (9), (14) and from the condition (6) and from the existence of the number $f < \infty$ so that

$$R_{i\gamma} > f^{-1} > 0, \quad R_{1-i\gamma} > f^{-1} > 0, \; 1 \le i \le K,$$

we have

$$T_i \le n^{i\gamma} \exp\left(-\left[n^{(i-1)\gamma}\right]\left(n-n^{(i-1)\gamma}\right)n^{-a}\right)\frac{\exp(1/12n)\, f^{1+n^{i\gamma}} n^{n^{i\gamma}+i\gamma}}{\sqrt{2\pi\left(n^{i\gamma}-1\right)\left(1-n^{i\gamma-1}\right)}} \to 0,\quad n\to\infty. \tag{19}$$

Analogously the formulas (10), (18) and the conditions (6), (7) lead to

$$T_0 \le 27\exp\left(-\left[n^{K\gamma}\right]n^{-a}\left(n-n^{K\gamma}\right)+\frac{1}{12n}\right)2^{n-9/2}\sqrt{\frac{n^3}{\pi(1-2/n)}} \to 0\ ,n\to\infty. \tag{20}$$

Unite the formulas (8), (19), (20) we obtain that $T\to 0$, $n\to\infty$. Consequently from the formula (5) we have (1).

Assume now that $a>1$. If all arcs connected with the node 1 do not work then the nodes 1; 2 are disconnected and so

$$\overline{P}_n \ge \overline{P}_n(1,2) \ge (1-p)^{n-1} = \left(1-n^{-a}\right)^{n-1} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{\frac{n-1}{n^a}}. \tag{21}$$

As $\left(1-n^{-a}\right)^{n^a}\to\exp(-1)$, $n\to\infty$, and $a>1$, then $\overline{P}_n\to 1$, $n\to\infty$. The formula (2) is proved.

Theorem 2. It is obvious that $Q_n(b)$ is not smaller than the probability that the nodes $1,2,...\left[n^b\right]$ are isolated in random realization of the graph G_n. That is

$$Q_n(b) \ge \left(1-n^{-a}\right)^{n\left[n^b\right]} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{n^{1-a}\left[n^b\right]}. \tag{22}$$

Suppose that $1+b<a<2$, then from the formula $\left(1-n^{-a}\right)^{n^a}\to\exp(-1)$, $n\to\infty$, the **c**ondition $a>1+b$ and the formula (22) we obtain the inequality (3).

Assume that $a>2$ then

$$Q_n(1) \ge \left(1-n^{-a}\right)^{n^2} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{n^{2-a}}. \tag{23}$$

Consequently from the condition $a>2$ the formula $\left(1-n^{-a}\right)^{n^a}\to\exp(-1)$, $n\to\infty$, and the formula (23) we obtain the equality (4).

Remarks 1, 2. Remark 1 proof almost word by word repeats Theorems 1, 2 proofs. To prove Remark 2 it is enough to replace the inequality (5) by

$$\overline{P}_n \le T \le \sum_{0<k\le[n/2]} C_n^k q^{k([Cn]-1-n/2)},$$

the inequality (21) by

$$\overline{P}_n \ge \overline{P}_n(1,2) \ge (1-p)^{[Cn]-1} = \left(1-n^{-a}\right)^{Cn-1} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{\frac{Cn-1}{n^a}},$$

the inequality (22) by

$$Q_n(b) \ge \left(1-n^{-a}\right)^{Cn^{b+1}} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{Cn^{b+1-a}},$$

the inequality (23) by

$$Q_n(1) \ge \left(1-n^{-a}\right)^{Cn^2} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{Cn^{2-a}}.$$

4. CONCLUSION

This paper is written using complicated numerical calculations. It is obvious that further for a realization of these calculations it is necessary to use supercomputers.

The author thanks A. Losev and G. Grenkin for large help in a realization of auxiliary numerical experiments.

REFERENCES

1. Prigojin I., Stengers I. 1986. Order from Chaos. *Moscow: Progress.* (In Russian).
2. Barlow R.E., Proschan F. 1965. Mathematical Theory of Reliability. *London and New York: Wiley*.
3. Ushakov I.A. et al. 1985. Reliability of Technical Systems: Handbook. *Moscow: Radio and Communication.* (In Russian).
4. Popkov V.K. 2006. Mathematical Models of Connectivity. *Novosibirsk: Institute for Computing Mathematics and Mathematical Geophysics, Siberian Branch of RAS.* (In Russian).
5. Riabinin I.A. 2007. Reliability and Safety of Structural Complicated Systems. *Sankt-Petersberg: Edition of Sankt-Petersberg university.* (In Russian).
6. Lomonosov M.V., Polesskiy V.P.1971. Upper Bound of Reliability of Information Networks. *Problems of information transmission* . Vol. 7. Numb. 4. Pp. 78-81. (In Russian).
7. Polesskiy V.P. 1971. Estimates of connectivity probability of random graph. *Problems of information transmission.* Vol. 7. Numb. 2. Pp. 88-96. (In Russian).
8. Lomonosov M.V., Polesskiy V.P. 1972. Low Estimate of Network Reliability. *Problems of information transmission.* Vol. 26. Numb. 1. Pp. 47-53. (In Russian).
9. Polesskiy V.P. 1990. Estimates of Connectivity Probability of Random Graph. *Problems of information transmission.* Vol. 26. Numb. 1. Pp. 90-98. (In Russian).
10. Polesskiy V.P. 1993. Low Estimates of Connectivity Probability for Some Classes of Random Graphs. *Problems of information transmission.* Vol. 29. Numb. 2. Pp. 85-95. (In Russian).
11. Polesskiy V.P. 1992. Low Estimates of Connectivity Probability in Random Graphs Generated by Doubly-Connected Graphs with Fixed Base Spectrum. *Problems of information transmission.* Vol. 28. Numb. 2. Pp. 86-95. (In Russian).
12. Tanguy C. 2007. What Is the Probability of Connecting Two Points? *J. Phys. A: Math. Theor.* V. 40. Pp. 14099-14116.
13. Tanguy C. 2009. Asymptotic Dependence of Average Failure Rate and MTTF for a Recursive Meshed Network Architecture. *Reliability and risk analysis: theory and applications.* V. 40. Pp. 14099- 14116.
14. Solojentsev E.D. 2003. Specific of Logic-Probability Risk Theory with Groups of Incompatible Events. *Automatics and remote control.* Numb. 7. Pp. 187-203. (In Russian)
15. Rodionov A.S. 2011. About Accelaration in Calculation of Reliability Polynom of Random Graph. *Automatics and remote control.* Vol. 7. Pp. 134-146. (In Russian).
16. Gertsbakh I., Shpungin Y. 2010. Models of Network Reliability. Analysis Combinatorics and Monte-Carlo. *CRC Press. Taylor and Francis Group.*
17. Shiriaev A.N. 1989. Probability. *Moscow: Science.* (In Russian).
18. Feller W. Introduction to Probability Theory and Its Applications. Vol. 1. 1984. *Moscow: World.* (In Russian).