AVAILABILITY AND RELIABILITY MEASURES FOR MULTISTATE SYSTEM BY USING MARKOV REWARD MODEL

M. A. El-Damcese and N. S. Temraz

•

Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt

ABSTRACT

This paper describes some models and measures of reliability for multistate systems. The expected cumulative reward for the continuous time Markov reward models are used for deriving the structure function for a multistate system where the system and its components can have different performance levels ranging from perfect functioning to complete failure. The suggested approach presents with respect to the non-homogeneous and homogeneous Markov reward model of two stochastic process for computation of these availability and reliability measures. A particular case for three levels is analyzed numerically by assuming Weibull and exponential distributions for failure and repair times.

Keywords: Markov reward model, demand, multistate system, availability and reliability measures.

1. INTRODUCTION

Traditional binary-state reliability models allow for a system and its components only two possible states: perfect functioning (up) and complete failure (down). However, a system can have a finite number of performance rates. And, many real-world systems are composed of components that in their turn can have different performance levels and for which one cannot formulate an "all or nothing" type of failure criterion. Failures of some system elements lead, in these cases, only to performance degradation. Such systems are called multi-state systems (MSS) [11]. Traditional reliability theory, which is based on a binary approach, has recently been extended by allowing components and systems to have an arbitrary finite number of states.

According to the generic multi-state system model [8], any system element $j \in \{1, 2, \ldots, n\}$ can have k different states corresponding to the performance rates, represented by the set $g_j = \{g_{j1}, g_{j2}, \ldots, g_{jk}\}$, where g_{ji} is the performance rate of element j in the state $i, i \in \{1, 2, \ldots, k\}$. The performance rate $G_j(t)$ of element j at any instant $t \geq 0$ is a discrete-state continuous-time stochastic process that takes its values from $g_j : G_j(t) \in g_j$. The system structure function $G(t) = \phi(G_1(t), \ldots, G_n(t))$ produces the stochastic process corresponding to the output performance of the entire MSS. In practice, a desired level of system performance (demand) also can be represented by a discrete-state continuous-time stochastic process $W(t)$. The relation between the MSS output performance and the demand represented by two corresponding stochastic processes should be studied in order to define reliability measures for the entire MSS. For reliability assessment, MSS output performance and the desired performance level (demand) are often assumed to be independent stochastic processes. In practice, the most commonly used MSS reliability measures are probability of failure-free operation during time interval $[0, t]$ or MSS reliability function $R(t)$, MSS availability, mean time to MSS failure, mean accumulated performance deficiency for a fixed time interval $[0, t]$, and so on.

Many technical systems are subjected during their lifetime to aging and degradation. After any failure, maintenance is performed by a repair team. Maintenance and repair problems have been widely investigated in the literature. [1], [4], [16] survey and summarize theoretical developments and practical applications of maintenance models. Aging is usually considered as a process which results in an age-related increase of the failure rate. The most common shapes of failure rates have been observed in [12], [18]. An interesting approach was introduced in [7], where it was shown that aging is not always manifested by the increasing failure rate.

After each corrective maintenance action or repair, the aging system's failure rate $\lambda(t)$ can be expressed as:

$$\lambda(t) = q\,\lambda(0) + (1-q)\,\lambda^{*}(t),$$

where q is an improvement factor that characterizes the quality of the overhauls $(0 \le q \le 1)$ and $\lambda^{*}(t)$ is the aging system's failure rate before repair [20]. If $q = 1,$ it means that the maintenance action is perfect (system becomes "as good as new" after repair). If $q = 0,$ it means that the failed system is returned back to a working state by minimal repair (system stays "as bad as old" after repair), in which failure rate of the system is nearly the same as before. The minimal repair is usually appropriate for multi-state systems. In such situation, the failure pattern can be described by non-homogeneous Poisson process (NHPP). Incorporating the time-varying failure intensity into existing Markov model was suggested in [17] for reliability modeling of hardware/software systems. More details and interesting examples one can find in [19]. Based on this, the extended approach is suggested, which incorporates the time-varying failure intensity of aging component into Markov reward model that is using for general reliability measures evaluation of non-aging MSS [7]. Such unified model will be called as a non-homogeneous Markov reward model.

This paper considers measures of availability and reliability for a multi-state system where the system and its components can have different performance levels ranging from perfect functioning to complete failure. In section 2 a general approach is presented for the computation of main MSS reliability measures. This approach is based on the application of the Markov reward model. The main MSS reliability measures can be found by corresponding reward matrix definitions for this model and then by using a standard procedure for finding expected accumulated rewards during a time interval $[0, t]$ as a solution of a system of differential equations. In section 3 a general approach is presented for computing reliability measures for aging MSS under corrective maintenance with minimal repair. This approach is based on non-homogeneous Markov reward model, where specific reward matrix is determined for finding any reliability measure. This chapter is based on [9], [11], and presents a model representing demand as a continuous-time Markov chain with three logic levels. In section 4 we introduce illustrative example in order to illustrate the approaches.

2. MARKOV REWARD MODEL FOR MULTI-STATE SYSTEM

2.1. Generalized MSS Reliability Measures

The MSS behavior is characterized by its evolution in the space of states. The entire set of possible system states can be divided into two disjoint subsets corresponding to acceptable and unacceptable system functioning. MSS entrance into the subset of unacceptable states constitutes a failure. The MSS reliability can be defined as its ability to remain in the acceptable states during the operation period. The system state acceptability depends on the relation between the MSS output performance and the desired level of this performance (demand $W(t)$) that is determined outside the system. Often the demand $W(t)$ is also a random process that can take discrete values from the set

$w = \{w_1, \ldots, w_M\}$. The desired relation between the system performance and the demand at any time instant t can be expressed by the acceptability function $\Phi(G(t), W(t))$. In many practical cases, the MSS performance should be equal to or exceed the demand. So, in such cases, the acceptability function takes the following form:

$$\Phi(G(t), W(t)) = G(t) - W(t) \tag{1}$$

and the criterion of state acceptability can be expressed as: $\Phi(G(t), W(t)) \geq 0$.
A general expression defining MSS reliability measures can be written in the following form:

$$R = E\{F\left[\Phi(G(t), W(t))\right]\}, \tag{2}$$

where $E =$ expectation symbol, $F =$ functional that determines corresponding type of reliability measure, and $\Phi =$ acceptability function. Many important MSS reliability measures can be derived from the expression (2) depending on the functional F that may be determined in different ways. For example, it may be a probability $\Pr\{\Phi(G(t), W(t)) \geq 0\}$ throughout a specified time interval $[0, t]$ and the acceptability function (1) will be nonnegative. In this case, this probability characterizes MSS availability. It may be also an expectation of an appropriate function up to the time of the MSS's initial entrance into the set of unacceptable states, where $\Phi(G(t), W(t)) < 0$ is the number of such entrances within time interval $[0, t]$ and so on. For a power system where the available generating capacity at time instant t is $G(t)$ and the corresponding load demand is $W(t)$, if the acceptability function is defined as:

$$\Phi(G(t), W(t)) = \begin{cases} W(t) - G(t), & \text{if } W(t) > G(t) \\ 0, & \text{if } W(t) \leq G(t) \end{cases}$$

A function,

$$F\left[\Phi(G(t), W(t))\right] = \int_0^T \Phi(G(t), W(t))\, dt,$$

will characterize an accumulated performance deficiency during time interval [0, T].

2.2. Markov Reward Model: General Description

The general Markov reward model was introduced in [6]. It considers the continuous-time Markov chain $\{X(t) \mid t \geq 0\}$ with a set of states $\{1, \ldots, k\}$ and a transition intensity matrix $A = [a_{ij}],\ i, j = 1, \ldots, k$. It is assumed that while the process is in any state i during any time unit, some money r_{ii} should be paid. It is also assumed that if there is a transition from state i to state j the amount r_{ij} will be paid. The amounts r_{ii} and r_{ij} are called rewards. Rewards can be negative while representing a loss or penalty. Such a reward process associated with its states or/and transitions is called a Markov process with rewards. For such processes, in addition to the transition intensity matrix, a reward matrix $r = [r_{ij}],\ i, j = 1, \ldots, k$ should be determined. The main problem is to find the total expected reward accumulated up to time instant t under specified initial conditions.

Let $V_i(t)$ denotes the total expected reward accumulated up to time t at state i. The following system of differential equations must be solved under the initial conditions: $V_i(0) = 0,\ i = 1, \ldots, k$ in order to find the total expected reward.

$$\frac{dV_i(t)}{dt} = r_{ii} + \sum_{\substack{j=1,\\ j\neq i}}^{k} a_{ij} r_{ij} + \sum_{j=1}^{k} a_{ij} V_j(t), \quad i = 1, \ldots, k \tag{3}$$

Markov reward models are widely used in financial calculations and operations research [5]. General Markov reward models for system dependability and performability analysis one can find in [2], [14], and [10]. Here the new approach is presented where the main MSS reliability measures can be found by determination of the corresponding reward matrix. Such an idea was primarily introduced for a binary-state system and constant demand in [15]. In this chapter, the approach is extended for multi-state systems and variable demand.

2.3. Rewards Determination for MSS Reliability Computation

MSS instantaneous (point) availability $A(t)$ is the probability that the MSS at instant $t > 0$ is in one of the acceptable states: $A(t) = \Pr\{\Phi(G(t), W(t)) \geq 0\}$.

The MSS average availability $\overline{A}(t)$ is defined in [13] as a mean fraction of time when the system resides in the set of acceptable states during the time interval [0, t], $\overline{A}(t) = \frac{1}{t}\int_0^t A(t)\, dt$.

In order to assess $\overline{A}(t)$ for MSS the rewards in matrix r for the MSS model should be determined in the following manner:

- The rewards associated with all acceptable states should be defined as one.
- The rewards associated with all unacceptable states should be zeroed as well as all rewards associated with all transitions.

The mean reward $V_i(t)$ accumulated during interval $[0, t]$ will define a time that MSS will be in the set of acceptable states in the case when the state i is the initial state. This reward should be found as a solution of the system (3). After solving (3) and finding $V_i(t)$, MSS average availability can be obtained for every initial state $i = 1, \ldots, k$, $\overline{A}_i(t) = (V_i(t))/t$.

Usually, the initial state is assumed as the best state.

Mean number $N_f(t)$ of MSS failures during time interval [0, t] measure can be treated as the mean number of MSS entrances to the set of unacceptable states during time interval [0, t]. For its computation the rewards associated with each transition from the set of acceptable states to the set of unacceptable states should be defined as one. All other rewards should be zeroed. In this case mean accumulated reward $V_i(t)$ will define the mean number of entrances in the unacceptable area during time interval [0, t]: $N_f(t) = V_i(t)$.

Mean time to failure (MTTF) is the mean time up to the instant when the MSS enters the subset of unacceptable states for the first time. For its computation the combined performance-demand model should be transformed; all transitions that return MSS from unacceptable states should be forbidden, because for this case all unacceptable states should be treated as absorbing states. In order to assess MTTF for MSS the rewards in matrix r for the transformed performance-demand model should be determined in the following manner:

- The rewards associated with all acceptable states should be defined as one.
- The rewards associated with unacceptable (absorbing) states should be zeroed as well as rewards associated with transitions.

In this case mean accumulated reward $V_i(t)$ will define the mean time accumulated up to the first entrance into the subset of unacceptable states or MTTF.

Probability of MSS failure during time interval [0, t]: The model should be transformed as in the previous case; all unacceptable states should be treated as absorbing states, and therefore all transitions that return MSS from unacceptable states should be forbidden. Rewards associated with all transitions to the absorbing states should be defined as one. All other rewards should be zeroed. Mean accumulated reward $V_i(t)$ will define for this case the probability of MSS failure during time interval [0, t] if the state i is the initial state. Therefore, the MSS reliability function can be obtained as: $R_i(t) = 1 - V_i(t)$, where $i = 1, \ldots, k$.

3. NON-HOMOGENEOUS MARKOV REWARD MODEL FOR AGING MULTI-STATE SYSTEM UNDER MINIMAL REPAIR

3.1. Model Description

The MSS output performance $G(t)$ at any instant $t \geq 0$ is a continuous-time Markov chain that takes its values from the set $g = \{g_1, \ldots, g_k\}, G(t) \in g$, where g_i is the MSS output performance in state $i, i = 1, \ldots, k$. For Markov MSS transition rates (intensities) a_{ij} between states i and j are defined by the corresponding system failure λ_{ij} and repair μ_{ij} rates. The minimal repair is a corrective maintenance action that brings the aging equipment to the conditions it was in just before the failure occurrence. Aging MSS subject to minimal repairs experiences reliability deterioration with the operating time, i.e., there is a tendency toward more frequent failures. In such situations, the failure pattern can be described by a Poisson process whose intensity function monotonically increases with t. A Poisson process with a non-constant intensity is called non-homogeneous, since it does not have stationary increments [4]. It was shown (see, for example, [20]) that NHPP model can be integrated into the Markov model with time-varying transition intensities $a_{ij}(t) = \lambda_{ij}(t)$. Therefore, for aging MSS transition intensities corresponding to failures of aging components will be functions of time $a_{ij}(t)$.

3.2. Non-Homogeneous Markov Reward Model

For non-homogeneous Markov model a system's state at time t can be described by a continuous-time Markov chain with a set of states $\{1, \ldots, k\}$ and a transition intensity matrix $A(t) = [a_{ij}(t)], \quad i, j = 1, \ldots, k$, where each transition intensity may be a function of time t. For such model, in addition to the transition intensity matrix, a reward matrix $r = [r_{ij}]$, $i, j = 1, \ldots, k$ should be determined [2].

Let $V_i(t)$ be the expected total reward accumulated up to time t given the initial state of the process at time instant $t = 0$ is state i. Howard differential equations [14] with time-varying transition intensities $a_{ij}(t)$ should be solved under specified initial conditions in order to find the total expected rewards:

$$\frac{dV_i(t)}{dt} = r_{ii} + \sum_{\substack{j=1,\\ j\neq i}}^{k} a_{ij}(t) r_{ij} + \sum_{j=1}^{k} a_{ij}(t) V_j(t), \quad i = 1, \ldots, k \tag{4}$$

In the most common case, MSS begins to accumulate rewards after time instant $t = 0$, therefore, the initial conditions are:

$$V_i(0) = 0, \quad i = 1, \ldots, k \tag{5}$$

If for example the state k with the highest performance level is defined as the initial state, the value $V_k(t)$ should be found as a solution of the system (4).

It was shown in [7] and [11] that many important reliability measures for non-aging MSS can be found by determination of rewards in a corresponding reward matrix. Here this approach is extended for aging MSS under minimal repair. And, notice that the approach is applied only for minimal repair.

3.3. Rewards Determination for Computation of Different Reliability Measures for Aging MSS

The reliability measures can be determined by the same manner as it was indicated in section 2.3.

4. ILLUSTRATIVE EXAMPLE

Consider the air-conditioning system used in a hospital. The system consists of three identical air conditioners which are connected in parallel. Demand is a continuous-time Markov chain with three levels: peak, middle, and low. The state-space diagram for this system is presented in figure (1).

There are 12 states. States from 1 to 4 associated with the low demand period, states from 5 to 8 associated with the middle demand period, and states from 9 to 12 associated with the peak demand period.

States 12, 8, and 4 indicate all components work, the system performance is $g_{12} = g_8 = g_4 = 3$. States 11, 7, and 3 indicate two components work and the third component failed, the system performance is $g_{11} = g_7 = g_3 = 2$. States 10, 6, and 2 indicate that one component only works, the system performance is $g_{10} = g_6 = g_2 = 1$. States 9, 5, and 1 indicate full system failure, the system performance is $g_9 = g_5 = g_1 = 0$. If in the peak-demand period the required demand level is w = 3, in the middle-demand period the required demand level is w = 2, and in the low-demand period the required demand level is w = 1, then there are six acceptable states: 12, 8, 4, 7, 3, and 2. States: 11, 10, 6, 9, 5, and 1 are unacceptable.

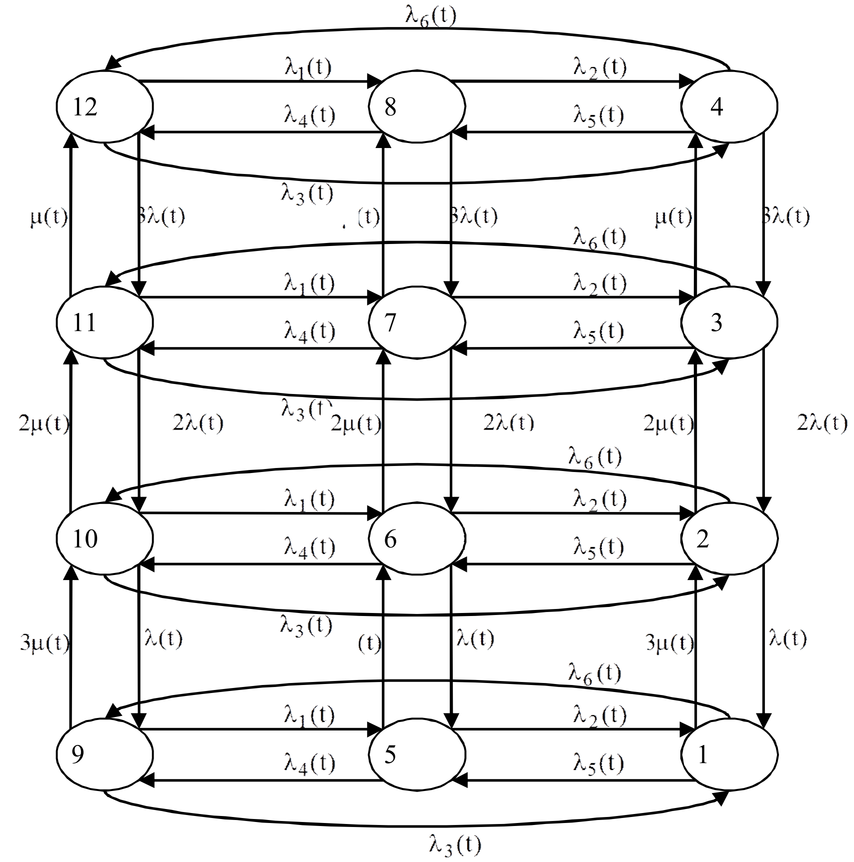

Figure (1): The state-space diagram for a system with three identical air conditioners

The transitions from state 12 to state 11, from state 8 to state7, and from state 4 to state 3 are associated with the failure of one of the three conditioners and have an intensity of $3\lambda(t)$. The transitions from state 11 to state 10, from state 7 to state 6, and from state 3 to state 2 are associated with the failure of the second conditioner and have intensity of $2\lambda(t)$. The transitions from state 10 to state 9, from state 6 to state 5, and from state 2 to state 1 are associated with the failure of the third conditioner and have intensity of $\lambda(t)$.

The transitions from state 1 to state 2, from state 5 to state 6, and from state 9 to state 10 are associated with repair of one of the three failed conditioners and have intensity of $3\mu(t)$. The transitions from state 2 to state 3, from state 6 to state 7, and from state 10 to state 11 are associated with repair of one of the two failed conditioners and have intensity of $2\mu(t)$. The transitions from state 3 to state 4, from state 7 to state 8, and from state 11 to state 12 are associated with repair of the failed conditioner and have intensity of $\mu(t)$.

The transitions from state 12 to state 8, from state 11 to state 7, from state 10 to state 6, and from state 9 to state 5 are associated with a variable demand and have intensity of $\lambda_1(t)$. The transitions from state 8 to state 4, from state 7 to state 3, from state 6 to state 2, and from state 5 to state 1 are associated with a variable demand and have intensity of $\lambda_2(t)$. The transitions from state 12 to state 4, from state 11 to state 3, from state 10 to state 2, and from state 9 to state 1 are associated with a variable demand and have intensity of $\lambda_3(t)$. The transitions from state 8 to state

12, from state 7 to state 11, from state 6 to state 10, and from state 5 to state 9 are associated with a variable demand and have intensity of $\lambda_4(t)$. The transitions from state 4 to state 8, from state 3 to state 7, from state 2 to state 6, and from state 1 to state 5 are associated with a variable demand and have intensity of $\lambda_5(t)$. The transitions from state 4 to state 12, from state 3 to state 11, from state 2 to state 10, and from state 1 to state 9 are associated with a variable demand and have intensity of $\lambda_6(t)$.

In order to find the MSS average availability $\overline{A}(t)$ we should present the reward matrix r_A in the following form:

$$r_A = [r_{ij}] = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \quad (6)$$

In this matrix, rewards associated with all acceptable states are defined as one and rewards associated with all unacceptable states are zeroed as well as all rewards associated with all transitions.

The system of differential equations (7) can be written in order to find the expected total rewards $V_i(t), i = 1, \ldots, 12.$ The initial conditions are: $V_i(0) = 0, i = 1, \ldots, 12.$
After solving this system and finding $V_i(t),$ MSS average availability can be obtained as follows: $\overline{A}(t) = V_{12}(t)/t,$ where the 12-th state is the initial state.

$$
\begin{cases}
\frac{dV_1(t)}{dt} = -C_1(t)V_1(t) + 3\,\mu(t)V_2(t) + \lambda_5(t)V_5(t) + \lambda_6(t)V_9(t) \\
\frac{dV_2(t)}{dt} = 1 - C_2(t)V_2(t) + \lambda(t)V_1(t) + 2\,\mu(t)V_3(t) + \lambda_5(t)V_6(t) \\
\qquad + \lambda_6(t)V_{10}(t) \\
\frac{dV_3(t)}{dt} = 1 - C_3(t)V_3(t) + 2\lambda(t)V_2(t) + \mu(t)V_4(t) + \lambda_5(t)V_7(t) \\
\qquad + \lambda_6(t)V_{11}(t) \\
\frac{dV_4(t)}{dt} = 1 - C_4(t)V_4(t) + 3\lambda(t)V_3(t) + \lambda_5(t)V_8(t) + \lambda_6(t)V_{12}(t) \\
\frac{dV_5(t)}{dt} = -C_5(t)V_5(t) + 3\,\mu(t)V_6(t) + \lambda_2(t)V_1(t) + \lambda_4(t)V_9(t) \\
\frac{dV_6(t)}{dt} = -C_6(t)V_6(t) + \lambda(t)V_5(t) + 2\,\mu(t)V_7(t) + \lambda_2(t)V_2(t) \\
\qquad + \lambda_4(t)V_{10}(t) \\
\frac{dV_7(t)}{dt} = 1 - C_7(t)V_7(t) + 2\lambda(t)V_6(t) + \mu(t)V_8(t) + \lambda_2(t)V_3(t) \\
\qquad + \lambda_4(t)V_{11}(t) \\
\frac{dV_8(t)}{dt} = 1 - C_8(t)V_8(t) + 3\lambda(t)V_7(t) + \lambda_2(t)V_4(t) + \lambda_4(t)V_{12}(t) \\
\frac{dV_9(t)}{dt} = -C_9(t)V_9(t) + 3\,\mu(t)V_{10}(t) + \lambda_1(t)V_5(t) + \lambda_3(t)V_1(t) \\
\frac{dV_{10}(t)}{dt} = -C_{10}(t)V_{10}(t) + \lambda(t)V_9(t) + 2\,\mu(t)V_{11}(t) + \lambda_1(t)V_6(t) \\
\qquad + \lambda_3(t)V_2(t) \\
\frac{dV_{11}(t)}{dt} = -C_{11}(t)V_{11}(t) + 2\lambda(t)V_{10}(t) + \mu(t)V_{12}(t) + \lambda_1(t)V_7(t) \\
\qquad + \lambda_3(t)V_3(t) \\
\frac{dV_{12}(t)}{dt} = 1 - C_{12}(t)V_{12}(t) + 3\lambda(t)V_{11}(t) + \lambda_1(t)V_8(t) + \lambda_3(t)V_4(t)
\end{cases}
\tag{7}
$$

where,

$$
\begin{cases}
C_1(t) = 3\,\mu(t) + \lambda_5(t) + \lambda_6(t) \\
C_2(t) = \lambda(t) + 2\,\mu(t) + \lambda_5(t) + \lambda_6(t) \\
C_3(t) = 2\lambda(t) + \mu(t) + \lambda_5(t) + \lambda_6(t) \\
C_4(t) = 3\lambda(t) + \lambda_5(t) + \lambda_6(t) \\
C_5(t) = 3\,\mu(t) + \lambda_2(t) + \lambda_4(t) \\
C_6(t) = 2\,\mu(t) + \lambda(t) + \lambda_2(t) + \lambda_4(t) \\
C_7(t) = \mu(t) + 2\lambda(t) + \lambda_2(t) + \lambda_4(t) \\
C_8(t) = 3\lambda(t) + \lambda_2(t) + \lambda_4(t) \\
C_9(t) = 3\,\mu(t) + \lambda_1(t) + \lambda_3(t) \\
C_{10}(t) = \lambda(t) + 2\,\mu(t) + \lambda_1(t) + \lambda_3(t) \\
C_{11}(t) = 2\lambda(t) + \mu(t) + \lambda_1(t) + \lambda_3(t) \\
C_{12}(t) = 3\lambda(t) + \lambda_1(t) + \lambda_3(t)
\end{cases}
\tag{8}
$$

In order to find the mean total number of system failures $N_f(t)$ we should present the reward matrix r_N in the form (9). In this matrix the rewards associated with each transition from the set of acceptable states to the set of unacceptable states should be defined as one. All other rewards should be zeroed.

$$r_N = [r_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{9}$$

The following system of differential equations (10) can be written in order to find the expected total rewards $V_i(t), i = 1, \dots, 12$.

$$\begin{cases} \dfrac{dV_1(t)}{dt} = -C_1(t)V_1(t) + 3\mu(t)V_2(t) + \lambda_5(t)V_5(t) + \lambda_6(t)V_9(t) \\ \dfrac{dV_2(t)}{dt} = \lambda(t) + \lambda_5(t) + \lambda_6(t) - C_2(t)V_2(t) + \lambda(t)V_1(t) \\ \qquad + 2\mu(t)V_3(t) + \lambda_5(t)V_6(t) + \lambda_6(t)V_{10}(t) \\ \dfrac{dV_3(t)}{dt} = \lambda_6(t) - C_3(t)V_3(t) + 2\lambda(t)V_2(t) + \mu(t)V_4(t) \\ \qquad + \lambda_5(t)V_7(t) + \lambda_6(t)V_{11}(t) \\ \dfrac{dV_4(t)}{dt} = -C_4(t)V_4(t) + 3\lambda(t)V_3(t) + \lambda_5(t)V_8(t) + \lambda_6(t)V_{12}(t) \\ \dfrac{dV_5(t)}{dt} = -C_5(t)V_5(t) + 3\mu(t)V_6(t) + \lambda_2(t)V_1(t) + \lambda_4(t)V_9(t) \\ \dfrac{dV_6(t)}{dt} = -C_6(t)V_6(t) + \lambda(t)V_5(t) + 2\mu(t)V_7(t) + \lambda_2(t)V_2(t) \\ \qquad + \lambda_4(t)V_{10}(t) \\ \dfrac{dV_7(t)}{dt} = 2\lambda(t) + \lambda_4(t) - C_7(t)V_7(t) + 2\lambda(t)V_6(t) + \mu(t)V_8(t) \\ \qquad + \lambda_2(t)V_3(t) + \lambda_4(t)V_{11}(t) \\ \dfrac{dV_8(t)}{dt} = -C_8(t)V_8(t) + 3\lambda(t)V_7(t) + \lambda_2(t)V_4(t) + \lambda_4(t)V_{12}(t) \\ \dfrac{dV_9(t)}{dt} = -C_9(t)V_9(t) + 3\mu(t)V_{10}(t) + \lambda_1(t)V_5(t) + \lambda_3(t)V_1(t) \\ \dfrac{dV_{10}(t)}{dt} = -C_{10}(t)V_{10}(t) + \lambda(t)V_9(t) + 2\mu(t)V_{11}(t) + \lambda_1(t)V_6(t) \\ \qquad + \lambda_3(t)V_2(t) \\ \dfrac{dV_{11}(t)}{dt} = -C_{11}(t)V_{11}(t) + 2\lambda(t)V_{10}(t) + \mu(t)V_{12}(t) + \lambda_1(t)V_7(t) \\ \qquad + \lambda_3(t)V_3(t) \\ \dfrac{dV_{12}(t)}{dt} = 3\lambda(t) - C_{12}(t)V_{12}(t) + 3\lambda(t)V_{11}(t) + \lambda_1(t)V_8(t) \\ \qquad + \lambda_3(t)V_4(t) \end{cases} \tag{10}$$

Here $C_1, \ldots, C_{12}$ are calculated via formulas (8).

The initial conditions are: $V_i(0) = 0, i = 1, \ldots, 12$. After solving this system and finding $V_i(t)$, the mean total number of system failures $N_f(t)$ can be obtained as follows: $N_f(t) = V_{12}(t)$, where the 12-th state is the initial state.

In order to calculate the mean time to failure (MTTF), the initial model should be transformed; all transitions that return MSS from unacceptable states should be forbidden and all unacceptable states should be treated as absorbing states. The transformed model is shown in figure (2).

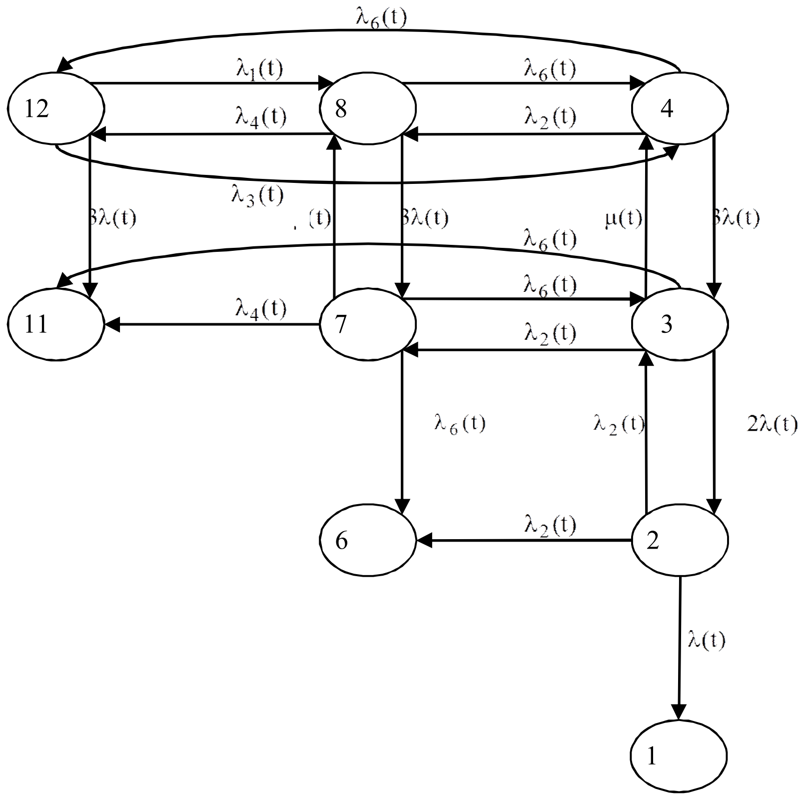

Figure (2): The state-space diagram for the transformed system with three identical air conditioners with absorbing states

In order to assess MTTF for MSS, the rewards in matrix r for the transformed model should be determined in the following manner. The rewards associated with all acceptable states should be defined as one and the rewards associated with unacceptable (absorbing) states should be zeroed as well as all rewards associated with transitions.

The reward matrix r for this system is as follows:

$$r=[r_{ij}]=\begin{bmatrix}0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1\end{bmatrix} \tag{11}$$

The following system of differential equations can be written in order to find the expected total rewards $V_i(t)$, $i=1, 2, 3, 4, 6, 7, 8, 11, 12$.

$$\begin{cases}\frac{dV_1(t)}{dt}=0\\ \frac{dV_2(t)}{dt}=1-C_2(t)V_2(t)+\lambda(t)V_1(t)+2\,\mu(t)V_3(t)+\lambda_5(t)V_6(t)\\ \frac{dV_3(t)}{dt}=1-C_3(t)V_3(t)+2\lambda(t)V_2(t)+\mu(t)V_4(t)+\lambda_5(t)V_7(t)\\ \qquad +\lambda_6(t)V_{11}(t)\\ \frac{dV_4(t)}{dt}=1-C_4(t)V_4(t)+3\lambda(t)V_3(t)+\lambda_5(t)V_8(t)+\lambda_6(t)V_{12}(t)\\ \frac{dV_6(t)}{dt}=0\\ \frac{dV_7(t)}{dt}=1-C_7(t)V_7(t)+\lambda_2(t)V_3(t)+\mu(t)V_8(t)+2\lambda(t)V_6(t)\\ \qquad +\lambda_4(t)V_{11}(t)\\ \frac{dV_8(t)}{dt}=1-C_8(t)V_8(t)+3\lambda(t)V_7(t)+\lambda_2(t)V_4(t)+\lambda_4(t)V_{12}(t)\\ \frac{dV_{11}(t)}{dt}=0\\ \frac{dV_{12}(t)}{dt}=1-C_{12}(t)V_{12}(t)+\lambda_1(t)V_8(t)+\lambda_3(t)V_4(t)+3\lambda(t)V_{11}(t)\end{cases} \tag{12}$$

where,

$$\begin{cases}C_2(t)=\lambda(t)+2\mu(t)+\lambda_5(t)\\ C_3(t)=2\lambda(t)+\mu(t)+\lambda_5(t)+\lambda_6(t)\\ C_4(t)=3\lambda(t)+\lambda_5(t)+\lambda_6(t)\\ C_7(t)=\lambda_2(t)+\mu(t)+2\lambda(t)+\lambda_4(t)\\ C_8(t)=\lambda_2(t)+\lambda_4(t)+3\lambda(t)\\ C_{12}(t)=\lambda_1(t)+\lambda_3(t)+3\lambda(t)\end{cases} \tag{13}$$

The initial conditions are: $V_i(0) = 0, i = 1, 2, 3, 4, 6, 7, 8, 11, 12.$

After solving this system and finding $V_i(t)$, the MTTF for MSS can be obtained as $V_{12}(t)$, where the 12-th state is the initial state.

To calculate the probability of MSS failure during time interval [0, t] the model should be transformed as in the previous case: all unacceptable states should be treated as absorbing states and, therefore, all transitions that return MSS from unacceptable states should be forbidden. Rewards associated with all transitions to the absorbing state should be defined as one. All other rewards should be zeroed.

The reward matrix r for this system is as follows:

$$r = [r_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (14)$$

Mean accumulated reward $V_i(t)$ will define the probability $Q(t)$ of MSS failure during time interval [0, t].

The following system of differential equations can be written in order to find the expected total rewards $V_i(t), i = 1, 2, 3, 4, 6, 7, 8, 11, 12.$

$$\begin{cases} \frac{dV_1(t)}{dt} = 0 \\ \frac{dV_2(t)}{dt} = \lambda(t) + \lambda_5(t) - C_2(t)V_2(t) + \lambda(t)V_1(t) + 2\mu(t)V_3(t) \\ \qquad + \lambda_5(t)V_6(t) \\ \frac{dV_3(t)}{dt} = \lambda_6(t) - C_3(t)V_3(t) + 2\lambda(t)V_2(t) + \mu(t)V_4(t) \\ \qquad + \lambda_5(t)V_7(t) + \lambda_6(t)V_{11}(t) \\ \frac{dV_4(t)}{dt} = -C_4(t)V_4(t) + 3\lambda(t)V_3(t) + \lambda_5(t)V_8(t) + \lambda_6(t)V_{12}(t) \\ \frac{dV_6(t)}{dt} = 0 \\ \frac{dV_7(t)}{dt} = 2\lambda(t) + \lambda_4(t) - C_7(t)V_7(t) + \lambda_2(t)V_3(t) + \mu(t)V_8(t) \\ \qquad + 2\lambda(t)V_6(t) + \lambda_4(t)V_{11}(t) \\ \frac{dV_8(t)}{dt} = -C_8(t)V_8(t) + 3\lambda(t)V_7(t) + \lambda_2(t)V_4(t) + \lambda_4(t)V_{12}(t) \\ \frac{dV_{11}(t)}{dt} = 0 \\ \frac{dV_{12}(t)}{dt} = 3\lambda(t) - C_{12}(t)V_{12}(t) + \lambda_1(t)V_8(t) + \lambda_3(t)V_4(t) \\ \qquad + 3\lambda(t)V_{11}(t) \end{cases} \qquad (15)$$

Here $C_i, i = 1, 2, 3, 4, 6, 7, 8, 11, 12$ are calculated via formulas (13). The initial conditions are:

$$V_i(0) = 0, i = 1, 2, 3, 4, 6, 7, 8, 11, 12.$$

After solving this system and finding $V_i(t)$, MSS reliability function can be obtained as $R(t) = 1 - V_{12}(t)$, where the 12-th state is the initial state.

Now, we consider two types of the parameters as follows:

(i) The air conditioners failure and repair rates are time-varying

As a particular case, we assume that the working time and the repair time of each conditioner are both Weibully distributed. We can then write:

$$\lambda_1(t) = 1.1\, t^{0.1} \qquad \lambda_5(t) = 1.7\, t^{0.7}$$

$$\lambda_2(t) = 1.2\, t^{0.2} \qquad \lambda_6(t) = 1.8\, t^{0.8}$$

$$\lambda_3(t) = 1.4\, t^{0.4} \qquad \lambda(t) = 1.5\, t^{0.5}$$

$$\lambda_4(t) = 1.6\, t^{0.6} \qquad \mu(t) = 1.9\, t^{0.9}$$

Using MAPLE program, the MSS average availability $\overline{A}(t)$ against time is illustrated in figure (3) with numerical solutions based on Runge-Kutta method.

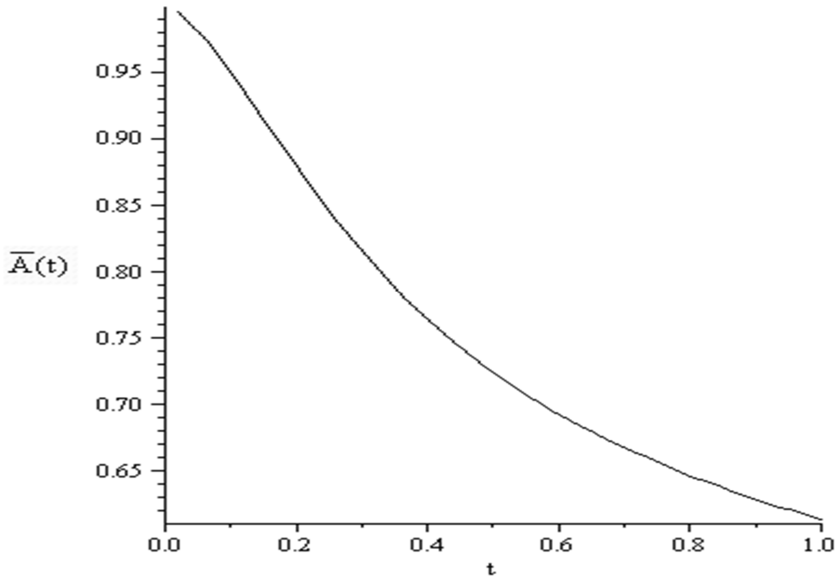

Figure (3): The average availability $\overline{A}(t)$ versus the time t (case i)

Similarly, the mean total number of system failures $N_f(t)$, the MTTF for MSS, and the MSS reliability function $R(t)$ against time are illustrated in figures (4), (5), and (6), respectively.

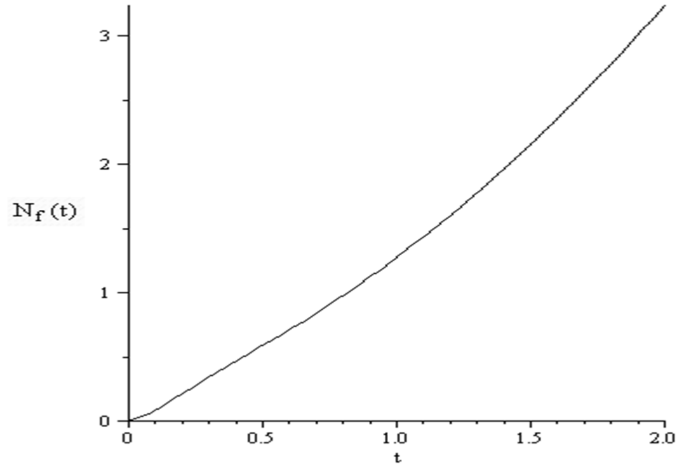

Figure (4): The mean total number of system failures $N_f(t)$ versus the time t (case i)

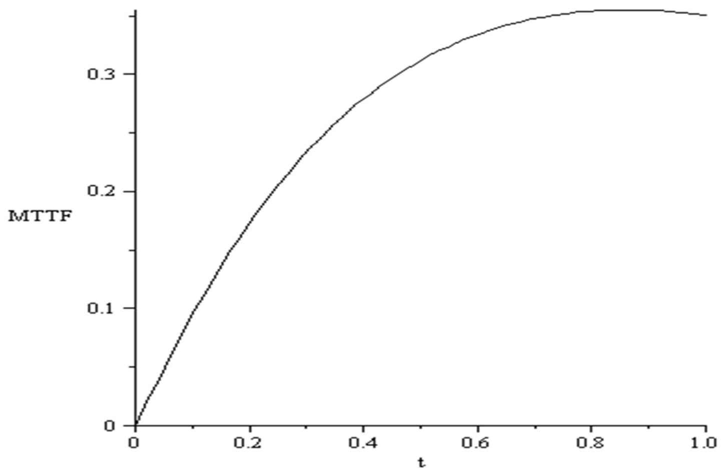

Figure (5): The MTTF for MSS versus the time t (case i)

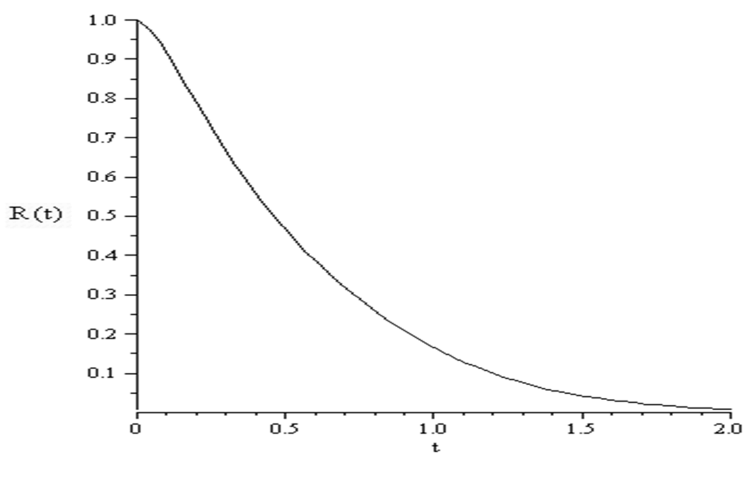

Figure (6): The MSS reliability function $R(t)$ versus the time t (case i)

(ii) The air conditioners failure and repair rates are constant:

As a particular case, we assume that the working time and the repair time of each conditioner are both exponentially distributed. We can then write:

$$\lambda_1(t) = \lambda_1 = 0.5 \qquad \lambda_5(t) = \lambda_5 = 0.25$$
$$\lambda_2(t) = \lambda_2 = 0.25 \qquad \lambda_6(t) = \lambda_6 = 0.2$$
$$\lambda_3(t) = \lambda_3 = 0.25 \qquad \lambda(t) = \lambda = 0.3$$
$$\lambda_4(t) = \lambda_4 = 0.2 \qquad \mu(t) = \mu = 0.6$$

Using MAPLE program, the MSS average availability $\overline{A}(t)$ against time is illustrated in figure (7) with solutions based on Laplace transform method.

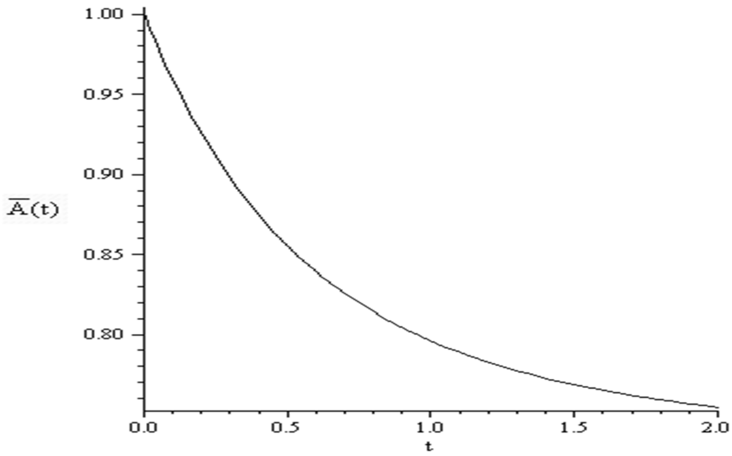

Figure (7): The average availability $\overline{A}(t)$ versus the time t (case ii)

Similarly, the mean total number of system failures $N_f(t)$, the MTTF for MSS, and the MSS reliability function $R(t)$ against time are illustrated in figures (8), (9), and (10), respectively.

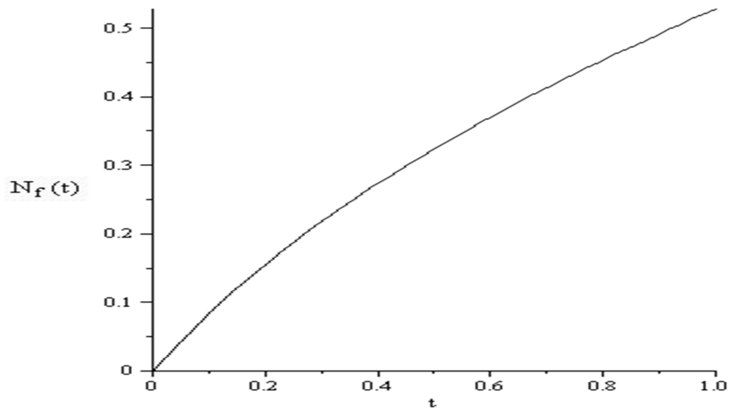

Figure (8): The mean total number of system failures $N_f(t)$ versus the time t (case ii)

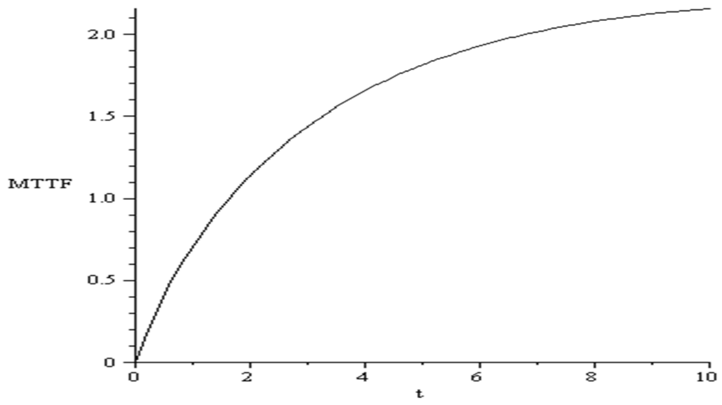

Figure (9): The MTTF for MSS versus the time t (case ii)

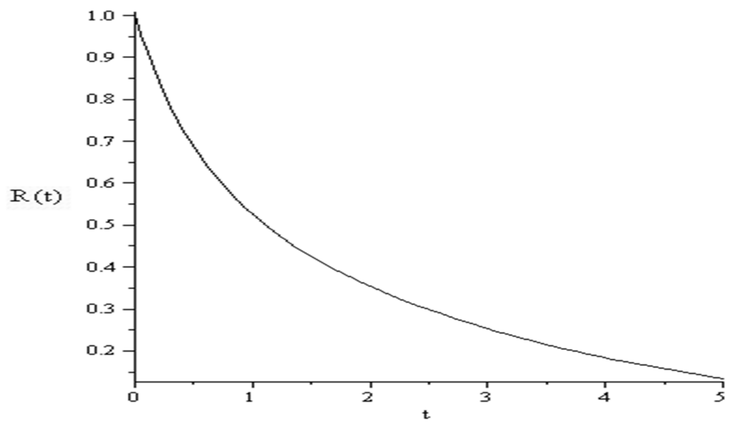

Figure (10): The MSS reliability function $R(t)$ versus the time t (case ii)

time to MSS failure, mean accumulated performance deficiency for a fixed

5. CONCLUSIONS

1. Extension of continuous-time Markov chain to Markov reward models make them even more useful.

2. A Markov reward models was developed as the basis for the generalized computation of availability and reliability measures.

3. The method has been suggested for the computation of MSS reliability and availability measures based on a different reward matrix determination for the Markov reward model.

4. A Markov reward models is well formalized and suitable for practical application in reliability engineering.

5. The numerical results are presented in order to illustrate the suggested model.

REFERENCES

[1] Barlow, R. E. and Proschan, F., *Statistical theory of reliability and life testing*, Holt, Rinehart and Winston, New York, 1975.
[2] Carrasco, J., *Handbook of reliability engineering*, Springer-Verlag, Berlin, 2003.
[3] Finkelstein, M. S., On the shape of the mean residual lifetime function, *Journal of Applied Stochastic Models in Business and Industry*, 18(2), 135-146, 2002.
[4] Gertsbakh, I., *Reliability theory with applications to preventive maintenance*, Springer-Verlag, Berlin, 2000.
[5] Hiller, F. and Lieberman, G., *Introduction to operation research*, McGraw-Hill, New York, 1995.
[6] Howard, R., *Dynamic programming and Markov processes*, MIT Press, Cambridge, 1960.
[7] Lisnianski, A., The Markov reward model for a multi-state system reliability assessment with variable demand, *Journal of Quality Technology & Quantitative Management*, 4(2), 265-278, 2007.
[8] Lisnianski, A. and Levitin, G., *Multi-state system reliability: Assessment, optimization and applications*, World-Scientific, Singapore, 2003.
[9] Lisnianski, A. and Frenkel, I., Non-homogeneous Markov reward model for aging multi-state system under minimal repair, *International Journal of Performability Engineering*, 5(4), 303-312, 2009.
[10] Lisnianski, A., Frenkel, I., and Khvatskin, L., *Safety and reliability for managing risk*, Taylor Francis, London, 2006.
[11] Lisnianski, A., Frenkel, I., Khvatskin, L., and Ding, Y., *Statistical models and methods for biomedical and technical systems*, Birkhäuser, Boston, 2008.
[12] Meeker, W. and Escobar, L., *Statistical methods for reliability data*, Wiley, New York, 1998.
[13] Modarres, M., Kaminskiy, M., and Krivtsov, V., *Reliability engineering and risk analysis: A practical guide*, Marcel Dekker, New York, 1999.
[14] Sahner, R., Trivedi, K., and Poliafito, A., *Performance and reliability analysis of computer systems: An example-based approach using the SHARPE software package*, Kluwer Academic, Boston, 1996.
[15] Volik, B. G., Buyanov, B. B., Lubkov, N. V., Maximov, V. I., and Stepanyants, A. S., *Methods of analysis and synthesis of control systems structures*, Moscow, Energoatomizdat (in Russian), 1988.
[16] Wang, H., A survey of maintenance policies of deteriorating systems, *European Journal of Operational Research*, 139, 469-489, 2002.
[17] Welke, S., Johnson, B., and Aylor, J., Reliability modeling of hardware/software systems, *Journal of IEEE Transactions on Reliability*, 44(3), 413-418, 1995.
[18] Wendt, H. and Kahle, W., *Probability, statistics and modelling in public health*, Berlin, Springer Science & Business Media, 2006.
[19] Xie, M., Dai, Y-S., Poh, K-L., Computing systems reliability: Models and analysis, Kluwer Academic/Plenum Publishers, New York, 2004.
[20] Zhang, F. and Jardine, A. K. S., Optimal maintenance models with minimal repair, periodic overhaul and complete renewal, Journal of IIE Transactions, 30, 1109-1119, 1998.