ASYMPTOTIC APPROACH TO RELIABILITY OF LARGE COMPLEX SYSTEMS

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"All the results presented in the paper would not be possible to develop without Gnedenko's origin result on limit distributions of minimum and maximum statistics"

1 INTRODUCTION

The paper is concerned with the application of limit reliability functions to the reliability evaluation of large complex systems. Two-state and multi-state ageing large complex systems composed of independent components are considered.

Many technical systems belong to the class of complex systems as a result of the large number of components they are built of and their complicated operating processes. This complexity very often causes evaluation of system reliability and safety to become difficult. As a rule these are series systems composed of large number of components. Sometimes the series systems have either components or subsystems reserved and then they become parallel-series or series-parallel reliability structures. We meet large series systems, for instance, in piping transportation of water, gas, oil and various chemical substances. Large systems of these kinds are also used in electrical energy distribution. A city bus transportation system composed of a number of communication lines each serviced by one bus may be a model series system, if we treat it as not failed, when all its lines are able to transport passengers. If the communication lines have at their disposal several buses we may consider it as either a parallel-series system or an "m out of n" system. The simplest example of a parallel system or an "m out of n" system may be an electrical cable composed of a number of wires, which are its basic components, whereas the transmitting electrical network may be either a parallel-series system or an "m out of n"-series system. Large systems of these types are also used in telecommunication, in rope transportation and in transport using belt conveyers and elevators. Rope transportation systems like port elevators and ship-rope elevators used in shipyards during ship docking and undocking are model examples of series-parallel and parallel-series systems.

Taking into account the importance of the safety and operating process effectiveness of such systems it seems reasonable to expand the two-state approach to multi-state approach in their reliability analysis. The assumption that the systems are composed of multi-state components with reliability states degrading in time gives the possibility for more precise analysis of their reliability, safety and operational processes' effectiveness. This assumption allows us to distinguish a system reliability critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operational process effectiveness. Then, an important system reliability characteristic is the time to the moment of exceeding the system reliability critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system multi-state reliability function that is a basic characteristic of the multi-state system.

In the case of large systems, the determination of the exact reliability functions of the systems and the system risk functions leads us to very complicated formulae that are often useless for reliability practitioners. One of the important techniques in this situation is the asymptotic approach to system reliability evaluation. In this approach, instead of the preliminary complex formula for the system reliability function, after assuming that the number of system components tends to infinity and finding the limit reliability of the system, we obtain its simplified form.

The mathematical methods used in the asymptotic approach to the system reliability analysis of large systems are based on limit theorems on order statistics distributions considered in very wide literature (Barndorff-Nielsen 1963, Berman 1962, Berman 1964, Fisher, Tippett 1928, Frechet 1927, Galambos 1975, Gniedenko 1943, Gumbel 1935, Gumbel 1962, Leadbetter 1974, Von Mises 1936). These theorems have generated the investigation concerned with limit reliability functions of the systems composed of two-state components. The main and fundamental results on this subject that determine the three-element classes of limit reliability functions for homogeneous series systems and for homogeneous parallel systems have been established by Gniedenko in (Gniedenko 1943). These results are also presented, sometimes with different proofs, for instance in subsequent works (Barlow, Proschan 1975, Castillo 1988, Chernoff, Teicher 1965, De Haan 1970, Kołowrocki 1993c). The generalizations of these results for homogeneous "*m* out of *n*" systems have been formulated and proved by Smirnow in (Smirnow 1949), where the seven-element class of possible limit reliability functions for these systems has been fixed. Some partial results obtained by Smirnow may be found in (Kołowrocki 2001b). As it has been done for homogeneous series and parallel systems classes of limit reliability functions have been fixed by Chernoff and Teicher in (Chernoff, Teicher 1965) for homogeneous series-parallel and parallel-series systems. Their results were concerned with so-called "quadratic" systems only. They have fixed limit reliability functions for the homogeneous series-parallel systems with the number of series subsystems equal to the number of components in these subsystems, and for the homogeneous parallel-series systems with the number of parallel subsystems equal to the number of components in these subsystems. These results may also be found for instance in later works (Barlow, Proschan 1975) and (Kołowrocki 1993d).

All the results so far described have been obtained under the linear normalization of the system lifetimes. Of course, there is a possibility to look for limit reliability functions of large systems under other than linear standardization of their lifetimes. In this context, the results obtained by (Pantcheva1984) and (Cichocki 2001) are exemplary. Pantcheva in (Pantcheva 1984) has fixed the seven-element classes of limit reliability functions of homogeneous series and parallel systems under power standardization for their lifetimes. Cichocki in (Cichocki 2001) has generalized Pantcheva's results to hierarchical series-parallel and parallel-series systems of any order.

The paper contains the results described above and their newest generalizations for large two-state systems and their exemplary developments for multi-state systems' asymptotic reliability analysis under the linear standardization of the system lifetimes and the system sojourn times in the state subsets, respectively.

Generalizations presented here of the results on limit reliability functions of two-state homogeneous series, and parallel systems for these systems in case they are non-homogeneous, are mostly taken from (Kołowrocki 1994c) and (Kołowrocki 2001b). A more general problem is concerned with fixing the classes of possible limit reliability functions for so-called "rectangular" series-parallel and parallel-series systems. This problem for homogeneous series-parallel and parallel-series systems of any shapes, with different number of subsystems and numbers of components in these subsystems, has been progressively solved in (Kołowrocki 1993a,b,c,d), (Kołowrocki 1994c) and (Kołowrocki 1994e). The main and new result of these works was the determination of seven new limit reliability functions for homogeneous series-parallel systems as well as for parallel-series systems. This way, new ten-element classes of all possible limit reliability functions for these systems have been fixed. Moreover, in these works it has been pointed out that the type of the system limit reliability function strongly depends on the system shape. These results allow us to evaluate reliability characteristics of homogeneous series-parallel and parallel-series systems with regular reliability structures, i.e. systems composed of subsystems having the same numbers of components. The extensions of these results for non-homogeneous series-parallel and parallel-series systems have been formulated and proved successively in (Kołowrocki 1993d), (Kołowrocki 1994d,e, Kołowrocki 1995a,b) and (Kołowrocki 2001b). These generalizations additionally allow us to evaluate reliability characteristics of the series-parallel and parallel-series systems with non-regular structures, i.e. systems with subsystems having different numbers of components. In some of the cited works, as well as the theoretical considerations and solutions, numerous practical applications of the asymptotic approach to real technical system reliability evaluation may also be found (Daniels 1945), (Harlow, Phoenix 1991, Harlow 1997, Harris 1970, Kaufman, Dugan , Johnson 1999, Kołowrocki 1994a, Kołowrocki 1995a, Kołowrocki 1998, Smith 1982, Smith 1983, Soszyńska 2006a, Watherhold 1987).

More general and practically important complex systems composed of multi-state and degrading in time components are considered among others in (Xue 1985, 1995a,b). An especially important role they play in the evaluation of technical systems reliability and safety and their operating process effectiveness is defined in the paper for large multi-state systems with degrading components. The most important results regarding generalizations of the results on limit reliability functions of two-state systems dependent on transferring them to multi-state systems with degrading components are given in (Kołowrocki 1999a,b, Kołowrocki 2000a,b,c, Kołowrocki 2001a,b,c, Kołowrocki 2003a,b). Some of these publications also contain practical applications of the asymptotic approach to the reliability evaluation of various technical systems (Kołowrocki 1999a,b, Kołowrocki 2000a,b,c, Kołowrocki 2001a,b, Kołowrocki 2003a,b).

The results concerned with the asymptotic approach to system reliability analysis have become the basis for the investigation concerned with domains of attraction for the limit reliability functions of the considered systems (Kołowrocki 2004). In a natural way they have led to investigation of the speed of convergence of the system reliability function sequences to their limit reliability functions (Kołowrocki 2004). These results have also initiated the investigation of limit reliability functions of "*m* out of *n*"-series, series-"*m* out of *n*" systems and systems with hierarchical reliability structures as well as investigations on the problems of the system reliability improvement and optimization (Cichocki 2001, Kołowrocki 2004, Soszyńska 2007).

The aim of the paper is to present the state of art on the method of asymptotic approach to reliability evaluation for as wide as possible a range of large systems. The paper describes current theoretical results of the asymptotic approach to reliability evaluation of large two-state and multi-state systems. Additionally, some recent partial results on the asymptotic approach to reliability evaluation of large systems reliability analysis in their operation processes called complex technical systems are presented in the paper (Kołowrocki, Soszyńska 2009, Kołowrocki, Soszyńska 2010a,b, Kołowrocki, Soszyńska-Budny 2011, Soszyńska 2004a,b, Soszyńska 2006a,b,c, Soszyńska 2007, Soszyńska 2008, Soszyńska 2010).

2 BASIC NOTIONS

Considering the reliability of two-state systems we assume that the distributions of the component and the system lifetimes T do not necessarily have to be concentrated in the interval $<0,\infty)$. It means that a reliability function

$$R(t) = P(T > t),\ t \in (-\infty, \infty),$$

does not have to satisfy the usually demanded condition

$$R(t) = 1 \quad \text{for } t \in (-\infty, 0).$$

This is a generalisation of the normally used concept of a reliability function. This generalisation is convenient in the theoretical considerations. At the same time, from the achieved results on the

generalised reliability functions, for particular cases, the same properties of the normally used reliability functions appear.
From that assumption it follows that between a reliability function $R(t)$ and a distribution function

$$F(t) = P(T \le t), t \in (-\infty, \infty),$$

there exists a relationship given by

$$R(t) = 1 - F(t) \text{ for } t \in (-\infty, \infty).$$

Thus, the following corollary is obvious.

Corollary 2.1
A reliability function $R(t)$ is non-increasing, right-continuous and moreover

$$R(-\infty) = 1, R(+\infty) = 0.$$

Definition 2.1
A reliability function $R(t)$ is called degenerate if there exists $t_0 \in (-\infty, \infty)$, such that

$$R(t) = \begin{cases} 1, & t < t_0 \\ 0, & t \ge t_0. \end{cases}$$

The asymptotic approach to the reliability of two-state systems depends on the investigation of limit distributions of a standardised random variable

$$(T - b_n) / a_n,$$

where T is the lifetime of a system and $a_n > 0$, $b_n \in (-\infty, \infty)$, are suitably chosen numbers called normalising constants.
Since

$$P((T - b_n) / a_n > t) = P(T > a_n t + b_n) = \boldsymbol{R}_n(a_n t + b_n),$$

where $\boldsymbol{R}_n(t)$ is a reliability function of a system composed of n components, then the following definition becomes natural.

Definition 2.2
A reliability function $\mathfrak{R}(t)$ is called a limit reliability function or an asymptotic reliability function of a system having a reliability function $\boldsymbol{R}_n(t)$ if there exist normalising constants $a_n > 0$, $b_n \in (-\infty, \infty)$, such that

$$\lim_{n \to \infty} \boldsymbol{R}_n(a_n t + b_n) = \mathfrak{R}(t) \text{ for } t \in C_{\mathfrak{R}}. \quad (2.1)$$

Thus, if the asymptotic reliability function $\mathfrak{R}(t)$ of a system is known, then for sufficiently large n, the approximate formula

$$\boldsymbol{R}_n(t) \cong \mathfrak{R}((t - b_n)/a_n), \; t \in (-\infty, \infty). \quad (2.2)$$

may be used instead of the system exact reliability function $\boldsymbol{R}_n$(t).
From the condition

$$\lim_{n \to \infty} \boldsymbol{R}_n(a_n t + b_n) = \mathfrak{R}(t) \text{ for } t \in C_{\mathfrak{R}},$$

it follows that setting

$$\alpha_n = a a_n, \; \beta_n = b a_n + b_n,$$

where $a > 0$ and $b \in (-\infty, \infty)$, we get

$$\lim_{n\to\infty} \boldsymbol{R}_n(\alpha_n t + \beta_n) = \lim_{n\to\infty} \boldsymbol{R}_n(a_n(at+b)+b_n) = \mathfrak{R}(at+b) \text{ for } t \in C_{\mathfrak{R}}.$$

Hence, if $\mathfrak{R}(t)$ is the limit reliability function of a system, then $\mathfrak{R}(at + b)$ with arbitrary $a > 0$ and $b \in (-\infty, \infty)$ is also its limit reliability function. That fact, in a natural way, yields the concept of a type of limit reliability function.

Definition 2.3
The limit reliability functions $\mathfrak{R}_0(t)$ and $\mathfrak{R}(t)$ are said to be of the same type if there exist numbers $a > 0$ and $b \in (-\infty, \infty)$ such that

$\mathfrak{R}_0(t) = \mathfrak{R}(at + b)$ for $t \in (-\infty, \infty)$.

3 RELIABILITY OF LARGE TWO-STATE SYSTEMS

3.1 RELIABILITY EVALUATION OF TWO-STATE SERIES SYSTEMS

The investigations of limit reliability functions of homogeneous two-state series systems are based on the following auxiliary theorem.

Lemma 3.1
If

(i) $\overline{\mathfrak{R}}(t) = \exp[-\overline{V}(t)]$ is a non-degenerate reliability function,

(ii) $\overline{\boldsymbol{R}}_n(t)$ is the reliability function of a homogeneous two-state series system defined by (2.1) (Kołowrocki 2004)

(iii) $a_n > 0,\ b_n \in (-\infty, \infty)$,

then

$$\lim_{n\to\infty} \overline{\boldsymbol{R}}_n(a_n t + b_n) = \overline{\mathfrak{R}}(t) \text{ for } t \in C_{\overline{\mathfrak{R}}}$$

if and only if

$$\lim_{n\to\infty} nF(a_n t + b_n) = \overline{V}(t) \text{ for } t \in C_{\overline{V}}$$

Lemma 3.1 is an essential tool in finding limit reliability functions of two-state series systems. Its various proofs may be found in (Barlow, Proschan 1975, Gniedenko 1943) and (Kołowrocki 1993d). It also is the basis for fixing the class of all possible limit reliability functions of these systems. This class is determined by the following theorem proved in (Barlow, Proschan 1975, Gniedenko 1943) and (Kołowrocki 1993d).

Theorem 3.1
The only non-degenerate limit reliability functions of the homogeneous two-state series system are:

$$\overline{\mathfrak{R}}_1(t) = \exp[-(-t)^{-\alpha}] \text{ for } t < 0,\ \overline{\mathfrak{R}}_1(t) = 0 \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_2(t) = 1 \text{ for } t < 0,\ \overline{\mathfrak{R}}_2(t) = \exp[-t^{\alpha}] \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_3(t) = \exp[-\exp[t]] \text{ for } t \in (-\infty, \infty).$$

The next auxiliary theorem is an extension of Lemma 3.1 to non-homogeneous two-state series systems.

Lemma 3.2
If

(i) $\overline{\mathfrak{R}}'(t) = \exp[-\overline{V}'(t)]$ is a non-degenerate reliability function,

(ii) $\overline{\mathbf{R}}'_n(t)$ is the reliability function of a non-homogeneous two-state series system defined by (2.8) (Kołowrocki 2004a),

(iii) $a_n > 0,\ b_n \in (-\infty, \infty)$,

then

$$\lim_{n\to\infty} \overline{\mathbf{R}}'_n(a_n t + b_n) = \overline{\mathfrak{R}}'(t) \text{ for } t \in C_{\overline{\mathfrak{R}}'}$$

if and only if

$$\lim_{n\to\infty} n \sum_{i=1}^{a} q_i F^{(i)}(a_n t + b_n) = \overline{V}'(t) \text{ for } t \in C_{\overline{V}'}.$$

The proof of Lemma 3.2 is given in (Kołowrocki 1993d). From the latest lemma, as a particular case, it is possible to derive the next auxiliary theorem that is a more convenient tool than Lemma 3.2 for finding limit reliability functions of non-homogeneous series systems and the starting point for fixing limit reliability functions for these systems.

Lemma 3.3
If

(i) $\overline{\mathfrak{R}}'(t) = \exp[-\overline{V}'(t)]$ is a non-degenerate reliability function,

(ii) $\overline{\mathbf{R}}'_n(t)$ is the reliability function of a non-homogeneous two-state series system defined by (2.8) (Kołowrocki 2004a),

(iii) $a_n > 0,\ b_n \in (-\infty, \infty)$,

(iv) $F(t)$ is one of the distribution functions $F^{(1)}(t), F^{(2)}(t),\ldots,F^{(a)}(t)$ defined by (2.7) (Kołowrocki 2004a), such that

(v) $\exists\, N\ \forall\, n > N\ F(a_n t + b_n) = 0$ for $t < t_0$ and $F(a_n t + b_n) \neq 0$ for $t \geq t_0$, where $t_0 \in <-\infty, \infty)$,

(vi) $\lim_{n\to\infty} \frac{F^{(i)}(a_n t + b_n)}{F(a_n t + b_n)} \leq 1$ for $t \geq t_0$, $i = 1,2,\ldots,a$,

and moreover there exists a non-decreasing function

(vii) $$\bar{d}(t) = \begin{cases} 0 & \text{for } t < t_o \\ \lim_{n\to\infty} \sum_{i=1}^{a} q_i \bar{d}_i(a_n t + b_n) & \text{for } t \geq t_o, \end{cases} \quad (3.1)$$

where

(viii) $$\bar{d}_i(a_n t + b_n) = \frac{F^{(i)}(a_n t + b_n)}{F(a_n t + b_n)}, \quad (3.2)$$

then

$$\lim_{n\to\infty} \overline{\boldsymbol{R}}'_n(a_n t + b_n) = \overline{\mathfrak{R}}'(t) \text{ for } t \in C_{\overline{\mathfrak{R}}'}$$

if and only if

$$\lim_{n\to\infty} nF(a_n t + b_n)\bar{d}(t) = \bar{V}'(t) \text{ for } t \in C_{\bar{V}'}.$$

On the basis of Theorem 3.1 and Lemma 3.3 in (Kołowrocki 1993d), the class of limit reliability functions for non-homogeneous two-state series systems has been fixed. The members of this class are specified in the following theorem (Kołowrocki 1993d).

Theorem 3.2

The only non-degenerate limit reliability functions of the non-homogeneous two-state series system, under the assumptions of Lemma 3.3, are:

$$\overline{\mathfrak{R}}'_1(t) = \exp[-\bar{d}(t)(-t)^{-\alpha}] \text{ for } t < 0, \ \overline{\mathfrak{R}}'_1(t) = 0 \text{ for } t \geq 0, \ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_2(t) = 1 \text{ for } t < 0, \ \overline{\mathfrak{R}}'_2(t) = \exp[-\bar{d}(t)t^{\alpha}] \text{ for } t \geq 0, \ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_3(t) = \exp[-\bar{d}(t)\exp[t]] \text{ for } t \in (-\infty, \infty),$$

where $\bar{\bar{d}}(t)$ is a non-decreasing function dependent on the reliability functions of particular system components and their fractions in the system defined by (3.1)-(3.2).

3.2 RELIABILITY EVALUATION OF TWO-STATE PARALLEL SYSTEMS

The class of limit reliability functions for homogeneous two-state parallel systems may be determined on the basis of the following auxiliary theorem proved for instance in (Barlow, Proschan 1975, Gniedenko 1943) and (Kołowrocki 1993d).

Lemma 3.4

If $\overline{\mathfrak{R}}(t)$ is the limit reliability function of a homogeneous two-state series system with reliability functions of particular components $\bar{R}(t)$, then

$$\mathfrak{R}(t) = 1 - \overline{\mathfrak{R}}(-t) \text{ for } t \in C_{\overline{\mathfrak{R}}}$$

is the limit reliability function of a homogeneous two-state parallel system with reliability functions of particular components

$$R(t) = 1 - \bar{R}(-t) \text{ for } t \in C_{\bar{R}}.$$

At the same time, if (a_n, b_n) is a pair of normalising constants in the first case, then $(a_n, -b_n)$ is such a pair in the second case.

Applying the above lemma it is possible to prove an equivalent of Lemma 3.1 that allows us to justify facts on limit reliability functions for homogeneous parallel systems. Its form is as follows (Barlow, Proschan 1975, Gniedenko 1943, Kołowrocki 1993d).

Lemma 3.5
If

(i) $\mathfrak{R}(t) = 1 - \exp[-V(t)]$ is a non-degenerate reliability function,

(ii) $\boldsymbol{R}_n(t)$ is the reliability function of a homogeneous two-state parallel system defined by (2.2) [42],

(iii) $a_n > 0,\ b_n \in (-\infty, \infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n t + b_n) = \mathfrak{R}(t) \text{ for } t \in C_{\mathfrak{R}},$$

if and only if

$$\lim_{n\to\infty} nR(a_n t + b_n) = V(t) \text{ for } t \in C_V.$$

By applying Lemma 3.5 and proceeding in an analogous way to the case of homogeneous series systems it is possible to fix the class of limit reliability functions for homogeneous two-state parallel systems. However, it is easier to obtain this result using Lemma 3.4 and Theorem 3.1. Their application immediately results in the following issue.

Theorem 3.3
The only non-degenerate limit reliability functions of the homogeneous parallel system are:

$$\mathfrak{R}_1(t) = 1 \text{ for } t \le 0,\ \mathfrak{R}_1(t) = 1 - \exp[-t^{-\alpha}] \text{ for } t > 0,\ \alpha > 0,$$

$$\mathfrak{R}_2(t) = 1 - \exp[-(-t)^{\alpha}] \text{ for } t < 0,\ \mathfrak{R}_2(t) = 0 \text{ for } t \ge 0,\ \alpha > 0,$$

$$\mathfrak{R}_3(t) = 1 - \exp[-\exp[-t]] \text{ for } t \in (-\infty, \infty).$$

The next lemma is a slight modification of Lemma 3.5 proved in (Kołowrocki 1993d). It is also a particular case of Lemma 2, which is proved in (Kołowrocki 1995a).

Lemma 3.6
If $\bar{\mathfrak{R}}'(t)$ is the limit reliability function of a non-homogeneous two-state series system with reliability functions of particular components

$$\bar{R}^{(i)}(t),\ i = 1,2,...,a,$$

then

$$\mathfrak{R}'(t) = 1 - \bar{\mathfrak{R}}'(-t) \text{ for } t \in C_{\bar{\mathfrak{R}}'}$$

is the limit reliability function of a non-homogeneous two-state parallel system with reliability functions of particular components

$$R^{(i)}(t) = 1 - \bar{R}^{(i)}(-t) \text{ for } t \in C_{\bar{R}^{(i)}},\ i = 1,2,...,a.$$

At the same time, if (a_n, b_n) is a pair of normalising constants in the first case, then $(a_n, -b_n)$ is such a pair in the second case.

Applying the above lemma and Theorem 3.2 it is possible to arrive at the next result (Kołowrocki 1993d, Kołowrocki 1995b).

Lemma 3.7
If

(i) $\mathfrak{R}'(t) = 1 - \exp[-V'(t)]$ is a non-degenerate reliability function,

(ii) $\boldsymbol{R}'_n(t)$ is the reliability function of a non-homogeneous two-state parallel system defined by (2.10) (Kołowrocki 2004a),

(iii) $a_n > 0,\ b_n \in (-\infty,\infty)$,

then

$$\lim_{n\to\infty} \boldsymbol{R}'_n(a_n t + b_n) = \mathfrak{R}'(t) \text{ for } t \in C_{\mathfrak{R}'}$$

if and only if

$$\lim_{n\to\infty} n\sum_{i=1}^{a} q_i R^{(i)}(a_n t + b_n) = V'(t) \text{ for } t \in C_{V'}.$$

The next lemma motivated in (Kołowrocki 1993d) that is useful in practical applications is a particular case of Lemma 3 proved in (Kołowrocki 1995b).

Lemma 3.8

If

(i) $\mathfrak{R}'(t) = 1 - \exp[-V'(t)]$ is a non-degenerate reliability function,

(ii) $\mathbf{R}'_n(t)$ is the reliability function of a non-homogeneous two-state parallel system defined by (2.10) (Kołowrocki 2004a),

(iii) $a_n > 0$, $b_n \in (-\infty,\infty)$,

(iv) $R(t)$ is one of the reliability functions $R^{(1)}(t), R^{(2)}(t),...,R^{(a)}(t)$ defined by (2.9) (Kołowrocki 2004a), such that

(v) $\exists N \,\forall n > N \; R(a_n t + b_n) \neq 0$ for $t < t_0$ and $R(a_n t + b_n) = 0$ for $t \geq t_0$,
where $t_0 \in (-\infty,\infty>$,

(vi) $\lim_{n\to\infty} \frac{R^{(i)}(a_n t + b_n)}{R(a_n t + b_n)} \leq 1$ for $t < t_0$, $i = 1,2,..., a$,

and moreover there exists a non-increasing function

(vii) $$d(t) = \begin{cases} \lim_{n\to\infty} \sum_{i=1}^{a} q_i d_i(a_n t + b_n) & for\ t < t_0 \\ 0 & for\ t \geq t_0, \end{cases} \tag{3.3}$$

where

(viii) $$d_i(a_n t + b_n) = \frac{R^{(i)}(a_n t + b_n)}{R(a_n t + b_n)}, \tag{3.4}$$

then

$$\lim_{n\to\infty} \mathbf{R}'_n(a_n t + b_n) = \mathfrak{R}'(t) \text{ for } t \in C_{\mathfrak{R}'}$$

if and only if

$$\lim_{n\to\infty} nR(a_n t + b_n)d(t) = V'(t) \text{ for } t \in C_{V'}.$$

Starting from this lemma it is possible to fix the class of possible limit reliability for non-homogeneous two-state parallel systems (Kołowrocki 1993d, Kołowrocki 1995b).

Theorem 3.4
The only non-degenerate limit reliability functions of the non-homogeneous two-state parallel system, under the assumptions of Lemma 3.8, are:

$$\mathfrak{R}'_1(t) = 1 \text{ for } t \leq 0,\ \mathfrak{R}'_1(t) = 1 - \exp[-d(t)t^{-\alpha}] \text{ for } t > 0,\ \alpha > 0,$$

$$\mathfrak{R}'_2(t) = 1 - \exp[-d(t)(-t)^{\alpha}] \text{ for } t < 0,\ \mathfrak{R}'_2(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}'_3(t) = 1 - \exp[-d(t)\exp[-t]] \text{ for } t \in (-\infty,\infty),$$

where $d(t)$ is a non-increasing function dependent on the reliability functions of particular system components and their fractions in the system defined by (3.3)-(3.4).

3.3 RELIABILITY EVALUATION OF TWO-STATE "*M* OUT OF *N*" SYSTEMS

The class of limit reliability function for homogeneous two-state "*m* out of *n*" systems may be established by applying the auxiliary theorems proved in (Smirnow 1949) and (Kołowrocki 1993d). The applications of these lemmas allow us to establish the class of possible limit reliability functions for homogeneous two-state "*m* out of *n*" systems pointed out in the following theorem (Kołowrocki 1993d, Smirnow 1949).

Theorem 3.5
The only non-degenerate limit reliability functions of the homogeneous two-state "*m* out of *n*" system are:

Case 1. $m = \text{constant}\,(m/n \to 0 \text{ as } n \to \infty)$.

$$\mathfrak{R}_1^{(0)}(t) = 1 \text{ for } t \le 0,\ \mathfrak{R}_1^{(0)}(t) = 1 - \sum_{i=0}^{m-1} \frac{t^{-i\alpha}}{i!} \exp[-t^{-\alpha}] \text{ for } t > 0,\ \alpha > 0,$$

$$\mathfrak{R}_2^{(0)}(t) = 1 - \sum_{i=0}^{m-1} \frac{(-t)^{i\alpha}}{i!} \exp[-(-t)^{\alpha}] \text{ for } t < 0,\ \mathfrak{R}_2^{(0)}(t) = 0 \text{ for } t \ge 0,\ \alpha > 0,$$

$$\mathfrak{R}_3^{(0)}(t) = 1 - \sum_{i=0}^{m-1} \frac{\exp[-it]}{i!} \exp[-\exp[-t]] \text{ for } t \in (-\infty, \infty).$$

Case 2. $m/n = \mu + o(1/\sqrt{n}),\ 0 < \mu < 1,\ (m/n \to \mu \text{ as } n \to \infty)$.

$$\mathfrak{R}_4^{(\mu)}(t) = 1 \text{ for } t < 0,\ \mathfrak{R}_4^{(\mu)}(t) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{ct^{\alpha}} e^{-\frac{x^2}{2}} dx \text{ for } t \ge 0,\ c > 0, \alpha > 0,$$

$$\mathfrak{R}_5^{(\mu)}(t) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-c|t|^{\alpha}} e^{-\frac{x^2}{2}} dx \text{ for } t < 0,\ \mathfrak{R}_5^{(\mu)}(t) = 0 \text{ for } t \ge 0,\ c > 0, \alpha > 0,$$

$$\mathfrak{R}_6^{(\mu)}(t) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-c_1|t|^{\alpha}} e^{-\frac{x^2}{2}} dx \text{ for } t < 0,\ c_1 > 0, \alpha > 0,$$

$$\mathfrak{R}_6^{(\mu)}(t) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{0}^{c_2 t^{\alpha}} e^{-\frac{x^2}{2}} dx \text{ for } t \ge 0,\ c_2 > 0, \alpha > 0,$$

$$\mathfrak{R}_7^{(\mu)}(t) = 1 \text{ for } t < -1,\ \mathfrak{R}_7^{(\mu)}(t) = \frac{1}{2} \text{ for } -1 \le t < 1,\ \mathfrak{R}_7^{(\mu)}(t) = 0 \text{ for } t \ge 0.$$

Case 3. $n - m = \overline{m} = \text{constant}\ (m/n \to 1 \text{ as } n \to \infty)$.

$$\overline{\mathfrak{R}}_8^{(1)}(t) = \sum_{i=0}^{\overline{m}} \frac{(-t)^{-i\alpha}}{i!} \exp[-(-t)^{-\alpha}] \text{ for } t < 0,\ \overline{\mathfrak{R}}_8^{(1)}(t) = 0 \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_9^{(1)}(t) = 1 \text{ for } t < 0,\ \overline{\mathfrak{R}}_9^{(1)}(t) = \sum_{i=0}^{\overline{m}} \frac{t^{i\alpha}}{i!}\exp[-t^{\alpha}] \text{ for } t \geq 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_9^{(1)}(t) = \sum_{i=0}^{\overline{m}} \frac{\exp[it]}{i!}\exp[-\exp[t]] \text{ for } t \in (-\infty,\infty).$$

3.4 RELIABILITY EVALUATION OF TWO-STATE SERIES-PARALLEL SYSTEMS

Prior to the formulation of the overall results for the classes of limit reliability functions for two-state regular series-parallel systems we should introduce some assumptions for all cases of the considered systems shapes. These assumptions distinguish all possible relationships between the number of their series subsystems k_n and the number of components l_n in these subsystems (Assumption 4.1, (Kołowrocki 2004)).
The proofs of the theorems on limit reliability functions for homogeneous regular series-parallel systems and methods of finding such functions for individual systems are based on the lemma given in (Kołowrocki 1993a) and (Kołowrocki 1993d). The results achieved in (Kołowrocki 1993a,b,c,d, Kołowrocki 1994c) and based on those lemmas may be formulated in the form of the following theorem (Kołowrocki 1993b, Kołowrocki 1994a, Kołowrocki 1995b).

Theorem 3.6
The only non-degenerate limit reliability functions of the homogeneous regular two-state series-parallel system are:

Case 1. $k_n = n$, $|l_n - c\log n| >> s$, $s > 0$, $c > 0$ (under Assumption 4.1 (Kołowrocki 2004)).

$$\mathfrak{R}_1(t) = 1 \text{ for } t \leq 0,\ \mathfrak{R}_1(t) = 1 - \exp[-t^{-\alpha}] \text{ for } t > 0,\ \alpha > 0,$$

$$\mathfrak{R}_2(\mathrm{t}) = 1 - \exp[-(-t)^{\alpha}] \text{ for } t < 0,\ \mathfrak{R}_2(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}_3(\mathrm{t}) = 1 - \exp[-\exp[-t]] \text{ for } t \in (-\infty,\infty),$$

Case 2. $k_n = n$, $l_n - c\log n \approx s$, $s \in (-\infty,\infty)$, $c > 0$.

$$\mathfrak{R}_4(t) = 1 \text{ for } t < 0,\ \mathfrak{R}_4(t) = 1 - \exp[-\exp[-t^{\alpha} - s/c]] \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}_5(t) = 1 - \exp[-\exp[(-t)^{\alpha} - s/c]] \text{ for } t < 0,\ \mathfrak{R}_5(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}_6(t) = 1 - \exp[-\exp[\beta(-t)^{\alpha} - s/c]] \text{ for } t < 0,$$

$$\mathfrak{R}_6(\mathrm{t}) = 1 - \exp[-\exp[-t^{\alpha} - s/c]] \text{ for t } \geq 0,\ \alpha > 0,\ \beta > 0,$$

$$\mathfrak{R}_7(t) = 1 \text{ for } t < t_1,\ \mathfrak{R}_7(t) = 1 - \exp[-\exp[-s/c]] \text{ for } t_1 \leq t < t_2,$$

$$\mathfrak{R}_7(t) = 0 \text{ for } t \geq t_2,\ t_1 < t_2,$$

Case 3. $k_n \to k$, $k > 0$, $l_n \to \infty$.

$$\mathfrak{R}_8(t) = 1 - [1 - \exp[-(-t)^{-\alpha}]]^k \text{ for } t < 0,\ \mathfrak{R}_8(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}_9(t) = 1 \text{ for } t < 0,\ \mathfrak{R}_9(t) = 1 - [1 - \exp[-t^{\alpha}]]^k \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}_{10}(t) = 1 - [1 - \exp[-\exp t]]^k \text{ for } t \in (-\infty,\infty).$$

The proofs of the facts concerned with limit reliability functions of non-homogeneous two-state series-parallel systems are based on the auxiliary theorems formulated and proved in (Kołowrocki 1993d, Kołowrocki 1994c) and (Kołowrocki 1995b). Theorem 3.6 and those lemmas determine the class of limit reliability functions for non-homogeneous regular series-parallel systems whose members are pointed out in the following theorem (Kołowrocki 1993d, Kołowrocki 1994a, Kołowrocki 1994d).

Theorem 3.7
The only non-degenerate limit reliability functions of the non-homogeneous regular two-state series-parallel system are:

Case 1. $k_n = n$, $|l_n - c \log n| >> s$, $s > 0$, $c > 0$ (under Assumption 4.1 (Kołowrocki 2004)).

$$\mathfrak{R}'_1(t) = 1 \text{ for } t \leq 0,\ \mathfrak{R}'_1(t) = 1 - \exp[-d(t)t^{-\alpha}] \text{ for } t > 0,\ \alpha > 0,$$

$$\mathfrak{R}'_2(t) = 1 - \exp[-d(t)(-t)^{\alpha}] \text{ for } t < 0,\ \mathfrak{R}'_2(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}'_3(t) = 1 - \exp[-d(t)\exp[-t]] \text{ for } t \in (-\infty,\infty),$$

Case 2. $k_n = n$, $l_n - c \log n \approx s$, $s \in (-\infty,\infty)$, $c > 0$ (under Assumption 3.1 (Kołowrocki 2004)).

$$\mathfrak{R}'_4(t) = 1 \text{ for } t < 0,\ \mathfrak{R}'_4(t) = 1 - \exp[-d(t)\exp[-t^{\alpha} - s/c]] \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}'_5(t) = 1 - \exp[-d(t)\exp[(-t)^{\alpha} - s/c]] \text{ for } t < 0,\ \mathfrak{R}'_5(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}'_6(t) = 1 - \exp[-d(t)\exp[\beta(-t)^{\alpha} - s/c]] \text{ for } t < 0,$$

$$\mathfrak{R}'_6(t) = 1 - \exp[-d(t)\exp[-t^{\alpha} - s/c]] \text{ for } t \geq 0,\ \alpha > 0,\ \beta > 0,$$

$$\mathfrak{R}'_7(t) = 1 \text{ for } t < t_1,\ \mathfrak{R}'_7(t) = 1 - \exp[-d(t)\exp[-s/c]] \text{ for } t_1 \leq t < t_2,\ \mathfrak{R}'_7(t) = 0 \text{ for } t \geq t_2,\ t_1 < t_2,$$

Case 3. $k_n \to k$, $k > 0$, $l_n \to \infty$ (under Assumption 4.1 (Kołowrocki 2004).

$$\mathfrak{R}'_8(t) = 1 - \prod_{i=1}^{a}[1 - d_i(t)\exp[-(-t)^{-\alpha}]]^{q_i k} \text{ for } t < 0,\ \mathfrak{R}'_8(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\mathfrak{R}'_9(t) = 1 \text{ for } t < 0,\ \mathfrak{R}'_9(t) = 1 - \prod_{i=1}^{a}[1 - d_i(t)\exp[-t^{\alpha}]]^{q_i k} \text{ for } t \geq 0,\ \alpha > 0, \quad (3.80)$$

$$\mathfrak{R}'_{10}(t) = 1 - \prod_{i=1}^{a}[1 - d_i(t)\exp[-\exp t]]^{q_i k} \text{ for } t \in (-\infty,\infty),$$

where $d(t)$ and $d_i(t)$ are non-increasing functions dependent on the reliability functions of the system's particular components and their fractions in the system defined in (Kołowrocki 2004).

3.5 RELIABILITY EVALUATION OF TWO-STATE PARALLEL-SERIES SYSTEMS

Prior to the formulation of the overall results for the classes of limit reliability functions for two-state regular parallel-series systems we should introduce some assumptions for all cases of the considered systems shapes. These assumptions distinguish all possible relationships between the number of their parallel subsystems k_n and the number of components l_n in these subsystems (Assumption 4.1 (Kołowrocki 2004)).
The class of limit reliability functions for homogeneous regular two-state parallel-series systems is successively fixed in (Kołowrocki 1993a,b,c,d, Kołowrocki 1994c) and (Kołowrocki 1994e,f, Kołowrocki 1995). The class of limit reliability functions for homogeneous regular two-state parallel-series system is pointed out in the following theorem (Kołowrocki 1993d, Kołowrocki 1994d).

Theorem 3.8
The only non-degenerate limit reliability functions of the homogeneous regular two-state parallel-series system are:

Case 1. $k_n = n$, $|l_n - c\log n| >> s$, $s > 0$, $c > 0$ (under Assumption 4.1 (Kołowrocki 2004).

$$\overline{\mathfrak{R}}_1(t) = \exp[-(-t)^{-\alpha}] \text{ for } t < 0,\ \overline{\mathfrak{R}}_1(t) = 0, \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_2(t) = 1 \text{ for } t < 0,\ \overline{\mathfrak{R}}_2(t) = \exp[-t^{\alpha}], \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_3(t) = \exp[-\exp[t]] \text{ for } t \in (-\infty,\infty),$$

Case 2. $k_n = n$, $l_n - c\log n \approx s$, $s \in (-\infty,\infty)$, $c > 0$;

$$\overline{\mathfrak{R}}_4(t) = \exp[-\exp[-(-t)^{\alpha} - s/c]] \text{ for } t < 0,\ \overline{\mathfrak{R}}_4(t) = 0 \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_5(t) = 1 \text{ for } t < 0,\ \overline{\mathfrak{R}}_5(t) = \exp[-\exp[t^{\alpha} - s/c]] \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_6(t) = \exp[-\exp[-(-t)^{\alpha} - s/c]] \text{ for } t < 0, \overline{\mathfrak{R}}_6(t) = \exp[-\exp[\beta t^{\alpha} - s/c]] \text{ for } t \ge 0,\ \alpha > 0,\ \beta > 0,$$

$$\overline{\mathfrak{R}}_7(t) = 1 \text{ for } t < t_1,\ \overline{\mathfrak{R}}_7(t) = \exp[-\exp[-s/c]] \text{ for } t_1 \le t < t_2, \overline{\mathfrak{R}}_7(t) = 0 \text{ for } t \ge t_2,\ t_1 < t_2,$$

Case 3. $k_n \to k$, $k > 0$, $l_n \to \infty$.

$$\overline{\mathfrak{R}}_8(t) = 1 \text{ for } t \le 0,\ \overline{\mathfrak{R}}_8(t) = [1 - \exp[-t^{-\alpha}]]^k \text{ for } t > 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_9(t) = [1 - \exp[-(-t)^{\alpha}]]^k \text{ for } t < 0,\ \overline{\mathfrak{R}}_9(t) = 0 \text{ for } t \ge 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}_{10}(t) = [1 - \exp[-\exp[-t]]]^k \text{ for } t \in (-\infty,\infty).$$

The class of limit reliability functions for non-homogeneous regular two-state parallel-series system is pointed out in the following theorem (Kołowrocki 1993d, Kołowrocki 1994d).

Theorem 3.9
The only non-degenerate limit reliability functions of the non-homogeneous regular two-state parallel-series system are:

Case 1. $k_n = n$, $|\, l_n - c \log n \,| >> s$, $s > 0$, $c > 0$ (under Assumption 4.1 (Kołowrocki 2004)).

$$\overline{\mathfrak{R}}'_1(t) = \exp[-\bar{d}(t)(-t)^{-\alpha}] \text{ for } t < 0,\ \overline{\mathfrak{R}}'_1(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_2(t) = 1 \text{ for } t < 0,\ \overline{\mathfrak{R}}'_2(t) = \exp[-\bar{d}(t)t^{\alpha}] \text{ for } t \geq 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_3(t) = \exp[-\bar{d}(t)\exp[t]] \text{ for } t \in (-\infty,\infty),$$

Case 2. $k_n = n$, $l_n - c \log n \approx s$, $s \in (-\infty,\infty)$, $c > 0$ (under Assumption 4.1 (Kołowrocki 2004)).

$$\overline{\mathfrak{R}}'_4(t) = \exp[-\bar{d}(t)\exp[-(-t)^{\alpha} - s/c]] \text{ for } t < 0,\ \overline{\mathfrak{R}}'_4(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_5(t) = 1 \text{ for } t < 0,\ \overline{\mathfrak{R}}'_5(t) = \exp[-\bar{d}(t)\exp[t^{\alpha} - s/c]] \text{ for } t \geq 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_6(t) = \exp[-\bar{d}(t)\exp[-(-t)^{\alpha} - s/c]] \text{ for } t < 0,$$

$$\overline{\mathfrak{R}}'_6(t) = \exp[-\bar{d}(t)\exp[\beta t^{\alpha} - s/c]] \text{ for } t \geq 0,\ \alpha > 0,\ \beta > 0,$$

$$\overline{\mathfrak{R}}'_7(t) = 1 \text{ for } t < t_1,\ \overline{\mathfrak{R}}'_7(t) = \exp[-\bar{d}(t)\exp[-s/c]] \text{ for } t_1 \leq t < t_2,$$

$$\overline{\mathfrak{R}}'_7(t) = 0 \text{ for } t \geq t_2,\ t_1 < t_2,$$

Case 3. $k_n \to k$, $k > 0$, $l_n \to \infty$ (under Assumption 4.1 (Kołowrocki 2004)).

$$\overline{\mathfrak{R}}'_8(t) = 1 \text{ for } t \leq 0,\ \overline{\mathfrak{R}}'_8(t) = \prod_{i=1}^{a}[1 - \bar{d}_i(t)\exp[-t^{-\alpha}]]^{q_i k} \text{ for } t > 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_9(t) = \prod_{i=1}^{a}[1 - \bar{d}_i(t)\exp[-(-t)^{\alpha}]]^{q_i k} \text{ for } t < 0,\ \overline{\mathfrak{R}}'_9(t) = 0 \text{ for } t \geq 0,\ \alpha > 0,$$

$$\overline{\mathfrak{R}}'_{10}(t) = \prod_{i=1}^{a}[1 - \bar{d}_i(t)\exp[-\exp(-t)]]^{q_i k} \text{ for } t \in (-\infty,\infty),$$

where $\bar{d}(t)$ and $\bar{d}_i(t)$ are non-decreasing functions dependent on the reliability functions of particular system components and their fractions in the system defined in (Kołowrocki 2004).

4 RELIABILITY OF LARGE MULTI-STATE SYSTEMS

In the multi-state reliability analysis to define systems with degrading (ageing) components we assume that:

– E_i, $i = 1,2,...,n$, are components of a system,
– all components and a system under consideration have the reliability state set $\{0,1,...,z\}$, $z \ge 1$,
– the reliability states are ordered, the state 0 is the worst and the state z is the best,
– $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the reliability state subset $\{u,u+1,...,z\}$, while they were in the reliability state z at the moment $t = 0$,
– $T(u)$ is a random variable representing the lifetime of a system in the reliability state subset $\{u,u+1,...,z\}$ while it was in the reliability state z at the moment $t = 0$,
– the system state degrades with time t,
– $e_i(t)$ is a component E_i reliability state at the moment t, $t \in <0,\infty)$, given that it was in the reliability state z at the moment $t = 0$,
– $s(t)$ is a system reliability state at the moment t, $t \in <0,\infty)$, given that it was in the reliability state z at the moment $t = 0$.

The above assumptions mean that the reliability states of the system with degrading components may be changed in time only from better to worse. The way in which the components and the system reliability states change is illustrated in Figure 4.1.

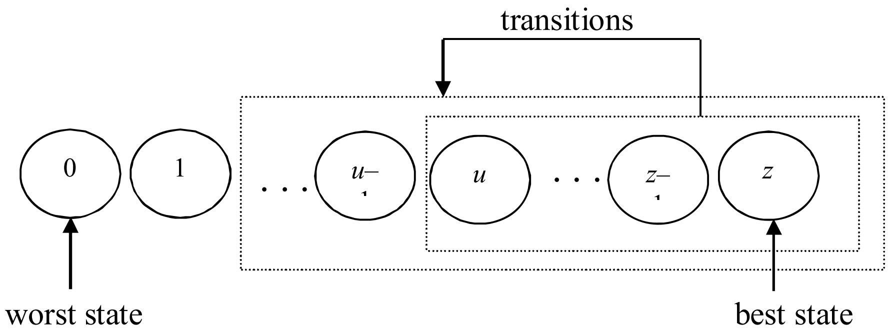

Figure. 4.1. Illustration of reliability states changing in system with ageing components

Definition 4.1
A vector

$$R_i(t,\cdot) = [R_i(t,0),R_i(t,1),...,R_i(t,z)],\ t \in <0,\infty),\ i = 1,2,...,n,$$

where

$$R_i(t,u) = P(e_i(t) \ge u \mid e_i(0) = z) = P(T_i(u) > t),\ t \in <0,\infty),\ u = 0,1,...,z,$$

is the probability that the component E_i is in the reliability state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multi-state reliability function of a component E_i.

Definition 4.2
A vector

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0), \boldsymbol{R}_n(t,1), ..., \boldsymbol{R}_n(t,z)], \; t \in (-\infty, \infty), \tag{4.1}$$

where

$$\boldsymbol{R}_n(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t), \; t \in <0,\infty), \; u = 0,1,...,z, \tag{4.2}$$

is the probability that the system is in the reliability state subset $\{u, u+1, ..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the reliability state z at the moment t = 0, is called the multi-state reliability function of a system.

Definition 4.3
A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \leq t), \; t \in <0,\infty),$$

that the system is in the subset of reliability states worse than the critical reliability state r, $r \in \{1,...,z\}$ while it was in the reliability state z at the moment t = 0 is called a risk function of the multi-state system.

Under this definition, from (4.1)-(4.2), we have

$$\boldsymbol{r}(t) = 1 - P(s(t) \geq r \mid s(0) = z) = 1 - \boldsymbol{R}_n(t,r), \; t \in <0,\infty).$$

and if τ is the moment when the risk exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta),$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$.

In the asymptotic approach to multi-state system reliability analysis we are interested in the limit distributions of a standardised random variable

$$(T(u) - b_n(u)) / a_n(u), \; u = 1,2,...,z,$$

where $T(u)$ is the lifetime of the system in the state subset $\{u, u+1, ..., z\}$ and

$$a_n(u) > 0, \; b_n(u) \in (-\infty, \infty), u = 1,2,...,z,$$

are some suitably chosen numbers, called normalising constants. And, since

$$P((T(u) - b_n(u)) / a_n(u) > t) = P(T(u) > a_n(u)t + b_n(u)) = \boldsymbol{R}_n(a_n(u)t + b_n(u), u), \; u = 1,2,...,z,$$

where

$$\boldsymbol{R}_n(t,\cdot) = [\boldsymbol{R}_n(t,0),\boldsymbol{R}_n(t,1),...,\boldsymbol{R}_n(t,z)],\ t \in (-\infty,\infty),$$

is the multi-state reliability function of the system composed of n components, then we assume the following definition.

Definition 4.4
A vector

$$\boldsymbol{\mathfrak{R}}(t,\cdot) = [1,\boldsymbol{\mathfrak{R}}(t,1),...,\boldsymbol{\mathfrak{R}}(t,z)],\ t \in (-\infty,\infty),$$

is called the limit multi-state reliability function of the system with reliability function $\boldsymbol{R}_n(t,\cdot)$ if there exist normalising constants $a_n(u) > 0$, $b_n(u) \in (-\infty,\infty)$, such that

$$\lim_{n\to\infty} \boldsymbol{R}_n(a_n(u)t + b_n(u),u) = \boldsymbol{\mathfrak{R}}(t,u) \text{ for } t \in C_{\boldsymbol{\mathfrak{R}}(u)},\ u = 1,2,...,z,$$

where $C_{\boldsymbol{\mathfrak{R}}(u)}$ is the set of continuity points of $\boldsymbol{\mathfrak{R}}(t,u)$.

Knowing the system limit reliability function allows us, for sufficiently large n, to apply the following approximate formula

$$[1,\boldsymbol{R}_n(t,1),...,\boldsymbol{R}_n(t,z)] \cong [1,\boldsymbol{\mathfrak{R}}(\frac{t-b_n(1)}{a_n(1)},1),...,\boldsymbol{\mathfrak{R}}(\frac{t-b_n(z)}{a_n(z)},z)],\ t \in (-\infty,\infty).$$

Similar as in Section 3, auxiliary theorems on limit reliability functions of multi-state systems, which are necessary for their approximate reliability evaluation, can be formulated and proved (Kołowrocki 2004). The classes of limit reliability functions for homogeneous and non-homogeneous series, parallel, series-parallel and parallel-series multi-state systems and for a homogeneous multi-state “m out of n” system can be fixed as well (Kołowrocki 2004).

5 RELIABILITY OF COMPLEX TECHNICAL SYSTEMS

Most real technical systems are structurally very complex and they often have complicated operation processes. The time dependent interactions between the systems’ operation processes operation states changing and the systems’ structures and their components reliability states changing processes are evident features of most real technical systems. The common reliability and operation analysis of these complex technical systems is of great value in the industrial practice. The convenient tools for analysing this problem are presented in (Kołowrocki, Soszyńska-Budny 2011) where the multistate system’s reliability modelling commonly used with the semi-Markov modelling of the systems operation processes, leads to the construction the joint general reliability models of the complex technical systems related to their operation process (Kołowrocki 2006, Kołowrocki 2007a,b, Kołowrocki, Soszyńska 2006, Kołowrocki, Soszyńska 2010a, Kołowrocki, Soszyńska 2011, Soszyńska 2004a,b, Soszyńska 2006, Soszyńska 2007, Soszyńska 2008). In the case of large complex technical systems, one of the important techniques is the asymptotic approach (Kołowrocki 2004, Kołowrocki 2008b, Soszńska 2004a,b, Soszyńska 2006a, Soszyńska 2007, Soszyńska 2008) to their reliability evaluation. .

5.1 RELIABILITY OF MULTISTATE SYSTEMS AT VARIABLE OPERATIONS CONDITIONS

We assume that the changes of the operation states of the system operation process have an influence on the system multistate components E_i, $i = 1,2,...,n,$ reliability and the system reliability structure as well. Consequently, we denote the system multistate component E_i, $i = 1,2,...,n,$ conditional lifetime in the reliability state subset $\{u, u+1,..., z\}$ while the system is at the operation state z_b, $b = 1,2,..., v,$ by $T_i^{(b)}(u)$ and its conditional reliability function by the vector

$$[R_i(t, \cdot)]^{(b)} = [1, [R_i(t, 1)]^{(b)}, ..., [R_i(t, z)]^{(b)}],$$

with the coordinates defined by

$$[R_i(t,u)]^{(b)} = P(T_i^{(b)}(u) > t | Z(t) = z_b)$$

for $t \in < 0, \infty)$, $u = 1,2,..., z$, $b = 1,2,..., v$.

The reliability function $[R_i(t,u)]^{(b)}$ is the conditional probability that the component E_i lifetime $T_i^{(b)}(u)$ in the reliability state subset $\{u, u+1,..., z\}$ is greater than *t*, while the system operation process is at the operation state z_b.

Similarly, we denote the system conditional lifetime in the reliability state subset $\{u, u+1,..., z\}$ while the system is at the operation state z_b, $b = 1,2,..., v,$ by $T^{(b)}(u)$ and the conditional reliability function of the system by the vector

$$[\boldsymbol{R}(t,\cdot)]^{(b)} = [1, [\boldsymbol{R}(t,1)]^{(b)}, ..., [\boldsymbol{R}(t, z)]^{(b)}], \quad (5.1)$$

with the coordinates defined by

$$[\boldsymbol{R}(t,u)]^{(b)} = P(T^{(b)}(u) > t | Z(t) = z_b)$$

(5.2)

for $t \in < 0, \infty)$, $u = 1,2,..., z$, $b = 1,2,..., v$.

The reliability function $[\boldsymbol{R}(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the reliability state subset $\{u, u+1,..., z\}$ is greater than *t*, while the system operation process is at the operation state z_b.

Further, we denote the system unconditional lifetime in the reliability state subset $\{u, u+1,..., z\}$ by $T(u)$ and the unconditional reliability function of the system by the vector

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1), ..., \boldsymbol{R}(t, z)], \quad (5.3)$$

with the coordinates defined by

$$\boldsymbol{R}(t,u) = P(T(u) > t)$$

for $t \in < 0, \infty)$, $u = 1,2,..., z$.

In the case when the system operation time is large enough, the coordinates of the unconditional reliability function of the system defined by (5.3) are given by

$$\boldsymbol{R}(t,u) \cong \sum_{b=1}^{v} p_b [\boldsymbol{R}(t,u)]^{(b)} \text{ for } t \geq 0,\ u = 1,2,...,z,$$

where $[\boldsymbol{R}(t,u)]^{(b)}$, $u = 1,2,...,z$, $b = 1,2,...,\nu$, are the coordinates of the system conditional reliability functions defined by (5.1)-(5.2) and p_b, $b = 1,2,...,\nu$, are the system operation process limit transient probabilities given by (2.22) (Kołowrocki, Soszyńska-Budny 2011).

5.2 ASYMPTOTIC APPROACH TO RELIABILITY OF LARGE MULTISTATE SYSTEMS AT VARIABLE OERATION CONDITIONS

In the case of large complex systems, the possibility of combining the results of the reliability joint models of complex technical systems and the results concerning the limit reliability functions of the considered systems is possible (Kołowrocki 2004, Kołowrocki 2008b, Soszyńska 2004a,b, Soszyńska 2006a, Soszyńska 2008). This way, the results concerned with asymptotic approach to estimation of non-repairable multi-state systems at variable operation conditions may be obtained. Main results concerning asymptotic approach to multi-state large system reliability with ageing components in the constant operation conditions are comprehensively presented in the work (Kołowrocki 2004) and some of these results' extentions to the systems operating at the variable conditions can be found in (Soszyńska 2004a,b, Soszyńska 2006a, Soszyńska 2007, Soszyńska 2008).

In order to combine the results on the reliability of multi-state systems related to their operation processes and the results concerning the limit reliability functions of the multistate systems, and to obtain the results on the asymptotic approach to the evaluation of the large multi-state systems reliability at the variable operation conditions, we assume the following definition (Soszyńska 2007).

Definition 5.1

A reliability function

$$\boldsymbol{\mathfrak{R}}(t,\cdot) = [1, \boldsymbol{\mathfrak{R}}(t,1),..., \boldsymbol{\mathfrak{R}}(t,z)],\ t \in (-\infty,\infty),$$

where

$$\boldsymbol{\mathfrak{R}}(t,u) = \sum_{b=1}^{v} p_b [\boldsymbol{\mathfrak{R}}(t,u)]^{(b)},\ u = 1,2,...,z,$$

is called a limit reliability function of a complex multistate system with the reliability function sequence

$$\boldsymbol{R}_n(t,\ \cdot) = [1, \boldsymbol{R}_n(t,1),..., \boldsymbol{R}_n(t,z)],\ t \in (-\infty,\infty),\ \ n \in N,$$

where

$$\boldsymbol{R}_n(t,u) \cong \sum_{b=1}^{v} p_b [\boldsymbol{R}_n(t,u)]^{(b)},\ u = 1,2,...,z,$$

if there exist normalizing constants

$$a_n^{(b)}(u) > 0,\ b_n^{(b)}(u) \in (-\infty, \infty),\ u = 1,2,...,z,\ b = 1,2,...,v,$$

such that

$$\lim_{n\to\infty}[\boldsymbol{R}_n(a_n^{(b)}(u)t + b_n^{(b)}(u), u)]^{(b)} = [\boldsymbol{\mathfrak{R}}(t,u)]^{(b)}$$

for all t from the sets of continuity points $C_{[\boldsymbol{\mathfrak{R}}(u)]^{(b)}}$ of the functions $[\boldsymbol{\mathfrak{R}}(t,u)]^{(b)}$, $u = 1,2,...,z$, $b = 1,2,...,v$.

Hence, for sufficiently large n, the following approximate formulae are valid

$$\boldsymbol{R}_n(t,\ \cdot) = [1,\ \boldsymbol{R}_n(t,1),...,\ \boldsymbol{R}_n(t,z)],\ t \in (-\infty, \infty),$$

where

$$\boldsymbol{R}_n(t,u) \cong \sum_{b=1}^{v} p_b [\boldsymbol{\mathfrak{R}}(\frac{t - b_n^{(b)}(u)}{a_n^{(b)}(u)}, u)]^{(b)},\ t \in (-\infty, \infty),\ u = 1,2,...,z.$$

The following theorems concerned with the large complex series-parallel and parallel-series exponential systems operating at the variable operation states are exemplary results that can be worked out on the basis of the results included in (Kołowrocki 2004, Kołowrocki 2008b, Soszyńska 2004b, Soszyńska 2007) for the large systems.

Theorem 5.1

If components of the multistate series-parallel regular system at the operation states z_b, $b = 1,2,...,v$, i.e., the system with the structure shape parameters such that

$$k = k_n^{(b)},\ l_1 = l_2 = ... = l_k = l_n^{(b)},\ b = 1,2,...,v,\ n \in N,$$

have the exponential reliability functions given by (3.15)-(3.16) in (Kolowrocki, Soszynska-Budny, 2011) are homogeneous, i.e.,

$$[\lambda_{ij}(u)]^{(b)} = [\lambda(u)]^{(b)},\ i = 1,2,...,k_n^{(b)},\ j = 1,2,...,l_n^{(b)},\ b = 1,2,...,v,$$

then the system unconditional multistate reliability function is given by the approximate formulae, respectively in the following cases of the system structure shape at the particular operation states:

i) $k_n^{(b)} = n,\ l_n^{(b)} > 0,$

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1),..., \boldsymbol{R}(t,z)]$$

where

$$\boldsymbol{R}(t,u) \cong 1 - \sum_{b=1}^{v} p_b \exp[-n\exp[-[\lambda(u)]^{(b)} l_n^{(b)} t]] \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,z;$$

ii) $k_n^{(b)} \to k^{(b)},\ l_n^{(b)} \to \infty,$

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1),..., \boldsymbol{R}(t,z)]$$

where

$$\boldsymbol{R}(t,u) \cong \begin{cases} 1 & \text{for } t < 0, \\ 1 - \sum_{b=1}^{v} p_b [1 - \exp[-[\lambda(u)]^{(b)} l_n^{(b)} t]]^{k^{(b)}} & \text{for } t \geq 0, \end{cases} \quad u = 1,2,...,z.$$

Theorem 5.2
If components of the multistate parallel-series regular system at the operation states z_b, $b = 1,2,...,v$, i.e., the system with the structure shape parameters such that

$$k = k_n^{(b)},\ l_1 = l_2 = ... = l_k = l_n^{(b)},\ b = 1,2,...,v,\ n \in N,$$

have the exponential reliability functions given by (3.15)-(3.16) in (Kolowrocki, Soszynska-Budny, 2011) are homogeneous, i.e.,

$$[\lambda_{ij}(u)]^{(b)} = [\lambda(u)]^{(b)},\ i = 1,2,...,k_n^{(b)},\ j = 1,2,...,l_n^{(b)},\ b = 1,2,...,v,$$

then the system unconditional multi-state reliability function is given by the approximate formulae, respectively in the following cases of the system structure shapes at the particular operation states:

i) $k_n^{(b)} = n,\ l_n^{(b)} \to l^{(b)},\ l^{(b)} > 0,$

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1),..., \boldsymbol{R}(t,z)]$$

where

$$\boldsymbol{R}(t,u) \cong \begin{cases} 1 & \text{for } t < 0, \\ \sum_{b=1}^{v} p_b \exp[-n([\lambda(u)]^{(b)} t)^{l^{(b)}}] & \text{for } t \geq 0, \end{cases} \quad u = 1,2,...,z.$$

ii) $k_n^{(b)} \to k^{(b)},\ l_n^{(b)} \to \infty,$

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1),..., \boldsymbol{R}(t,z)]$$

where

$$\boldsymbol{R}(t,u) \cong \sum_{b=1}^{v} p_b [1 - \exp[-l_n^{(b)} \exp[-[\lambda(u)]^{(b)} t]]]^{k^{(b)}} \text{ for } t \in (-\infty,\infty), u = 1,2,...,z.$$

It is possible to obtain similar and more general results for other complex multistate systems after some modification of the results included in (Kołowrocki 2004, Kołowrocki 2008b).

6 SUMMARY

In the paper, the asymptotic approach to the reliability evaluation of homogeneous and non-homogeneous series and parallel systems, homogeneous "*m* out of *n*" systems and homogeneous and non-homogeneous regular series-parallel and parallel-series systems has been presented. For these systems, in the case where their components are two-state as well in the case where they are multi-state, the classes of limit reliability functions can be fixed. Moreover, the auxiliary theorems useful for finding limit reliability functions of real technical systems composed of components

having any reliability functions can be formulated and motivated. The series-parallel and parallel-series systems have been considered in the case where their reliability structures are regular. However, this fact does not restrict the completeness of the performed analysis, since by conventional joining of a suitable number of failed components in parallel subsystems of the non-regular parallel-series systems we get the regular non-homogeneous parallel-series systems considered in the book. Similarly, conventional joining of a suitable number of components which do not fail, in series sub-systems of the non-regular series-parallel systems, leads us to the regular non-homogeneous series-parallel systems considered in the book. Thus the problem has been analysed exhaustively.
The results presented in the paper have become the basis of investigations on domains of attraction of system limit reliability functions and initiated the problem of the speed at which system reliability function sequences reach their limit reliability functions (Kołowrocki 2004).
Additionally, the results presented in the paper have initiated and become the basis for the investigations on limit reliability functions of practically important large series-"*m* out of *n*" and "*m* out of *n*"-series systems and hierarchical systems have been recently significantly developed (Kołowrocki 2004, Kołowrocki, Soszyńska 2007, Sun et al 2011). Some further consequences of these results are also given in (Kołowroci, Soszyńska–Budny 2011), where the comprehensive approach to the analysis, identification, evaluation, prediction and optimization of the complex technical systems operation, reliability, availability and safety is presented. Those all tools are useful in reliability, availability and safety optimization and operation cost analysis of a very wide class of real technical systems operating at the varying conditions that have an influence on changing their reliability and safety structures and their components reliability and safety characteristics.

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