JOINT IMPORTANCE MEASURES IN NETWORK SYSTEM

V. M. Chacko [1] **and M. Manoharan** [2]

•

[1]Department of Statistics, St.Thomas College, Thrissur-680001,Kerala, India
[1]Department of Statistics, University of Calicut, Kerala, India-673635
e-mail: chackovm@gmail.com, [1]

Abstract: Many real world systems (electric power, transportation, telecommunication, etc) are multistate systems composed of multistate components in which system reliability can be computed in terms reliabilities of its components. Such systems may be regarded as flow networks whose arcs (components) have independent, discrete, and multi-valued random capacities. An arc can, at different conditions, be characterized by different performance levels, causing network system to work with different levels of output performance. The criticality of such arcs must be measured with reference to their performance level and reliability, and its contribution to the overall system output performance measure(OPM). In this paper, we introduce a generalized concept of importance measures and joint importance measures for the flow network made up of multistate arcs with respect to output performance measures (expected performance, reliability and availability). An approach based on the universal generating function (UGF) for the evaluation of the proposed joint importance measures is introduced. An illustrative example is given.

Keywords: Network reliability, availability, discrete state arc, joint importance measure, UGF.
MSC 2000. 68M10, 90B25, 90B15.

1. Introduction

Since the very early times of reliability engineering, the network reliability is one of the main subjects of research. The network reliability theory has been applied extensively in many real-world systems such as computer and communication systems, power transmission and distribution systems, transportation systems, oil/gas production systems etc [8]. Network reliability evaluation approaches exploit a variety of tools for system modeling and reliability index calculation. Network reliability problems are generally classified based on the method used to transfer the flow (or signal) and how the flow conservation law is satisfied. Typically there are two categories; the multistate arc network (MAN) and the multistate node network (MNN). In MAN, each arc has a non-negative integer valued discrete random variable capacity (multistate arc) and all flows in the network obey the conservation law. Apparently in MNN, each node is a multistate node with discrete states determined by a set of nodes receiving the signal directly from it without satisfying conservation law. Both have their own applications; for example electrical power distribution system can be modeled by MAN, and computer networks or cellular phone networks can be modeled as MNN.

The standard mathematical and statistical theory of system reliability assumes both system and component behavior are of binary nature, functioning (state 1) and failed (state 0), [1]. However, in some systems, when components may be operating in a degraded state, the system may be operating in degraded state, and the system may still provide an acceptable level of service, [2]. The network reliability evaluation for complex designs relies on enumerative techniques, [12]. The flow reliability problem for the directed capacited-flow network in which the capacity of each arc has $M+1$ value from source to sink is generalized as a multistate system model, [10]. A graph theoretic method is used for the reliability evaluation of multistage interconnection networks with multistate elements, [14].

Importance measure (IM) quantifies the criticality of a particular component within a system design. They have been widely used as tools for identifying system weakness, and to prioritize reliability improvement activities, [6]. They can also provide valuable information for the safety and efficient operation of the system. In multistate system (MSS), IMs characterizes, for a given component, the most important component state with regard to its impact on system reliability. The knowledge about the IM can be used as a guide to provide redundancy so that system reliability is increased. Thus, measures that can differentiate such an impact are highly desirable. In general, there are two ways to improve the reliability of a binary system, 1) increase the reliability of individual components, and/or 2) add redundant components to the system. Composite importance measures are developed with the aim of identifying and ranking particular arcs (components) in a network system depending on their impact on the multistate network reliability behavior, [13]. Joint reliability importance (*JRI*) of two or more components is a quantitative measure of the interactions of two or more components or states of two or more components, [5]. It is investigated to provide information on the type and degree of interactions between two or more components by identifying the sign and size of it, [5,15]. The value of *JRI* represents the degree of interactions between two or more components with respect to system reliability. Joint structural importance (*JSI*) is used when the component reliabilities are not available, [15]. Joint structural and joint reliability importance measures for any number of multistate components in the MSS are useful for the design engineers, [5], [17]. For the MSS with multistate components, the problem related to MSS reliability improvement is still evolving. The problem of finding the joint importance of more than two arcs in a network system with various output performance measures (e.g. reliability, availability, etc) still remains unsolved. To solve this problem, methods dependent on the information obtained from multistate IMs and joint importance measure(JIM)s can be developed for efficient resource allocation. Many of the engineering systems are modeled by networks (electric power generation system, transportation system, telecommunication network system, etc) (see [7], [10], and [14]), hence the development of joint importance measures of two or more arcs with different output performances (e.g. productivity, capacity, etc) in a directed network with multistate performance levels is quite desirable. In order to answer this problem, we introduce the JIMs of two or more arcs in multistate directed network system with various output performance measures (expected performance, reliability and availability). We provide an algorithm based on universal generating function (UGF) for the evaluation of joint importance measure when network system has different output performances.

This paper is organized as follows. In section 2, we define the JIMs in network system with various output performance measures (expected performance reliability, and availability). Section 3 considers the application of UGF for the JIM evaluation. Illustration is given in section 4 followed by conclusions in the last section.

2. Joint importance measures of arcs in multistate network system

Consider a directed multistate network made up of n arcs. Each arc i may be in one of $M_i + 1$ states, $\{0,1,...,M_i\}$, $i \in \{1,2,...,n\}$. Let $W(t)$ output performance of the multistate network at time t which takes the values w_i, i=0, 1, 2, … M, where $M = \max_i \{M_i\}$, depending on the system state i at time t. The two vectors of the system performance realizations, w={w_i, $0 \le i \le M$}, and of the system state probabilities, p={p_i, $0 \le i \le M$}, define the system output performance distribution. Let $\phi(t)$ is the state of the MSS at time t.

We use some measures of the performance of a MSS for obtaining joint importance measures.

The steady-state the probability distribution of the system states is:

$$p_i = \lim_{t \to \infty} \Pr\{\phi(t) = i\} = \lim_{t \to \infty} \Pr\{W(t) = w_i\}, \quad 0 \le i \le M .$$

An associated simple measure of system output performance is its expected value of system state defined, in the steady-state, as:

$$E_s[\phi(t)] = \sum_i ip_i \ .$$

A similar measure of system output performance is its expected value of system output performance, in the steady-state, as:

$$E = \sum_i w_i p_i \ .$$

When applied to MSS, the concept of availability is related to the ability of the system to meet a required demand w_k, corresponding to state k. The general definition of instantaneous multistate system availability is, then:

$$A(t) = \Pr\{\phi(t) \geq k\} = \Pr\{W(t) \geq w_k\} \ .$$

If the system is under operation without break up to time t, then $A(t)$ is the system reliability:

$$R(t) = \Pr\{\phi(t) \geq k\} = \Pr\{W(t) \geq w_k\} \ .$$

The MSS stationary availability is defined as

$$A = \sum_{i=0}^{M} p_i \, 1(\mathrm{w_i} - w_k) \ .$$

Let $G = (N, A)$ represent a stochastic capacited network with known demand d from a specified source node s to a specified sink node t. N represents the set of all nodes and $A = \{a_i \mid 1 \leq i \leq n\}$ represents the set of all arcs. The current state (capacity) of arc a_i, represented by $x_i \in \{0,1,...,M_i\}$, the range of states of arc a_i. The vector $p_i = (p_{i0}, p_{i1}, p_{i2}, ..., p_{iM_i})$ represents the probability associated to each of the values taken by x_i. The system state vector $x = (x_1, x_2, ..., x_n)$ denotes the state of all the arcs of the network system. Function $\phi(x): Z^n \to Z$, where $Z = \{0,1,2,...,M\}$, $M = \max_i\{M_i\}$, maps the system state vector into system state. That is, $\phi(x)$, is the network capacity from source to sink under system state vector x, which represents a multistate structure function, [2]. Network reliability may be defined as the probability that a demand of d units can be supplied from source to sink through the multistate arcs. We shall make the following assumptions for the network reliability system.

1. Arc states are stochastically independent.
2. The structure function $\phi(x)$ is statistically coherent. That is, improving an arc performance cannot cause to degrade the performance of the network system and all arcs are relevant.

Joint reliability importance (*JRI*) of the two edges in an undirected network in binary nature is an extension of the marginal reliability importance (*MRI*) of edges, [6]. In an undirected network, reliability is the probability that source and terminal are connected by working edges, [6].

For an undirected stochastic network $G(N, E)$, where $E = \{e_i \mid 1 \leq i \leq n\}$ is set of all edges, N, the set of nodes, let $R(G)$ represents the probability that the source and terminal are connected by working edges and $q = (q_1, q_2, ..., q_n)$ where $q_i = Pr\{e_i \in E \text{ is working}\}$. *MRI* of edge e_i in an undirected network is defined as $I_G(i) = \frac{\partial R(G)}{\partial q_i}$, [6]. Again *JRI* of two edges is defined as follows.

Definition 1. The *JRI* of two edges e_i and e_j is the second order partial derivative of reliability of an undirected network with respect to reliabilities of both edges:

$$I_G(i,j) = \frac{\partial^2 R(G)}{\partial q_i \partial q_j}.$$

An explicit expression for this *JRI* of two edges is

$$I_G(i,j) = R(G^* i^* j) - R(G^* i - j) - R(G^* j - i) + R(G - i - j)$$

where $G^* i - j$ represents G with edge e_i contracted and edge e_j is deleted. The *JRI* is expressed in terms of *MRI* of edges in same sub-graphs as,

$$I_G(i,j) = I_{G^*j}(i) - I_{G-j}(i) \text{ and } I_G(i,j) = I_{G^*i}(j) - I_{G-i}(j).$$

Alternatively the following relationships are obtained.

$$I_G(i,j) = \frac{I_G(i) - I_{G-j}(i)}{p_j} \text{ and } I_G(i,j) = \frac{I_G(j) - I_{G-i}(j)}{p_i}.$$

We now proceed with the problem of measuring joint importance in the directed network system with respect to expected performance. First we find the *JRI* of any number of arcs in the network where the arc capacity is represented as finite discrete state in nature. That is each arc can take the value in a discrete state space $\{0,1,...,M_i\}$ where M_i represents the maximum flow (best state) through the arc *i*. For finding the *JRI* of more than two arcs, we follow the method for finding *JRI* in MSS, [5]. Suppose for instance the probability distribution of each arc is unknown, then we use the joint structural importance of multistate system (*JSIM*) of more than two components, [5]. The *JSIM* (*i*, *j*), for two components *i* and *j* is given by,

$$JSIM(i,l) = \sum_{m=1}^{M_i} \sum_{k=1}^{M_l} \{SIM(i,l;m,k) - SIM(i,l;m,k̃)\}$$

where $SIM(i,l;m,k) = \dfrac{\sum_{\vec{X}_{il}} \sum_{q=1}^{j} \chi(\phi(m_i,k_l,\vec{X}_{il}) = j, \phi(m̃_i,k_l,\vec{X}_{il}) = j-q)}{(M+1)^{n-2}}$, $m̃ = m-1$, and

$\overline{X}_{il} = (x_1, x_2, ..., m_i, ..., k_l, ..., x_n)$, the state space vector of system components.

Here χ(true)=1 and χ(false)=0, $\phi(m_i,k_l,\overline{X}_{il}) = j, \phi(m̃_i,k_l,\overline{X}_{il}) = j-q$ determines the critical path vector to the level j with state *m* of component *i*. The *JSIM* (*i,j,k*) for three components is

$$JSIM(i,l,r) = \sum_{k=1}^{M_i} \sum_{n=1}^{M_l} \sum_{m=1}^{M_r} \{JSIM(i,l,r;m,k,n) - JSIM(i,l,r;m,k,ñ)\},$$

where $JSIM(i,l,r;m,k,n) = SIM(i,l,r;m,k,n) - SIM(i,l,r;m,k̃,n)$. So in order to find the *JSIM* of three arcs we have to find *JSIM* of two arcs for each state of third arc and, take successive difference and total sum. Again the change in *JSIM* of three components with fourth component provides *JSIM* of four components. Thus proceeding like this we can find *JSIM* of any number of arcs.

Suppose that the arc probabilities are known. Then to find the joint reliability importance of more than two multistate arcs for the network, one may proceed as follows. The joint reliability importance (*JRIM*) of MSS, for *k* components is defined as follows, [5]. The joint reliability importance of state b_1 of component a_1, state b_2 of component a_2,..., state b_k of the component a_k $(k \le n)$ of the MSS is

$$JRIM(a_{1,},...,a_k;b_1,...,b_k)=\frac{\partial^k E_s}{\partial R_{a_1}b_1\partial R_{a_2}b_2...\partial R_{a_k}b_k},\ k=2,...,n,$$

where $E_s=\sum_{j=0}^{M}P(\phi(x)\geq j)$ is the expected performance of system and $\forall i\in\{1,2,...,k\}$, $R_{a_i}b_i=P(x_{a_i}\geq b_i)$ are the component reliabilities. Thus in finding the *JRIM* of *k* multistate arcs, we have to find the k^{th} partial derivative of the overall expected performance of network with respect to the reliabilities of each arc under consideration. For instance we consider the *JRIM* of states of three components. Let $P_{mkn}=P(\phi(m_i,k_l,n_r,\overline{X}_{ilr})\geq j)$ and $p_{im}=P(X_i\geq m)$. For *k*=3, i.e. differentiating E_s partially with respect to p_{im}, p_{lk} and p_{rn}, we get,

$$\frac{\partial^3 E_s}{\partial p_{im}\partial p_{lk}\partial p_{rn}}=[\,P_{mkn}-P_{m\bar{k}n}-P_{\bar{m}kn}+P_{\bar{m}\bar{k}n}\,]-[\,P_{mk\bar{n}}-P_{m\bar{k}\bar{n}}-P_{\bar{m}k\bar{n}}+P_{\bar{m}\bar{k}\bar{n}}\,].$$

But observe that $P_{mkn}-P_{m\bar{k}n}-P_{\bar{m}kn}+P_{\bar{m}\bar{k}n}$ is *JRIM* of states of two components when third component is in state *n*. Therefore *JRIM* of states three components are expressed in terms of *JRIM* of states of two components as follows:

$$\frac{\partial E_s^{\ 3}}{\partial R_i m\,\partial R_l k\partial R_r n}=[\,\frac{\partial^2 E_s}{\partial R_i m\,\partial R_l k}\,]_{n_r}-[\,\frac{\partial^2 E_s}{\partial R_i m\,\partial R_l k}\,]_{\bar{n}_r}$$

JRI working states for edges can be written as $\frac{\partial^2 R(G)}{\partial R_i 1\partial R_l 1}$ where $R_i 1=p_i$. It shows that the above result holds with binary nature of edges*, i. e., M*=1. Hence, the results of *JRI* of two edges in a binary network, [5], can be considered as a generalization of the results *JRIM*, [6], to any number of binary and multistate edges when considering undirected network system. Thus we have the following theorem for three arcs of a directed network system.

Theorem 1. The joint reliability importance of three arcs in a multistate network with multistate arcs is

$$\sum_{m=1}^{M_i}\sum_{k=1}^{M_l}\sum_{n=1}^{M_r}\frac{\partial E_s^{\ 3}}{\partial R_i m\,\partial R_l k\partial R_r n}=\sum_{m=1}^{M_i}\sum_{k=1}^{M_l}\sum_{n=1}^{M_r}([\frac{\partial^2 E_s}{\partial R_i m\,\partial R_l k}]_{n_r}-[\frac{\partial^2 E_s}{\partial R_i m\,\partial R_l k}]_{\bar{n}_r})$$

where E_s is the expected output performance of network, $R_i m$, $R_l k$, *and* $R_r n$ are the reliabilities of arcs *i, l,* and *r* with respect to performance level *m, k,* and *n* respectively.□

In the above discussed joint reliability importance measures and joint structural importance measures, we used the expected performance of the network as output performance measure. But in order to find the JIMs with respect to other output performance measures, reliability and availability, of the multistate network systems, we proceed as follows.

When the generic j-th multistate arc is considered, one can introduce a performance threshold α and divide the complete ordered set of its states into two ordered subsets corresponding to the arc performance above and below the level α, respectively. By so doing, we re-introduce a collectively binary logic for the arc's states. Let arc *j* be constrained to performance below α, while the rest of arcs of the network system are not constrained: we denote by $OS_j^{\leq\alpha}$ the network system OPM (reliability or availability) obtained in this situation. Similarly, we denote by $OS_j^{>\alpha}$ the network system OPM resulting from the dual situation in which arc j is constrained to performances above α. The network system performance measures so introduced rely on a restriction of the achievable performance of the network arcs. Different modeling assumptions in the enforcement of this restriction will lead to different performance values. Using the measures $OS_j^{\le\alpha}$ and $OS_j^{>\alpha}$, we can define Birnubaum importance measures for multistate elements

$$OS^{\alpha}{}_{j} = OS^{>\alpha}{}_{j} - OS^{\le\alpha}{}_{j}.$$

Suppose that $OS^{>\beta,\alpha}{}_{i,j}$ represents the Birnubaum importance of the component *i* when component *j* is restricted to the performance above level β. Similarly define $OS^{\le\beta,\alpha}{}_{i,j}$ the Birnubaum importance of the component *i* when component *j* is restricted to below level β. Thus we can define the joint importance of two components *i* and *j* to the network system performance as

$$OS^{\alpha,\beta}{}_{i,j} = OS^{\ge\beta,\alpha}{}_{i,j} - OS^{\le\beta,\alpha}{}_{i,j}.$$

Similarly we can obtain the higher order joint importance measures for more than two arcs, i.e., for example, to measure the improvement of joint importance of two arcs with respect to the interactive effect of more than two arcs, at first we shall calculate change in the joint importance of two arcs with respect to the change of third arc. If there is any change in the joint importance of two arcs due to change in performance of third arc from upper states to lower states we can say that there is an interactive effect for three arcs for the network OPM improvement. We shall find the joint importance measures at steady state system performance in the following section.

3. Application of UGF

In MSS modeled by networks with respect to various output performances, we modify the above joint importance measures. The UGF is found to be a useful tool in assessment of output performance measures of the network systems, [11]. The method of UGF generalizes the technique that is based on using a well known ordinary generating function. The basic ideas are introduced by Ushakov in 1987,. [11]. The approach proved to be very convenient for numerical realization. It requires relatively small computational resources for evaluating MSS reliability indices and, therefore, can be used in complex reliability operations. Importance measure evaluation in MSSs using UGF can be see in Ref. [9].

The MSS model includes the performance distribution of all arcs and the system structure function: x_j, $p_j, 1 \le j \le n$, $\phi(x_1, x_2, ..., x_n)$, where any system element *j* can have finite number, M_j of discrete states, and its performance distribution is represented by ordered sets $x_j : (x_{j1}, x_{j2}, ..., x_{jM_j})$ and $p_j : (p_{j1}, p_{j2}, ..., p_{jM_j})$ that relate the probability of each state with performance corresponding to this state.

The UGF of a discrete variable *X* corresponding to the state of an arc is defined as the polynomial

$$U(Z) = \sum_{k=0}^{M} p_k Z^{x_k}$$

where the discrete random variable *X* has *M* possible values and p_k is the probability that *X* is equal to x_k. In order to represent all the possible combinations of states of the two arcs a_1 and a_2, one has to relate the corresponding probabilities of states of two multistate arcs subsystem with values of the vector $\psi(x_{a_1}, x_{a_2})$ in these states. For these purpose, we consider a composition operator Ω over UGFs of individual multistate arcs which takes the following form for a pair (i, j) of multistate arcs.

$$U_{i,i+1}(Z) = \Omega(U_i(Z), U_{i+1}(Z)) = \Omega(\sum_{k=1}^{M_i} p_{ik} Z^{x_{ik}}, \sum_{n=1}^{M_{i+1}} p_{i+1,n} Z^{x_{i+1n}}) = \sum_{k=1}^{M_i} \sum_{n=1}^{M_{i+1}} p_{ik} p_{i+1n} Z^{\psi(x_{ik}, x_{i+1n})}$$

The resulting polynomial $U_{i,i+1}(Z)$ represents the probability distribution of the subsystem containing arcs *i* and *i*+1. Applying the operator Ω to all other arcs one by one we get the resulting polynomial that takes the form,

$$U_{1,2,\ldots,n}(Z) = \sum_{k=0}^{M} q_k Z^{y_k}$$

The polynomial represents distribution of state of connections between source and sink of the entire network. This polynomial relates the probabilities of all the possible states of whole network, q_k, with the output performance corresponding to these states, y_k. Thus we can obtain the reliability of network as

$$R = \sum_{k=0}^{M} q_k . I(demand\ of\ d\ units\ supplied\ from\ s\ to\ t)$$

where $I(.)$ is the indicator function. To evaluate the joint importance measures we need the steady state distribution of the observed performance of the network system under some constraints. In order to use the UGF in joint importance measure evaluation, we use the following approach.

Let O_{jk} be the output performance of multistate network system when arc *j* is in fixed state *k* while the rest of the arcs evolve stochastically among their corresponding states with steady-state performance distributions $\{x_{il}, p_{il}\}$, $1 \le i \le n, 0 \le l \le M_i$. Assume that the arc *j* is in one of its states, *k,* with performance not greater than α. We denote by $k_{j\alpha}$ the state in the ordered set of states of arc *j* whose performance $x_{jk_{j\alpha}}$ is equal or immediately below α, i.e., $x_{jk_{j\alpha}} \le \alpha < x_{jk_{j\alpha}+1}$. The conditional probability of the arc *j* being in a generic state *k* characterized by a performance $X_j = x_{jk}$ not greater than a pre-specified level threshold α $(k \le k_{j\alpha})$ is:

$$\Pr[X_j = x_{jk} \mid k \le k_{j\alpha}] = p_1{}^*{}_{jk} = \frac{p_{jk}}{\sum_{r=0}^{k_{j\alpha}} p_{jr}} = \frac{p_{jk}}{p^{\le\alpha}{}_j}.$$

Similarly, the conditional probability of arc *j* being in a state *k* when it is known that $k > k_{j\alpha}$ is

$$Pr[X_j = x_{jk} \mid k > k_{j\alpha}] = p_2{}^*{}_{jk} = \frac{p_{jk}}{Pr[k > k_{j\alpha}]} = \frac{p_{jk}}{\sum_{r=k_{j\alpha}+1}^{M_j} p_{jr}} = \frac{p_{jk}}{p^{>\alpha}{}_j}.$$

Now consider the joint probability distribution of two arcs *i* and *j*, for $X_i = x_{ik}, X_j = x_{jh}$ given four additional restrictions, (1) $k > k_{i\alpha}, h > h_{j\beta}$, (2) $k \le k_{i\alpha}, h > h_{j\beta}$, (3) $k > k_{i\alpha}, h \le h_{j\beta}$ and (4) $k \le k_{i\alpha}, h \le h_{j\beta}$. Thus now under the consideration that the arcs are independent we could arrive at probability distributions given below. That can be computed as earlier result for independent arcs as follows. Let

$$Pr[X_{ik} = x_{ik}, X_j = x_{jh} \mid k \le k_{i\alpha}, h \le h_{j\beta}] = p_1{}^{**}{}_{hk} = \frac{p_{ik} p_{jh}}{\sum_{r=0}^{k_{i\alpha}} p_{ir} \sum_{m=0}^{h_{j\beta}} p_{jm}},$$

$$Pr[\,X_i = x_{ik}\,, X_j = x_{jh} \mid k \le k_{i\alpha}\,, h > h_{j\beta}\,] = p_2{}^{**}{}_{hk} = \frac{p_{ik}\,p_{jh}}{\sum_{r=0}^{k_{i\alpha}} p_{ir} \sum_{m=h_{j\beta}+1}^{M_j} p_{jm}},$$

$$Pr[\,X_i = x_{ik}\,, X_j = x_{jh} \mid k > k_{i\alpha}\,, h \le h_{j\beta}\,] = p_3{}^{**}{}_{hk} = \frac{p_{ik}\,p_{jh}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir} \sum_{m=0}^{h_{j\beta\, j}} p_{jm}},$$

and

$$Pr[\,X_i = x_{ik}\,, X_j = x_{jh} \mid k > k_{i\alpha}\,, h > h_{j\beta}\,] = p_4{}^{**}{}_{hk} = \frac{p_{ik}\,p_{jh}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir} \sum_{m=h_{j\beta}+1}^{M_j} p_{jm}}.$$

Similarly we can find joint distributions of any number of arcs with the specified restrictions. We define $OS^{\le\alpha}{}_j$ as the network output performance measure (e.g. reliability or availability) obtained when arc j is forced to visit only states with performance not greater than α:

$$OS^{\le\alpha}{}_j = \sum_{k=0}^{k_{j\alpha}} \frac{p_{jk}}{p^{\le\alpha}{}_j} O_{jk}\ .$$

Similarly, we define as $OS^{>\alpha}{}_j$ the network output performance measure obtained under the condition that the arc j stays only in states with performance greater than α:

$$OS^{>\alpha}{}_j = \sum_{k=k_{j\alpha}+1}^{M_j} \frac{p_{jk}}{p^{>\alpha}{}_j} O_{jk}\ .$$

Thus the Birnubaum importance takes the form,

$$OS^{\alpha}{}_j = OS^{>\alpha}{}_j - OS^{\le\alpha}{}_j\ .$$

In order to compute the joint importance of two arcs i and j, i.e., let $OS^{\alpha,\beta}{}_{ij}$ denote the joint importance of two arcs with respect to the network output performance measure OS, then

$$OS^{\alpha,\beta}{}_{ij} = OS^{>\beta,\alpha}{}_{ij} - OS^{\le\beta,\alpha}{}_{ij}$$

i.e.,

$$OS^{\alpha,\beta}{}_{ij} = \sum_{k=0}^{k_{i\alpha}} \sum_{h=0}^{h_{j\beta}} p_1{}^{**}{}_{hk} O_{ik,jh} - \sum_{k=0}^{k_{i\alpha}} \sum_{h=h_{j\beta}+1}^{M_j} p_2{}^{**}{}_{hk} O_{ik,jh} - \sum_{k=k_{i\alpha}+1}^{M_i} \sum_{h=0}^{h_{j\beta}} p_3{}^{**}{}_{hk} O_{ik,jh} + \sum_{k=k_{i\alpha}+1}^{M_i} \sum_{h=h_{j\beta}+1}^{M_j} p_4{}^{**}{}_{hk} O_{ik,jh}$$

Similarly by finding change in joint importance of two arcs with respect to third arc, we get the joint importance of three arcs. Continuing like this we get the joint importance of any number of arcs with respect to network output performance measure and state space restricted probabilities of all arcs.

In order to obtain the state space restricted measures, one has to modify the UGF of arcs as follows,

$$U^{\le}{}_i(Z)=\sum_{k=0}^{k_{i\alpha}}\frac{p_{ik}}{p^{\le\alpha}{}_i}Z^{x_{ik}},\qquad U^{>}{}_i(Z)=\sum_{k=k_{i\alpha}+1}^{M_i}\frac{p_{ik}}{p^{>\alpha}{}_i}Z^{x_{ik}},\qquad U^{\le}{}_j(Z)=\sum_{h=0}^{h_{j\beta}}\frac{p_{jk}}{p^{\le\beta}{}_j}Z^{x_{jk}},$$

$$U^{>}{}_j(Z)=\sum_{h=h_{j\beta}+1}^{M_j}\frac{p_{jk}}{p^{>\beta}{}_j}Z^{x_{jk}},\ U^{\le,\le}{}_{ij}(Z)=\sum_{k=0}^{k_{i\alpha}}\frac{p_{ik}}{p^{\le\alpha}{}_i}Z^{x_{ik}}\sum_{h=0}^{h_{j\beta}}\frac{p_{jh}}{p^{\le\beta}{}_j}Z^{x_{jh}}=\sum_{k=0}^{k_{i\alpha}}\sum_{h=0}^{h_{j\beta}}\frac{p_{ik}p_{jh}}{p^{\le\alpha}{}_i p^{\le\beta}{}_j}Z^{\psi(x_{ik},x_{jh})},$$

$$U^{>,\le}{}_{ij}(Z)=\sum_{k=k_{i\alpha}+1}^{M_i}\frac{p_{ik}}{p^{>\alpha}{}_i}Z^{x_{ik}}\sum_{h=0}^{h_{j\alpha}}\frac{p_{jh}}{p^{\le\beta}{}_j}Z^{x_{jh}}=\sum_{k=k_{i\alpha}+1}^{M_i}\sum_{h=0}^{h_{j\beta}}\frac{p_{ik}p_{jh}}{p^{>\alpha}{}_i p^{\le\beta}{}_j}Z^{\psi(x_{ik},x_{jh})},$$

$$U^{>,>}{}_{ij}(Z)=\sum_{k=k_{i\alpha}+1}^{M_i}\frac{p_{ik}}{p^{>\alpha}{}_i}Z^{x_{ik}}\sum_{h=h_{j\beta}+1}^{M_j}\frac{p_{jh}}{p^{>\beta}{}_j}Z^{x_{jh}}=\sum_{k=k_{i\alpha}+1}^{M_i}\sum_{h=h_{j\beta}+1}^{M_j}\frac{p_{ik}p_{jh}}{p^{>\alpha}{}_i p^{>\beta}{}_j}Z^{\psi(x_{ik},x_{jh})}$$

and $$U^{\le,>}{}_{ij}(Z)=\sum_{k=0}^{k_{i\alpha}}\frac{p_{ik}}{p^{\le\alpha}{}_i}Z^{x_{ik}}\sum_{h=h_{j\beta}+1}^{M_j}\frac{p_{jh}}{p^{>\beta}{}_j}Z^{x_{jh}}=\sum_{k=0}^{k_{i\alpha}}\sum_{h=h_{j\beta}+1}^{M_j}\frac{p_{ik}p_{jh}}{p^{\le\alpha}{}_i p^{>\beta}{}_j}Z^{\psi(x_{ik},x_{jh})},$$

when evaluating UGF of $OS^{\le\alpha}{}_i,\ OS^{>\alpha}{}_i,\ OS^{\le\beta}{}_j,\ OS^{>\beta}{}_j,\ OS^{\le\beta,\alpha}{}_{ij},\ and\ OS^{>\beta,\alpha}{}_{ij}$. We use the following algorithm for evaluation of $OS^{\le\alpha}{}_i, OS^{>\alpha}{}_i, OS^{\le\beta}{}_j, OS^{>\beta}{}_j, OS^{\le\beta,\alpha}{}_{ij}, and\ OS^{>\beta,\alpha}{}_{ij}$.

Obtain the u-functions of all of the system elements. If the system contains a pair of elements connected in parallel or in series, replace this pair with an equivalent macro-element with u-function obtained by 'sum' or 'min' operator for $\psi(\cdot)$. If the system contains more than one element, do it again and again. Then, determine the u-function of the entire series-parallel system as the u-function of the remaining single equivalent macro-element. The system probability and performance distributions are represented by the coefficients and exponents of this u-function, corresponding to the state probabilities and performance levels, respectively. Compute the system OPM for the given level with the given vectors of the state probabilities and performance levels.

4. Illustrative example

For the network in figure 1, it is desired to obtain the probability that a demand of 10 units can be supplied from source to sink, [13]. Here the system can be considered as the MSS in Ref. [5]. Table 1 presents arc probabilities. In table 2, we computed *JRIM* of pair of arcs with α=2, β=2 which influences system most with respect to system output met the demand or not.

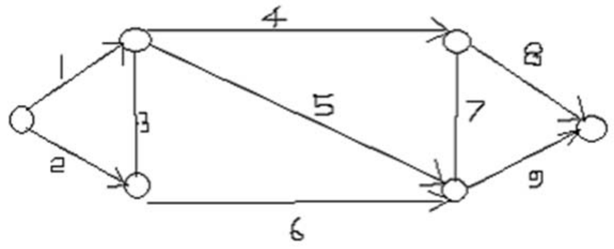

Fig1

It shows the pair (8,9) has largest *JRIM* with respect to system reliability.

4. Conclusion

In this paper joint importance measures of two or more arcs in multistate arc network with various output performance measures are developed. The procedure of evaluating joint importance measures using UGF is proposed. The proposed measures can be used in any systems modeled as multistate networks having various output performance measures with multistate arcs.

Table 1

Arc	State	State probabilities
1	0 3 4 8	0.005 0.005 0.01 0.98
2	0 3 4 6	0.02 0.01 0.015 0.955
3	0 3	0.02 0.98
4	0 3 4	0.01 0.015 0.975
5	0 3	0.02 0.98
6	0 3 6	0.005 0.02 0.975
7	0 3	0.01 0.99
8	0 3 4 6	0.01 0.015 0.005 0.97
9	0 3 4 8	0.02 0.01 0.01 0.96

Table 2

Pair	(1,2)	(8,9)	(1,4)	(2,6)
JRI	0.000394	0.089946	-0.894983	-0.01469

References

[1]. R. E. Barlow, and F. Proschan, Statistical theory of reliability and life testing, Holt Rinheart, NewYork,1975.
[2] R. E. Barlow, and A. Wu, Coherent system with multistate components, Math. Oper. Res., 3 (1978), 275-281.
[3] W. Birnbaum, On the importance of different components in a multi-component system, P R. Krishnaiah, (Ed.), *Multivariate analysis* II, 591-592, Acadamic Press, NewYork, 1969.
[4] V. C. Bueno, On the importance of components for multistate monotone systems, Stat. Prob. Lett., 7 (1989), 51-59.
[5] V. M. Chacko, and M. Manoharan, Joint importance measures for the multistate system, Advances in Performance and Safety of Complex systems, A. K.Verma, P. K. Kapur, and S. G. Ghadge (Eds), 308-314, Macmillan, India, 2008.
[6] J. S. Hong, and C. H. Lie, Joint reliability importance of two edges in an undirected network, IEEE Trans. Reliab., 47 (1993), 97-101.
[7] G. Levitin, Optimal reliability enhancement for multistate transmission networks with fixed transmission times, Reliab. Eng. Syst. Saft.,76 (2002) 287-299.
[8] G. Levitin, Reliability of acyclic multistate networks with constant transmission characteristics of lines, Reliab. Eng. Syst. Saft., 78 (2002) 297-305.
[9] G. Levitin, and A. Lisnianski, Importance and sensitivity analysis of multistate systems using the universal generating function, Reliab. Eng. Syst. Saf, 65 (1999), 271-282.
[10] Y. K. Lin, Using minimal cuts to study the system capacity for a stochastic flow network in two-commodity case, Computers and Oper. Reser., 30 (2003) 1595-1605.
[11] A. Lisnianski and G. Levitin, Multistate System Reliability-assesment, optimization, and evaluation, World Scientific, Singapore, 2003.
[12] S. Patra, and R. B. Misra, Evaluation of probability mass function of flow in a communication network considering a multistate model of network links, Microel.Reliab.,36 (1996) 415-621.
[13] J. E. Ramirez-Marquez, and D. W. Coit, Multistate component criticality analysis in multistate systems, Adv. Saf. Reliab. (Ed. K. Kolowrocki), 1671-1677, Taylor and Francis group, London, 2005.
[14] C. R. Thripathy, S. Patra, R. B. Misra, and N. Mahapatra, Reliability evaluation of multistate interconnection networks with multistate elements, Microel.Reliab., 36 (1996)423-428.
[15] S. Wu, Joint importance of multistate systems, Comp. Indust. Eng., 49 (2005), 63–75.
[16] W. C Yeh, The *k-out-of-n* acyclic multistate node networks reliability evaluation using universal generating function method, Reliab. Eng. Syst. Saf., 91 (2006), 800-808.
[17] V M Chacko and M. Manoharan, Joint importance measures for multistate reliability systems, OPSEARCH, Volume 48, Number 3, 257-278, DOI: 10.1007/s12597-011-0048-z