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This paper presents an innovative approach to solve probability distributions of a closed feed back loop type queuing system with general service time distribution. This model is applied to a multi-processors system where some of its nodes are performed a repair procedure during a node's malfunction condition. Our model is appropriate for a multiprocessor system that employs a common bus or for a multi-node system in computer network. A meticulous analysis of the system's model has been conducted and numerical results have been obtained to scrutinize the proposed model.

An analysis of the limit $\lim_{n\to\infty} P_n = A$ of connectivity probability (CP) P_n of complete graph with n nodes and independent arcs which have working probability n^{-a} is made. It is proved that for $0 < a < 1$ we have the equality $A = 1$ and for $1 < a$ the equality $A = 0$.

An expression for the function of readiness duplicated system in the Laplace transform and its significance is the steady state for arbitrary distributions of time to failure and recovery of constituent elements. In these expressions have introduced the parameter reliability monitoring the state of the elements after their refusal. The value of this option allows you to take into account the duration of the recovery elements after their refusal.
Because of this, a generalization of the result obtained B.V.Gnedenko, ready for the duplicated system with control of the state elements.

This paper describes some models and measures of reliability for multistate systems. The expected cumulative reward for the continuous time Markov reward models are used for deriving the structure function for a multistate system where the system and its components can have different performance levels ranging from perfect functioning to complete failure. The suggested approach presents with respect to the non-homogeneous and homogeneous Markov reward model of two stochastic process for computation of these availability and reliability measures. A particular case for three levels is analyzed numerically by assuming Weibull and exponential distributions for failure and repair times.
Keywords: Markov reward model, demand, multistate system, availability and reliability measures.

Using such a DSS can significantly reduce the time to evaluate the situation and making decisions on the organization of firefighting units to eliminate fire techniques to produce efficient thinking at training officers fire departments railroads, and develop knowledge base DSS.
Future directions of the development of DSS are its use in research networks, comprising some QS with different characteristics, and models that take into account the loss of combat vehicles and servants, and the sequence of different disciplines to focus on capabilities rolling stock, affected by fire.

In article generalization of a problem of optimum quantization of a casual signal with blanks between quanta is presented. Unlike known works the law of distribution of a casual signal with quanta is received. Instead of the integer decision of a problem the approached asymptotic decision is offered at a great number of quanta and the estimation of its accuracy is given. Besides, the decision of the given problem is received at fuzzy values of parameters of a blank and a population mean of an initial random variable with the normal law of distribution.
Keywords: a random variable, asymptotic the optimum decision, quantum, probability density, uniform and normal distributions, fuzzy parameter, accessory function.

The author presents a general analytical solution determining “the Occurrence probability of a sequence of events each following Poison Stochastic Process”. Generally, this probability is described under the form of an integral equation of order “n”. Where “n” is number of the elementary events in the examined sequence.
As far as the author can tell, the solution is original. It will be of a great interest to a wide range of system reliability problems such as: sequential calculations, dominos effects, dynamics fault trees, Markov systems, priority AND gates, events trees, stochastic optimisation, acceleration techniques for Monte-Carlo simulation, …
Key words: Ordered events, sequential events, Poisson stochastic Process, Markov, probability

THE RISK ANALYSIS OF SEISMIC ACTIVITY INCIDENCE IN ROMANIA

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•
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ABSTRACT

In Romania there is one of most powerful seismic activity region from Europe, known as Vrancea. In the past 300 years, a single major seismic event occurred with an epicenter outside this area (1916). This paper starts from going over all major seismic events, with a magnitude of over 6 degrees on Richter's scale, which were documented. Was tested the most plausible statistic behavioral model and was determined the probabilities for future large scale earthquakes, by different time horizons.

Key words: seismic risk, modeling, validation, prediction, statistic distribution

1 INTRODUCTIVE FEATURES

Europe, from the geologically point of view, confirms high seismic risk areas such as Italy, Turkey, Iceland, Serbia, Bulgaria, Greece and Romania, as in figure 1. The seismic intensity zones are marked by color code. So in Romania stand out the Eastern region of the country (Figure 2) the Carpathian Mountains. Agglomerations of black dots on the map represent earthquakes frequencies. One can remark a region of high concentration of earthquakes, which is known as the Vrancea area.

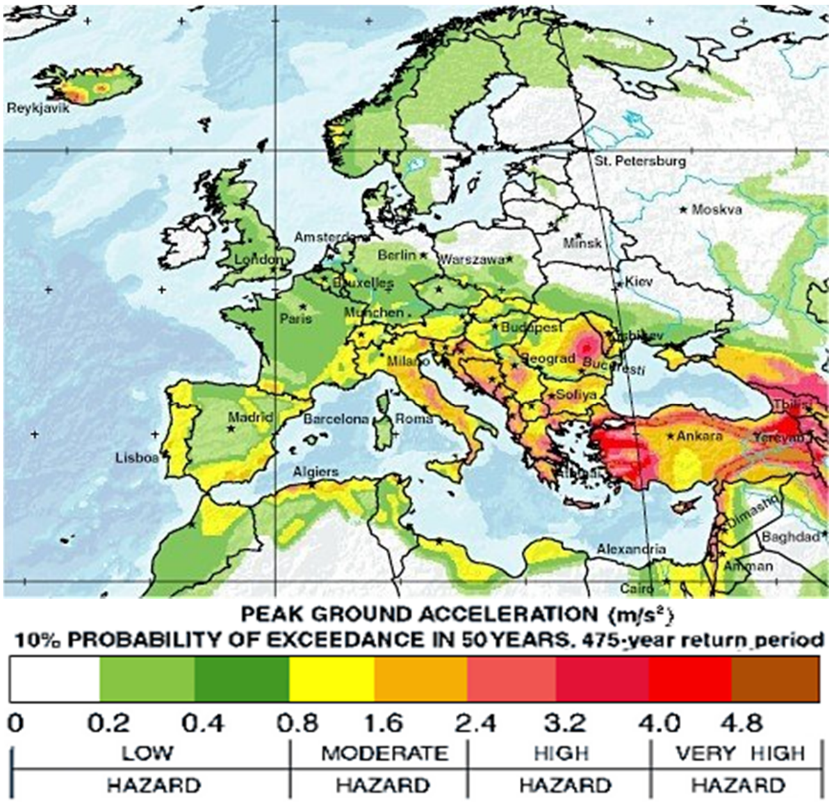
In general, is recognized that the occurrence of major seismic phenomena is a "rare event" from a statistical point of view. Due to the very large time horizon that can be taken into observation as against to registering events in artificial systems, as well as the non-periodicity of these events, there is the possibility of interpretation and statistical modeling of these seismic phenomena. In Romanian: Dragomir (2009), Lungu (1999), Lungu and Arion (2000), Radulescu (2004).

The statistical studies regarding the earthquakes usually start from the fact that rare events are best described using the exponential law – if considering the succession of time intervals between events, or Poisson's law – if it is intended to model the frequency of earthquakes (Săcuiu & Zorilescu, 1978; Johnson, Kotz & Balakrishnan, 1994; Evans, Hasting & Peacock, 2000).

The easiness of using these two distribution laws, distinct in nature, consists of the fact that they are defined by the same parameter, characterizing the same phenomenon – the behavior of a system in time, from both continuous and discrete points of view. A previous study made on seismic phenomena in Romania (Voda & Isaic-Maniu, 1983) covering the time period 1400-1977, has failed to confirm the hypothesis of an exponential behavior, the confirmed model being the bi-parametric Weibull model.

In the followings, we shall extend the area of investigation starting with the year 1100, with some additions to the identified supplementary information, as well as to the earthquake in 1977, the last one taken into account in the previous study.

We considered major seismic events those with a level of over 6 degrees on Richter's scale. Obviously, historical assessments are somewhat subjective, as the intensity was evaluated indirectly, since Mercalli (1931) and Richter's (1956) scales are more recent. The chronicles used to register that: "the earth had been shaken and the bells were ringing by themselves in Golia's tower" (n.n. Iasi – Romania), which indicates that an important seismic event took place. We used information in the profile literature (Constantinescu & Marza, 1980) as well as other official sources as those of the National Institute for the Physics of Earth (www.infp.ro).



Source http://geology.about.com/

Figure 1. The hazard of seismic activity in Europe

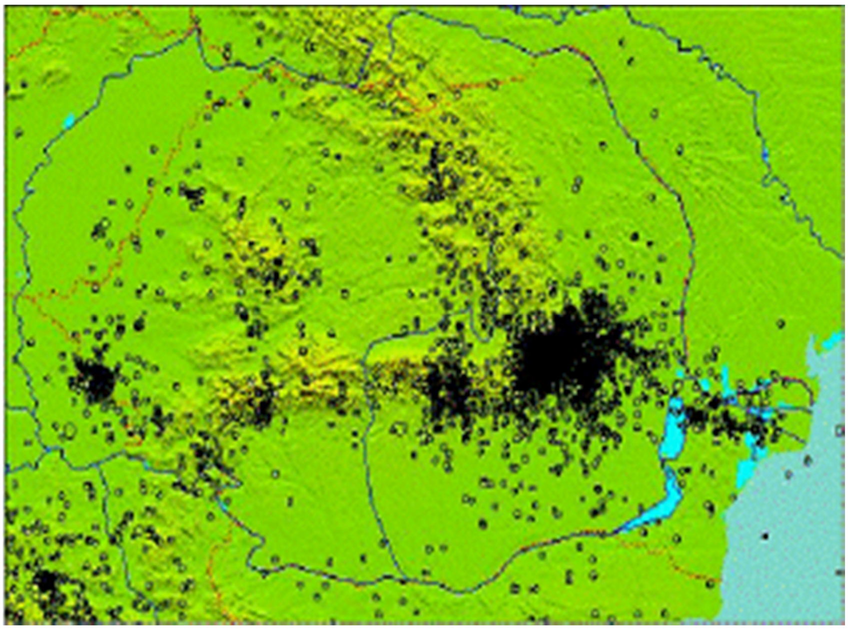


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Figure 2. The seismic activity in Romania

2 THE RECORD OF MAJOR SEISMIC ACTIVITY

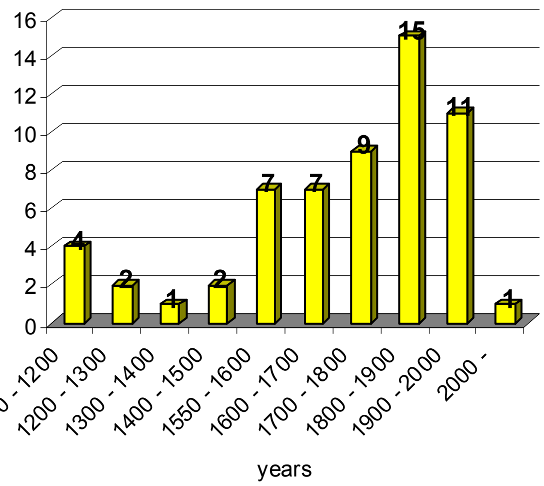
The main seismic events which occurred in Romania, and their characteristics, as they were recorded at the time in documents, or in modern and official registrations, were as follows in table 1.

Table 1. The main indicators of risk and reliability

November 5th, 1107	6.2 degrees Richter	January 18th, 1778	6.1 degrees Richter
August 8th, 1126	6.2 degrees Richter	March 18th, 1784	5.8 degrees Richter
April 1st, 1170	7.0 degrees Richter	April 6th, 1790	7-8 degrees Richter
February 13th	7.0 degrees Richter	December 8th, 1793	6.1 degrees Richter
May 10th, 1230	7.1 degrees Richter	October 26th, 1802	7.9 degrees Richter
year 1276	6.5 degrees Richter	March 5th, 1812	6.5 degrees Richter
year 1327	7.0 degrees Richter	January 5th, 1823	6.0 degrees Richter
October 10th, 1446	7.3 degrees Richter	November 26th, 1829	7.5 degrees Richter,
August 29th, 1471	7.1 degrees Richter	October 15th, 1834	6.0 degrees Richter
November 24th, 1516	7.2 degrees Richter	January 23rd, 1838	7.5 degrees Richter
July 19th, 1545	6.7 degrees Richter	October 15th, 1847	6.2 degrees Richter
October 16th, 1550	7.2 degrees Richter	October 17th, 1859	6.0 degrees Richter
November 2nd, 1558	6.1 degrees Richter	April 27th, 1865	6.4 degrees Richter
August 17th, 1569	6.7 degrees Richter	November 13th, 1868	6.0 degrees Richter
May 10th, 1590	6.5 degrees Richter	November 23rd, 1868	6.5 degrees Richter
August 10th, 1590	6.1 degrees Richter	November 26th 1868	6.1 degrees Richter
August 4th, 1599	6.1 degrees Richter	October 10th, 1879	6.2 degrees Richter
May 3rd, 1604	6.7 degrees Richter	August 31st, 1894	7.1 degrees Richter
November 24th 1605	6.7 degrees Richter	September 13th, 1903	6.3 degrees Richter
January 13th, 1606	6.4 degrees Richter	October 6th, 1908	7.1 degrees Richter
October 8th, 1620	7.9 degrees Richter	May 25th, 1912	6.3 degrees Richter
August 9th, 1679	6.8 degrees Richter	January 26th, 1916	6.4 degrees Richter
August 8th, 1681	6.7 degrees Richter	March 29th, 1934	6.9 degrees Richter
June 12th, 1701	7.1 degrees Richter	November 10th, 1940	7.7 degrees Richter
October 11th, 1711	6.1 degrees Richter	March 4th, 1977	7.4 degrees Richter
May 31st, 1738	7.0 degrees Richter	August 30th, 1986	7.1 degrees Richter
December 7th, 1746	6.5 degrees Richter	May 30th, 1990	6.9 degrees Richter
year 1750	6.0 degrees Richter	October 27th, 2004	6.0 degrees Richter

In the area of Vrancea (analyses of the area in Ivan, 2007; Ivan, 2011; Ardelean, 1999) there are registered almost daily earthquakes under 3 degrees.

Figure 3. The distribution of major seismic events in 100 years intervals in Romania



3 THE STATISTIC REPRESENTATION OF MAJOR SEISMIC ACTIVITY

The registered data were processed firs of all, statistically descriptive. The results as distribution series are presented in Table 2, the grouping being done in intervals of 100 years.

Table 2 - The distribution of major seismic events in 100 years intervals

No.	Interval(years)	Number of major seismic events
1	1100 – 1200	4
2	1200 – 1300	2
3	1300 – 1400	1
4	1400 – 1500	2
5	1500 – 1600	7
6	1600 – 1700	7
7	1700 – 1800	9
8	1800 – 1900	15
9	1900 – 2000	11
10	2000 –	1
	TOTAL	n = 59

The series (Table 2 and Figure 3) seems to suggest an acceleration of events in the last 250 years: in the first decade D_1 one earthquake was registered; $Q_2\left(M_e\right)=5.5$ earthquakes, and in $D_9 - 14.6$. This could be the effect of an energetic acceleration in the intensity of the activity of the terrestrial crust, but most probably it is the result of information inconsistencies in the medieval period which seem to suggest this seismic intensification. The maximum value in an interval of 100 years is 15 major seismic events (1800 – 1900). The total number of major earthquakes is 59. The average in a 100 year interval is 5.9, with a standard deviation of $\sigma = 4.75$ and a variation coefficient of CV = 0.805 which suggests a strong heterogeneity of the observation series. Standard error = 0.502.

Table 3 – Descriptive statistics

Statistic	Value
Sample Size	10
Range	14
Mean	5,9
Variance	22,544
Std. Deviation	4,7481
Coef. of Variation	0,80476
Std. Error	1,5015
Skewness	0,72385
Excess Kurtosis	-0,36372

The shape of the series is completed (table 3) with the values of the Skewness coefficient: $\beta_1 = \frac{\mu_3}{\left(s^2\right)^{3/2}} = 0.724$ (where μ_3 is the centered moment of rank 3, and s^2 – the centered moment of rank 2), and respectively $\hat{\beta}_2$ - Kurtosis coefficient: $\hat{\beta}_2 = \frac{\mu_4}{\left(s^2\right)^2} = -0.364$ (where μ_4 is the centered moment of rank 4). The minimum value in a 100 year interval was 1, and none of intervals frequencies were zero. The value of the first quartile was Q_1 = 1.75 and the third was Q_3 = 9.5 respectively.

4 THE STATISTIC MODEL OF THE SEISMIC INCIDENCE ACTIVITY

In order to analyse the process of earthquake occurrence, we tested several distribution laws, obviously starting with "the law of rare events" – Poisson, continuing with the exponential law (Evans, 2000) and Weibul (Isaic-Maniu, 1983). For the series of 50 years interval (Dragan & Isaic-Maniu, 2011), the best results were obtained for the log-logistic statistic model (Johnson, Kotz & Balakrishnan, 1995; Evans & Hastings, 2000; Stephens, 1979; Paiva, 1984; Ahmad, Sinclair & Werritty, 1988) by filtering three different selection tests.

Probability Density Function (PDF)

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1}\left(1+\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right)^{-2} \tag{1}$$

Cumulative Distribution Function (CDF):

$$F(x) = \left(1+\left(\frac{\alpha}{x-\gamma}\right)^{\beta}\right)^{-1} \tag{2}$$

For the distribution of 100 years interval, the best results were obtained for the Beta statistic model. The general formula for the probability density function of the beta distribution is:

$$f(x) = \frac{(x-a)^{\alpha_1-1}(b-x)^{\alpha_2-1}}{B(\alpha_1,\alpha_2)(b-a)^{\alpha_1+\alpha_2-1}}, a \le x \le b; \alpha_1, \alpha_2 > 0 \tag{3}$$

where α_1 and α_2 are the shape parameters, *a* and *b* are the lower and upper bounds, respectively, of the distribution, and $B(\alpha_1,\alpha_2)$ is the beta function. The beta function has the formula

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt \tag{4}$$

The case where $a = 0$ and $b = 1$ is called the standard beta distribution. The equation for the standard beta distribution is

$$f(x) = \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{B(\alpha_1,\alpha_2)}, 0 \le x \le 1; \alpha_1, \alpha_2 > 0 \tag{5}$$

Typically we define the general form of a distribution in terms of location and scale parameters. The beta is different in that we define the general distribution in terms of the lower and upper bounds. However, the location and scale parameters can be defined in terms of the lower and upper limits as follows: location = *a*; scale = *b* – *a*.

Since the general form of probability functions can be expressed in terms of the standard distribution, all subsequent formulas in this section are given for the standard form of the function.

The following (figure 4) is the plot of the beta probability density function for four different values of the shape parameters

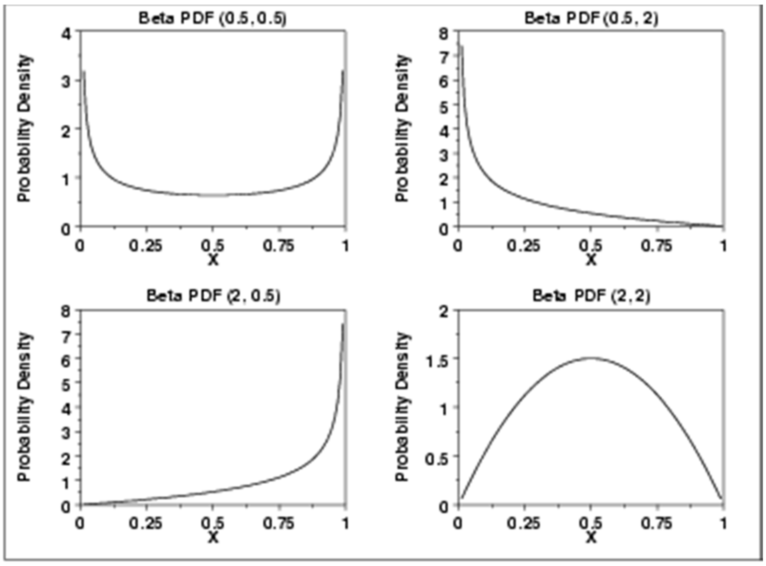


Figure 4. The beta probability density function

The formula for the cumulative distribution function of the beta distribution is also called the incomplete beta function ratio (commonly denoted by $\boldsymbol{I_x}$) and is defined as

$$F(x) = I_x(\alpha_1, \alpha_2) = \frac{\int_0^x t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}\, dt}{B(\alpha_1, \alpha_2)}, 0 \le x \le 1; \alpha_1, \alpha_2 > 0 \tag{6}$$

where B is the beta function defined above.

The formulas below are for the case where the lower limit is zero and the upper limit is one.

Mean	$\frac{\alpha_1}{\alpha_1 + \alpha_2}$
Mode	$\frac{\alpha_1 - 1}{\alpha_1 + \alpha_2 - 2}, \alpha_1, \alpha_2 > 1$
Range	0 to 1
Standard Deviation	$\sqrt{\frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2 (\alpha_1 + \alpha_2 + 1)}}$
Coefficient of Variation	$\sqrt{\frac{\alpha_2}{\alpha_1 (\alpha_1 + \alpha_2 + 1)}}$
Skewness	$\frac{2(\alpha_2 - \alpha_1)\sqrt{\alpha_1 + \alpha_2 + 1}}{(\alpha_1 + \alpha_2 + 2)\sqrt{\alpha_1 \alpha_2}}$

First consider the case where a and b are assumed to be known. For this case, the method of moments estimates are

$$\alpha_1 = \bar{x}\left(\frac{\bar{x}(1-\bar{x})}{s^2} - 1\right) \tag{7}$$

$$\alpha_2 = \left(1-\bar{x}\right)\left(\frac{\bar{x}\left(1-\bar{x}\right)}{s^2}-1\right) \tag{8}$$

where $\bar{x}$ is the sample mean and s^2 is the sample variance. If a and b are not 0 and 1, respectively, then replace $\bar{x}$ with $\dfrac{\bar{x}-a}{b-a}$ and s^2 with $\dfrac{s^2}{\left(b-a\right)^2}$ in the above equations.

For the case when a and b are known, the maximum likelihood estimates can be obtained by solving the following set of equations

$$\phi\left(\alpha_1\right)-\phi\left(\alpha_1+\alpha_2\right)=\frac{1}{n}\sum_{i=1}^{n}\lg\left(\frac{Y_i-a}{b-a}\right) \tag{9}$$

$$\phi\left(\alpha_2\right)-\phi\left(\alpha_1+\alpha_2\right)=\frac{1}{n}\sum_{i=1}^{n}\lg\left(\frac{b-Y_i}{b-a}\right) \tag{10}$$

The maximum likelihood equations for the case when a and b are not known are given in pages 221-235 of Volume II of Johnson, Kotz & Balakrishnan (1994).

5 FITTING THE DISTRIBUTION

In order to test the statistic nature of the distribution, we used the Kolmogorov-Smirnov, Anderson-Darling and Pearson-Fisher tests (Stephans, 1979; www.mathwave.com; www.vosesoftware.com).

Kolmogorov-Smirnov

The test is defined for the hypothesis

H_0: the distribution of earthquakes is Beta

H_1: the distribution of earthquakes is not Beta.

We compute the empirical distribution function $\hat{F}(x)$:

$$\hat{F}(x)=\frac{1}{n}\sum_{i=1}^{n}I_{X_{i\le x}} \tag{11}$$

where $I_{X_{i\le x}}$ is the indicator function, equal to 1 if $X_i \le x$ and equal to 0 otherwise.

The Kolmogorov-Smirnov statistic for a given cumulative distribution function $F(x)$ is

$$D_n=\sup_x\left|\hat{F}(x)-F(x)\right| \tag{12}$$

and F(x) the theoretical values of distribution.

The D_n computed value is compared to the maximum admitted equivalent.

The statistic computed value for the presented case resulted in $D_n = 0{,}19999$ is inferior to the critical level 0.48893 for a significance level of $\alpha = 0{,}01$, respectively inferior to value 0.40925 for 95%. The Beta distribution hypothesis is not rejected even for $\alpha = 0{,}2$.

The Anderson-Darling test – is also a distance test, proposed by Wilbur Anderson and Donald A. Darling in 1952.

The statistic of the test is

$$A^2 = -N - S \tag{13}$$

where:

$$S=\sum_{i=1}^{n}\frac{(2i-1)}{N}\left[\ln F(X_i)+\ln\left(1-F(X_{n+1-i})\right)\right] \tag{14}$$

in which F is the cumulative distribution function. For a significance level α, we validate one of the two hypotheses H_0 and H_1. The critical values for various specified distributions are computed by Stephens (1979).

The value of the statistics of the test: 4.3274 refute the Beta distribution for $\alpha = 0{,}01$, with critical value 3.9074, respectively 2.5018 for $\alpha = 0{,}05$

Pearson-Fisher Statistic

Chi Square or Pearson-Fisher (χ^2) test was proposed as a measure of random departure between observation and the theoretical model by Karl Pearson (Pearson 1900). The test was later corrected by Ronald Fisher trough decrease of the degrees of freedom by a unit (decrease duet of the existence of the equality relationship between the sum of observed frequencies and the sum of theoretical frequencies, (Fisher 1922), and by the number of 692 unknown parameters of the theoretical distribution when they come as estimated from measures of central tendency (Fisher 1924).

The chi-square test is used to test if a sample of data came from a population with a specific distribution. An attractive feature of the chi-square goodness-of-fit test is that it can be applied to any uni-variate distribution for which you can calculate the cumulative distribution function. The chi-square goodness-of-fit test is applied to binned data (i.e., data put into classes).

The test is defined for the hypothesis

H_0: The data follow a specific distribution

H_1: The data do not follow the specific distribution

The statistic is calculated as (in original):

$$\chi^2 = S\left\{\frac{(x-m)^2}{m}\right\} : \chi^2=\sum_{i=1}^{k}\frac{(O_i-E_i)^2}{E_i} \tag{15}$$

where O_i is the observed frequency for bin i and E_i is the expected frequency for bin i and is calculated by

$$E_i = N\left(F(Y_u)-F(Y_i)\right) \tag{16}$$

where F is the cumulative distribution function and Y_u and Y_i are the upper and lower limits for class i.

The test statistic follows, approximately, a chi-square distribution with *(k - c)* degrees of freedom where k is number of non-empty cells and c - the number of estimated parameters for the distribution +1.

Therefore, the hypothesis that data are from a population with the specified distribution is rejected if

$$\chi^2 > \chi^2_{\alpha,k-c}$$

where $\chi^2_{\alpha,k-c}$ is the chi-square percent point function with k - c degrees of freedom and a significance level of α.

The computations lead to a value of the $\chi^2_c = 7{,}3289\text{E-}8$ statistic inferior to the critical value $\chi^2_{0,01} = 6{,}6349$, so that the H_0 hypothesis is accepted with a probability of 99%. Either for different values of α (0.02; 0.05; 0.1) respectively 0.2 (critical value 1.6424) the Beta distribution hypothesis is not rejected.

Considering the three applied tests (Kolmogorov-Smirnov, Anderson-Darling and Pearson-Fisher), two of them confirm with a high confidence degree the Beta distribution, by parameters:

$\alpha_1 = 0.34225$

$\alpha_2 = 0.63562$

$a = 1.0$

$b = 15.0$

The Probability Density Function (pdf) for the estimated values of the parameters is presented in Figure 5 and The Hazard Function in Figure 6.

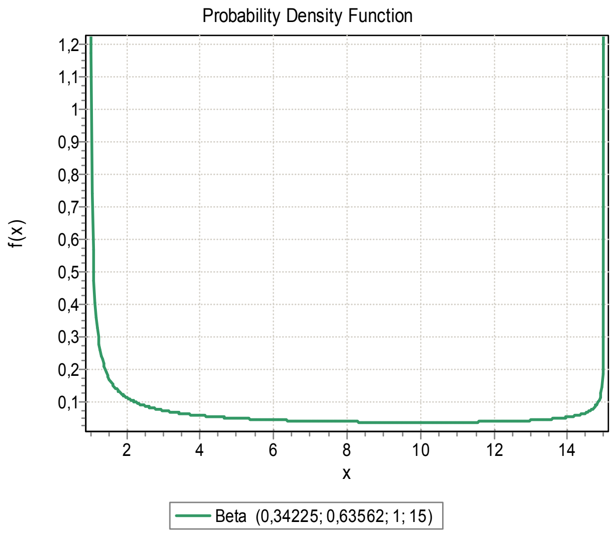


Figure 5. The Probability Density Function

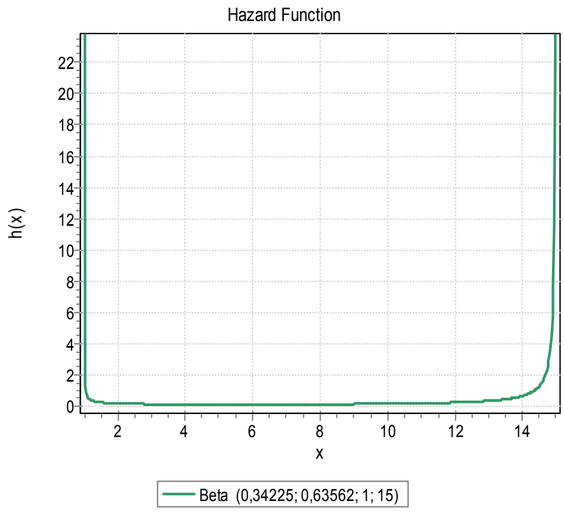


Figure 6. The Hazard Function

Table 4 presents the values of the main indicators of the Beta distribution for a number of x = 1, …, 15 events.

Table 4. The Values for pdf, CDF, h(x) şi S(x)

Statistic Functions	Values computed for **x** (earthquakes) equal to:														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
pdf- probability density function	-	0.115	0.075	0.059	0.051	0.045	0.042	0.040	0.039	0.038	0.039	0.040	0.044	0.054	-
CDF - cumulative distribution function	0.000	0.328	0.419	0.485	0.539	0.587	0.631	0.672	0.711	0.749	0.788	0.827	0.869	0.918	1.000
h(x) - hazard function	-	0.171	0.129	0.115	0.110	0.110	0.114	0.122	0.134	0.153	0.183	0.234	0.338	0.655	-
S(x) - distribution	1.000	0.672	0.581	0.515	0.461	0.413	0.369	0.328	0.289	0.251	0.212	0.173	0.131	0.082	0.000

6 CONCLUSIONS

In the followings, through simulation operations for the values of the Beta distribution, we formulate various hypotheses on the occurrence of seismic events, for the confirmed statistic model.

Thus, if we limit, for a 100 years interval, the number of major seismic events between $x_1 = 2$ and $x_2 = 10$ respectively, we have:

$P(x < x_1) = 32.51$ %

$P(x > x_1) = 67.19$ %

$P(x_1 < x < x_2) = 42.14$ %

$P(x < x_2) = 74.95$ %

P ($x > x_2$) = 25.05 %

It is an optimistic variant that the chances for less than two major seismic events to occur in a 100 year interval are around 33%, and for more than 10 major seismic events is reduced to 25%.

If we modify the limits to $x_1 = 1$ and $x_2 = 3$ major seismic events, then:

P ($x < x_1$) = 0 %
P ($x > x_1$) = 99.99 %
P ($x_1 < x < x_2$) = 41.89 %
P ($x < x_2$) = 41.89 %
P ($x > x_2$) = 58.11 %

So, there is a small probability that in Romania, less than 1 earthquake will occur, and slim chances that more than 3 earthquakes will occur. There is a probability of approximately 42% that in an interval of 100 years, between 1 and 3 events could occur.

If we modify the limits to $x_1 = 4$ and $x_2 = 12$ major seismic events, then:

P ($x < x_1$) = 48.5 %
P ($x > x_1$) = 51.5 %
P ($x_1 < x < x_2$) = 34.3 %
P ($x < x_2$) = 82.7 %
P ($x > x_2$) = 17.3 %

Romania represents an unique case in the world, from a seismic point of view: earthquakes of over 7 degrees Richter in magnitude which originate from Vrancea affect approximately 50% of the territory and approximately 60% of the population, including the capital, Bucharest. Nonetheless, the earthquake in 1977 was not the most powerful. It was only the fourth in magnitude among the earthquakes in the last 200 years. In Romania, there were 6 earthquakes of over 7 degrees Richter in the last 200 years. More technical details on the area Vrancea can be found in Ivan (2007, 2011).

In the case of Romania, the warning period for an earthquake is 25-30 second, which is relatively short in comparison to Mexico City - 60 seconds. However, it is enough to interrupt dangerous activities: nuclear reactors, heavy water production, chemical industry, gases, electricity and water. For trains and subways, stopping the electrical power is enough to stop the carriages.

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ESTIMATION LIKELIHOOD OF SPEED OF CHANGE OF DIAGNOSTIC PARAMETERS OF TRANSFORMERS

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ABSTRACT

Methods of classification retrospective data on independent groups of homogeneous data and estimations of reliability the assumption of constant speed of deterioration during normative service life are developed.
Keywords. *The transformer, diagnostics, criteria, speed of the change, the guaranteed estimations*

I. INSTRUCTION

Increase of efficiency of the control of conformity of a technical condition power transformers and autotransformers (further: TR) to shown requirements represents the important and difficult problem. Its importance caused by the high cost TR, expenses increasing in process of ageing TR for diagnostics, restoration of deterioration, and growth of influence of the human factor. Difficulty of the decision this problem connected with an insufficient computerization of process the analysis of retrospective data, including results of measurement diagnostic parameters (DP). Stochastic character of change DP, influence on DP numerous factors, deterministic the approach in methodology of the analysis of the technical condition TR, not considering these features, is a principal cause of observable discrepancy of results the analysis to real process.

Application of modern methods to research of technical condition TR demands automation of calculations. Considering, that number DP TR is estimated in tens, and number of versions of attributes of distinction TR - hundreds, application of computer technologies allows to solve not only challenges, but also extremely bulky.

"Tool" of practical realization of these technologies are the intellectual automated information systems (IAIS) with that difference from known AIS, that alongside with formalization and storage of retrospective data in special "database", ordering and a press of the information necessary for the analysis, they carry out this analysis and represent recommendations on maintenance service and repair TR.

As bright example of such approach, recommendations [1] serve at chromatographic analysis of the dissolved gases in oil TR. The essential contribution to perfection of system of the analysis of results of measurement DP brought with the researches [2] focused on use of expert systems. Authors of clause spend the researches for more severe constraints - when number experts is limited by units, and IAIS provides with their necessary information and the recommendations, allowing to prove made decisions with the set size of risk of the erroneous decision.

At the analysis of data of measurement DP, along with comparison DP with maximum permissible values, the great value has also the analysis of speed of change DP. This parameter calculated under the formula:

$$\vartheta\{\Pi,(t_2-t_1)\}=\vartheta[\Pi,t]=\frac{\Pi(t_2)-\Pi(t_1)}{(t_2-t_1)}; \quad (1)$$

where: $\Pi(t_2)$ and $\Pi(t_1)$ - accordingly current and preceded values DP (Π) during the moments of time t_2 and t_1. Its local character, which does not allow comparing with speed of change various DP (owing to distinction of dimensions) concerns to, lacks of this parameter.

The lack it is deprived speed of change of the relative values DP, calculated under the formula:

$$\vartheta\{Iz[\Pi,(t_2-t_1)]\}=\vartheta[Iz(\Pi,t)]=\frac{[Iz(\Pi,t_2)-Iz(\Pi,t_1)]}{(t_2-t_1)} \quad (2)$$

where: $Iz(\Pi,t)$ - relative size DP (Π), a describing degree of deterioration of property of a material of units TR during the moment t. In conformity with the developed practice, the size $Iz(\Pi,t)$ in abbreviated form named by "deterioration" during the moment t and calculated under the formula:

$$Iz(\Pi,t)=\frac{\Pi(t)-\Pi_o}{\Pi_\partial-\Pi_o} \quad (3)$$

where: Π_∂ and Π_o - accordingly, maximum permissible and initial values DP. Having substituted (3) in (2), we shall receive:

$$\vartheta\{Iz[\Pi,(t_2-t_1)]\} = \vartheta[Iz(\Pi,t)] = \frac{[\Pi(t_2)-\Pi(t_1)]}{(t_2-t_1)(\Pi_\partial-\Pi_o)} = \frac{\vartheta[\Pi,(t_2-t_1)]}{\Pi_\partial-\Pi_o} \tag{4}$$

So that to pass to relative values of speed of change DP, it is necessary to divide absolute value of speed of change DP on $(\Pi_\partial - \Pi_o)$. In some cases, the size Π_o ignored. It is inadmissible, if in process of deterioration size DP decreases. If in process of deterioration size DP increases, the error depends on a paritynd Π_∂. This parity is more the error of calculations is more. The estimation of size $\vartheta[Iz(\Pi,t)]$ is not end in itself. According to [3] $\vartheta[Iz(\Pi,t)]$ it compared to precede value.

So, according to [1] change of speed for concrete DP more, than on 10 % a month testifies to presence of quickly developing defect in TR. In other words:

$$\delta\vartheta(\Pi,t) = \left\{\frac{\vartheta\{\Pi,(t_3-t_2)\}-\vartheta\{\Pi,(t_2-t_1)\}}{\vartheta\{\Pi,(t_2-t_1)\}}\right\} < 0{,}1 \tag{5}$$

If $(t_3-t_2)=(t_2-t_1)$ the formula (5) becomes simpler and looks like:

$$\delta\vartheta(\Pi,t) = \left\{\frac{[\Pi(t_3)-\Pi(t_2)]}{[\Pi(t_2)-\Pi(t_1)]}\right\} < 1{,}1 \tag{6}$$

On fig.1 law of change DP according to four measurements, accordingly, during the moments t_0, t_1, t_2, t_3 where t_0 - the moment of measurement Π_o is resulted.

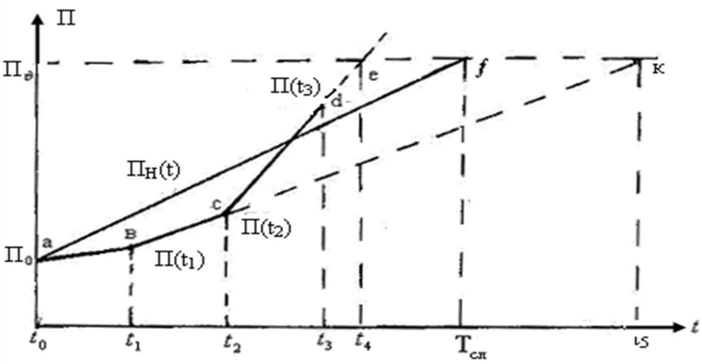

Fig.1. Graphic illustration of change DP before restoration of deterioration

As follows from fig.1, the given measurements during the moment t_3 testify to that that $\Pi(t_3)<\Pi_\partial$. However, speed of change DP does not satisfy to a condition (6). Speed on a site $(t_2 \div t_3)$ essentially is more, than on a site $(t_1 \div t_2)$, $\delta\vartheta(\Pi,t) > 1{,}1$. If to extrapolate line cd, that is to assume, that at $t > t_3$ speed of change DP remains constant it will appear, that $\Pi(t)$ will be equal Π_∂ during the moment of time t_4.

A seeming simplicity of these calculations is deceptive. Process of deterioration TR far not always corresponds a broken curve $abcdke$. The analysis of features of real process of deterioration, the account of these features is an indispensable condition of objectivity of the automated calculations.

If deterioration of the transformer connected with growth DP, in process of increase in service life, TR observed not only natural continuous increase in numerical value DP, but also its discrete reduction at use of those or other forms of restoration of deterioration or discrete increase at influence of operational factors. For example [1], at decontaminations of oil, addition of the decontaminated oil and of some other ways of improvement quality of oil TR, concentration of the gases dissolved in oil decreases. Moreover, at refusal of system of cooling, influence of through currents of short circuit, concentration of the gases dissolved in oil sharply increases and at absence of defects TR during one - two months decreases. Dependence of many DP from temperature of oil known.

Let us consider algorithm of ordering of data of speed of change DP. Let in empirical table ET (Π) databases the sequence of results of tests of park TR of a power supply system is placed.

1. We spend sample of measurement set DP;
2. From this sample systematized given measurements DP for concrete TR on time. These data include:

 2.1. Initial data (result of measurement DP at input of unit TR in work during the moment t_0

For separate units the moment of implementation coincides with the moment of implementation TR. Chances when these moments are various).

2.2. Results of measurement DP in process of increase in service life $t_j > t_0$; $j = p, M_i$; M_i - number of measurements DP for i-th TR.

3. Under the formula (2) speed of change of relative values DP during the moment t_{j-1} and t_j is calculated with $j = 1, M_i$;
4. The negative values $\vartheta[Iz(\Pi,t)]$ corresponding this or that form of restoration of deterioration are excluded from consideration.

Results of calculation are brought in empirical table ET (Π) together with data about service life ($t_{сл,j} = (t_{j-1} + t_j)/2$), design features TR and conditions of operation.

Enter concept of rated speed of change DP and designate as $\vartheta_н(\Pi)$. Further assume, that $\vartheta_н(\Pi) = (\Pi_д - \Pi_o)/T_{sl}$, where T_{sl} - normative service life TR. Hypothetical law of change DP thus corresponds to a line af fig.1. It is obvious, that $t_4 < T_{sl}$. And that it has not occurred, necessary to restore deterioration TR in an interval $t_3 \div t_4$. The parity speed of change DP on sites $(t_0 \div t_1)$ and $(t_1 \div t_2)$ does not satisfy to a condition (6), but equality $\Pi(t)$ and $\Pi_д$ occurring the moment $t_5 >> T_{sl}$. In other words, a condition (6) and $\Pi(t) < \Pi_д$ are often inconsistent.

As fuller characteristic of conformity of technical condition TR shown requirements are served with a condition not excess of size of relative change $\vartheta[\Pi(t_2 - t_1)]$ of unit. Calculations spent under the formula:

$$\delta\vartheta(\Pi,t) = \vartheta(\Pi,t)/\vartheta_н(\Pi) \tag{7}$$

If now in the formula (7) to substitute values $\vartheta(\Pi,t)$ and $\vartheta_н(\Pi)$ and to consider (3), receive:

$$\delta\vartheta[Iz(\Pi,t)] = \delta\vartheta(\Pi,t) = \left[\frac{T_{sl}}{(t_2 - t_1)}\right] \cdot \left[\frac{\Pi(t_2) - \Pi(t_1)}{(\Pi_д - \Pi_o)}\right] = T_{sl} \cdot \vartheta[\Pi,(t_2 - t_1)] < 1 \tag{8}$$

What as much as possible admissible value DP should be after restoration of deterioration during the moment t_3 to provide non-failure operation of work TR till the moment of time T_{sl} at speed of deterioration on an interval $(t_{sl} \div t_3)$ no more $\vartheta[\Pi,(t_3 - t_2)]$. Designate a size as well as $\Pi^*(t_3)$ and calculate it under the formula:

$$\Pi^*(t_3) = \{\Pi(t_3) - (t_{sl} - t_3)\frac{[\Pi(t_3) - \Pi(t_2)]}{(t_3 - t_2)}\} \tag{9}$$

Thus, shown, that:

- not excess current value DP of maximum permissible size DP does not testify yet to absence of defect TP. The reasons of such discrepancy are or the overestimated (underestimated) value $\Pi_д$, or the underestimated (overestimated) value of admissible change of speed $\vartheta(\Pi,t)$. This conclusion based on known in the theory of reliability process of deterioration of materials («a curve life») when after the normal period (speed of deterioration is constant) there comes the period of ageing and catastrophic deterioration (speed of deterioration nonlinear increases);
- Excess of speed of change DP more than on 10% in comparison with preceded value is not necessarily connect with occurrence of local defect. It speaks casual character of change $\vartheta(\Pi)$ and essential influence of some factors (design features and conditions of operation);
- Essential growth speed of change DP and not excess of predicted value of residual service life of normative size is a significant attribute of presence of the defect demanding restoration;
- Relative value of speed of change DP $\delta\vartheta[Iz(\Pi,t)]$ in view of reference value DP allows to compare with speeds of change various DP.

Noted above a ratio have been received in the assumption of constant speed of deterioration on an interval of service life TR (T_{sl}) and not excess DP of maximum permissible value ($\Pi_д$). In real conditions of operation, TR can appear that this assumption is erroneous. A principal cause to that is heterogeneity of set results of measurement DP and noted above discrepancy of limiting values DP and speeds of change DP to real process of deterioration.

So that to raise accuracy of the forecast of a residual operating time to excess DP of maximum permissible value it is necessary to consider first of all stochastic character of deterioration TR and to develop:

1. The method of classification of retrospective data of speed of change DP on groups of variety of attributes (VP).
2. The method of an estimation of reliability of the assumption of constant speed of deterioration on an interval of time T_{sl};

As a matter fact, the first method provides an opportunity of application of the second method. According to the terminology accepted in mathematical statistics, agree to name set of data of calculation relative speed of deterioration park TR of a power supply system a final data set (FDS), and a data set, chosen of FDS on the set version of one or of some attributes - sample.

Agree that data of relative values of speed of change DP TR collected and placed in the empirical table (ET). In columns ET the serial number of measurement, numerical values $\delta\vartheta[Iz(\Pi,t)]$, the name of distinctive attributes are consistently registered. To distinctive attributes concern not only nameplate data TR, i.e. its design features, but also attributes of conditions of operation TR, such as: service life, an operating time after major overhaul, the name of the enterprise and substation, etc.

Designate number of considered distinctive attributes through n, and number of variety of attributes (VP) - through r_i with $i=1,n$. Set of results of calculation $\vartheta[Iz(\Pi)]$ concrete DP, forming FDS, we shall designate as $\{\delta\vartheta[Iz(\Pi_{v,j})]\}_\Sigma$, where $v=1,k$; k - number DP; $j=1,M_v$; M_v - number of realizations for j-th DP, and set of realizations $\vartheta[Iz(\Pi)]$ of sample with set VP - as $\{\delta\vartheta[Iz(\Pi_{v,j})]\}_B$ with $v=1,k$ and $j=1,M_{v,B}$; $M_{v,B}$ - number of realizations $\vartheta[Iz(\Pi)]$ for v-th VP in sample (B).

II. QUALITY MONITORING OF IMPOSING APPEARANCE OF SAMPLE OF REALIZATIONS $\{\delta\vartheta[Iz(\Pi_{v,j})]\}$.

The method is based on a following axiom: if sample of realizations $\delta\vartheta[Iz(\Pi,t)]$ for some from set VP, having the greatest absolute value of the maximal divergence of statistical function of distribution (s.f.d.) from s.f.d. FDS, it is representative, other samples of set of versions of considered attributes are representative also all.

Under representative, we shall understand sample, the maximal divergence s.f.d. Which from s.f.d. FDS satisfies to a condition:

$$\alpha_k < \{1 - F_m^*[\Delta F_Э^*(\delta)]\} >> F_m^*[\Delta F_Э^{**}(\delta)] \tag{10}$$

where: $\delta_{v,j} = \delta\vartheta[Iz(\Pi_{v,j},t)]$- symbolic notation; $\alpha_k = 1-R$; α_k - mistake of the first sort; $\Delta F_э^*(\delta)$ - the greatest divergence between s.f.d. FDS (designate it as $F_\Sigma^*(\delta)$) and s.f.d. Samples (designate its $F_B^*(\delta)$). Calculated under the formula:

$$\Delta F_э^*(\delta) = max[\Delta F^*(\delta_j)] \tag{11}$$

where: $j=1,M_{v,B}$

Let's agree size $\Delta F_э^*(\delta)$ to name the greatest deviation empirical distributions $\Delta F_\Sigma^*(\delta_j)$ and $\Delta F_B^*(\delta_j)$.

$$\Delta F^*(\delta_j) = |F_\Sigma^*(\delta_j) - F_B^*(\delta_j)| \tag{12}$$

$$F_{\Sigma,i}^*(\delta) = i/M_v \tag{13}$$

$$F_{B,i}^*(\delta) = i/M_{v,B} \tag{14}$$

$F_m^*[\Delta F_m^*(\delta)]$- s.f.d. The greatest divergence between s.f.d. FDS $F_\Sigma^*(\delta)$ and modeled on $F_\Sigma^*(\delta)$ s.f.d. Samples $F_{B,m}^*(\delta)$.

$$\Delta F_m^*(\delta) = max[\Delta F_m^*(\delta_j)] \tag{15}$$

where: $j=1,M_v$, and

$$\Delta F_m^*(\delta_j) = |F_{\Sigma,m}^*(\delta_j) - F_{B,m}^*(\delta_j)| \tag{16}$$

$$F_{m,i}^*[\Delta F_{m,i}^*(\delta)] = i/N \tag{17}$$

where: N - number of modeled realizations of the greatest divergence between $F_\Sigma^*(\delta)$ and $F_{B,m}^*(\delta)$.

$F_m^*[\Delta F_m^{**}(\delta)]$ - s.f.d. The greatest divergence between s.f.d. FDS $F_\Sigma^*(\delta)$ and modeled on $F_B^*(\delta)$ s.f.d. Samples

$F^{**}_{B,m}(\delta)$.

$$\Delta F^{**}_{m}(\delta)=max\left[\Delta F^{**}_{m}(\delta_j)\right] \tag{18}$$

where: $j=1,M_v$, and

$$\Delta F^{**}_{m}(\delta_j)=\left|F^{**}_{\Sigma,m}(\delta)-F^{**}_{B,m}(\delta_j)\right| \tag{19}$$

$$F^{*}_{m,i}\left[\Delta F^{**}_{m,i}(\delta)\right]=i/N \tag{20}$$

To check of a condition (10) precede:

1. Formation of sample of measurements for each variety of considered attributes;
2. Under formulas (11÷14) the greatest empirical divergence of all versions i-th an attribute is calculated with $i=1,n$;
3. The greatest divergence among $\Delta F^{*}_{Э,i}(\delta)$ Is calculated with $i=1,n$. Designate its $\Delta F^{*}_{Э,m}(\delta)$.
4. The hypothesis about imposing appearance of the sample corresponding size $\Delta F^{*}_{Э,m}(\delta)$. Is checked. if sample is representative, according to an axiom other samples for set VP are representative also all. In other words, FDS it is homogeneous, and it is non-uniform - otherwise.

III. METHOD OF CHECK OF THE ASSUMPTION OF CONSTANT SPEED OF DETERIORATION

Casual character of speed of change DP essential influence on this size of operational factors cause difficulties of recognition on retrospective data of law of change in time. Construction of confidential area with the set factor of trust does not allow to solving a task in view since so as speed of deterioration on an interval T_{sl} is constant, appear assumptions of nonlinear laws of its change are fair.

Below the method of check of the assumption of constant speed of change DP, based, as well as a method of classification of data, on statistical modeling units and theories of check statistical hypotheses is resulted. The integrated block diagram of algorithm promoting representation about a method is resulted on fig.2. We shall consider some features of program realization of algorithm.

Block 1. FDS is formed of realizations $\delta\vartheta[Iz(\Pi,t)]$ ET (δ). Designate this FDS as $\{\delta\vartheta[Iz(\Pi,t)]\}_\Sigma=\delta_\Sigma$;

Block 2. S.f.d. $F^{*}_{\Sigma}(\delta)$ Pays off under the formula:

$$F^{*}_{\Sigma,i}(\delta)=i/M_\Sigma \tag{21}$$

where: M_Σ - number of lines ET (δ).

Block 3. Sample (δ_B) is formed from (δ_Σ), which realizations satisfy to a condition:

$$0{,}75\,T_{sl}<t_j\le T_{sl} \tag{22}$$

with $j=1,M_B$, where: M_B - number of sample units (B).

Block 4. S.f.d. $F^{*}_{B}(\delta)$ Pays off under the formula:

$$F^{*}_{B,i}(\delta)=i/M_B \tag{23}$$

Block 5. The greatest empirical divergence s.f.d. $F^{*}_{\Sigma}(\delta)$ also $F^{*}_{B}(\delta)$ calculated under the formula:

$$\Delta F_Э(\delta)=max\left\{\left|F^{*}_{\Sigma}(\delta_j)-F^{*}_{B}(\delta_j)\right|\right\} \tag{24}$$

Block 6. Average values of speed of change DP FDS and samples under formulas pay off:

$$M^{*}_{\Sigma}(\delta)=\left\{\sum_{j=1}^{M_\Sigma}\vartheta_j\left[Iz(\Pi,t)\right]\right\}/M_\Sigma \tag{25}$$

$$M^{*}_{B}(\delta)=\left\{\sum_{j=1}^{M_B}\vartheta_j\left[Iz(\Pi,t)\right]\right\}/M_B \tag{26}$$

Further management is transferred to modeling (m) s.f.d. The greatest divergence N of realizations $F^{*}_{\Sigma,m}(\delta)$ and $F^{*}_{B,m}(\delta)$ (block 7). It is originally modeled s.f.d. $F^{*}_{B,m}(\delta)$ A method of "inverse functions" on the basis of M_B random numbers with uniform distribution in an interval [0,1] and s.f.d. $F^{*}_{\Sigma}(\delta)$, accommodations of sample of M_B realizations of speed of change DP in ascending order and calculation of probability of display of these realizations under the formula (23). It is formed FDS and s.f.d. $F^*_{\Sigma,m}(\delta)$, under the formula (16) the greatest divergence $\Delta F^*_m(\delta_j)$ is calculated. These calculations repeat N time, is under construction s.f.d. $F^*_m\left[\Delta F^*_m(\delta)\right]=R(\delta)$ and at last is calculated s.f.d. $\alpha(\delta)=1-R(\delta)$ (block 8).

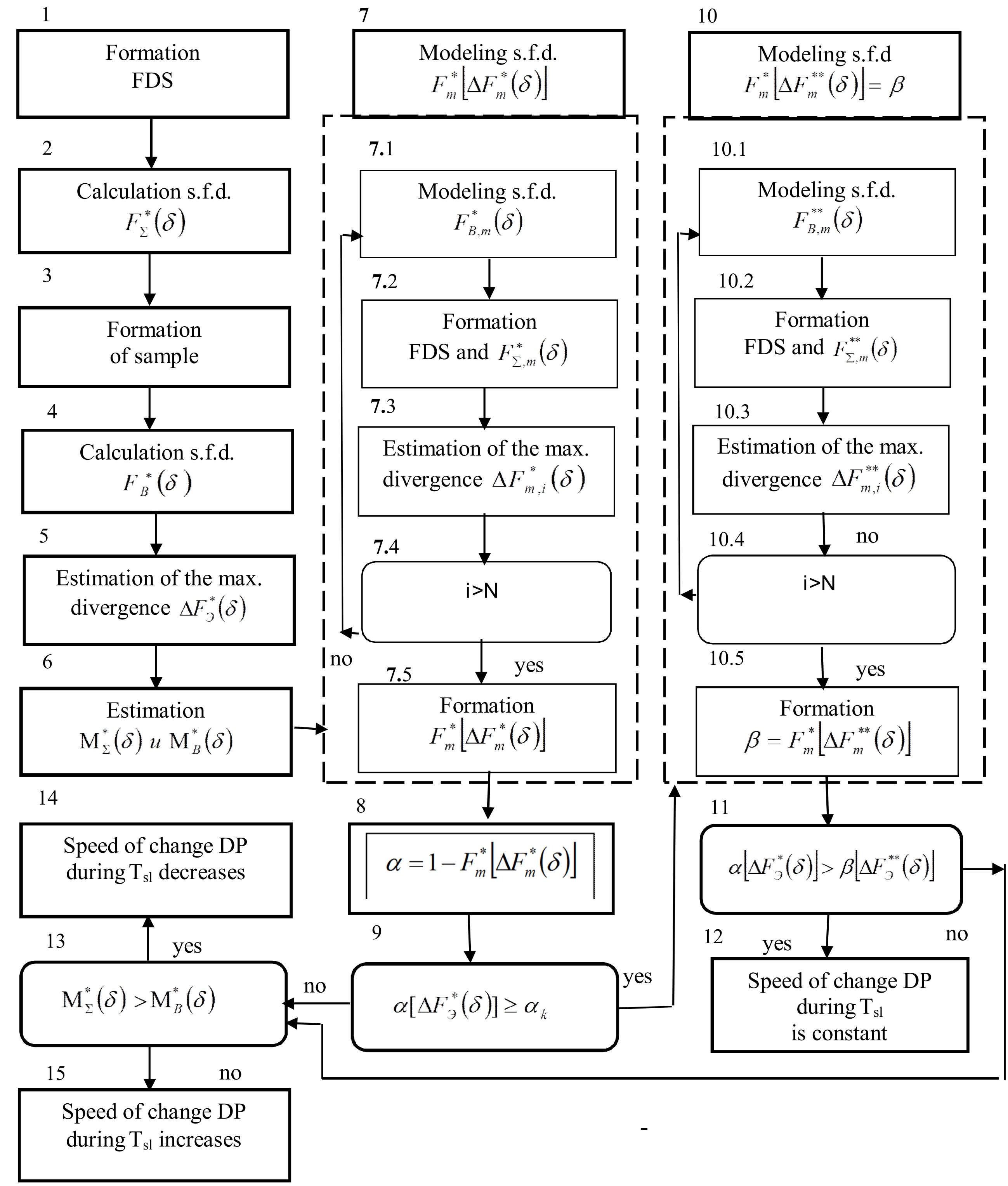

Fig.2 Integrated block diagram of algorithm of the statistical analysis of speed of change DP in current T_{sl}.

If it will appear, that $\alpha\left[\Delta F_Э(\delta)\right]<\alpha_k$, where α_k - critical value of a significance value it means, that on an interval [0,75T_{sl} ;T_{sl}] relative speed of change DP not casually differs from the average characteristic for FDS.

Moreover, the parity corresponds to decreasing speed of change DP (block 14), and a parity - to increasing speed of change DP (block 15).

If had data do not contradict a hypothesis H_o of constant speed of deterioration, i.e. $\alpha\left[\Delta F^*_Э(\delta)\right] \geq \alpha_k$, where α_k - critical size of a significance value management transferred the block 10 which is similar to the block 7 with that difference, that modeling of sample is carried out on distribution $F^*_B(\delta)$. In other words, the hypothesis H_2 about constant speed of deterioration on an interval normalized service life is checked.

If a mistake of the second sort for an empirical greatest divergence s.f.d. $F^*_\Sigma(\delta)$ Also $F^*_B(\delta)$ it appears it is less, than a mistake of the first sort it is possible to accept a hypothesis H_1 with the certain degree of confidence. If the return parity (the hypothesis H_2 is fair) takes place, management is transferred to the block 11 for an estimation of character of change of speed of deterioration.

CONCLUSION.

1. Operating experience, literary data testify to necessity to show care at use of criterion not excess of speed of deterioration of preceded value, and a diagnostic parameter - maximum permissible size. A principal cause to that is not the account of stochastic character of process of deterioration TR.
2. Methods of classification of retrospective data on independent groups of homogeneous data and estimations of reliability of the assumption of constant speed of deterioration during normative service life are developed. Consistency of data about constant speed of deterioration allows raise objectivity of the forecast of technical condition TR on the basis the guaranteed estimations.

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THE CONCEPTUAL FRAMEWORK FOR CONSTRUCTION PROJECT RISK ASSESSMENT

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ABSTRACT

The environment in which the project schedule will be executed is far from being static. Projects are subject to various uncertainties that have negative effect on activity durations. This is most apparent in the case of construction projects. The frequency and impact of risks depend on project-specific, contractor-specific and location-specific conditions. Identifying critical sources of risk is crucial to minimize disturbance in project development and assure success. The paper presents risk analysis and assessment framework. For the risk evaluation, the AHP was adopted in the paper. The proposed risk model is based on evaluating and weighting the particular project's characteristics and expected conditions. The method to assist planners in determining activity duration distribution parameters according to risk level is presented. This approach, combined with simulation technique, is argued to improve project planning and evaluation of risk mitigation alternatives.

1 INTRODUCTION

Construction projects are influenced by a variety of risk factors, e.g. weather, soil conditions, qualifications and productivity of the staff, crew and subcontractors, accidents, resource shortages, unreliable deliveries, defects. A schedule that is optimal with respect to project duration or cost may largely be affected by disruptions and uncontrollable factors. The available statistical knowledge of the uncertainties should be used while building the project schedule.

Risk in construction and engineering has been defined in various ways: the chance of injury, damage, or loss (Mehr & Cammack 1966), any exposure to the possibility of loss or damage (Papageorge 1988), the uncertainty and the result of uncertainty (Hertz & Thomas 1983), or the variation in the possible outcomes, a property of an entire probability distribution, whereas there is a separate probability for each outcome (Williams & Heins 1971). The risk factors have a significant impact on the outcome of a project especially in terms of duration and cause schedule delays.

To control the level of risk and mitigate its effects, risk management should be applied. The project risk management process requires risk identification, analysis and assessment, as the first steps for planning and implementing risk handling (response) strategies.

As a result of disturbances caused by risk factors, the activities' duration is a random variable. To determine a construction process' duration distribution types and parameters, a considerable number of time measurements would be necessary to make the results statistically sound. This might be too costly, time consuming and in some cases unjustified as, due to the unique character of construction projects and processes, statistical data from the past may be of little use in the future.

Many models have been proposed to describe and predict activity and project durations or work productivity on the basis of risk analysis: simple analytical, neural network based (e.g. Kog et al. 1999, Chua et al. 1997, Zayed & Halpin 2005, Shi 1999, AbouRizk et al. 2001, Sonmez & Rowings 1998), Bayesian belief network based (Nasir et al. 2003), fuzzy set besed (e.g. Lee &

Halpin 2003), regression (e.g. Hanna & Gunduz 2005, Jaselskis & Ashley 1991) and simulation models (e.g. Dawood 1997, Schatteman et al. 2008).

Most of the quantitative models assume that particular factors affect the processes independently. No model is considered to be superior as providing more reliable solutions than the other models. However, there is little evidence of extended practical use of the models developed to date.

2 PROJECT RISK ASSESSMENT FRAMEWORK

2.1 The proposed concept of project risk assessment

As a result of uncertainties, the project duration is a random variable. The probability density function of project duration reflects the project risk and enables to assess the probability of not meeting the contractual project due date. The proposed procedure for generate the pdf and predict the project risk consists of three main steps showed on Figure 1 and explained in the next paragraphs.

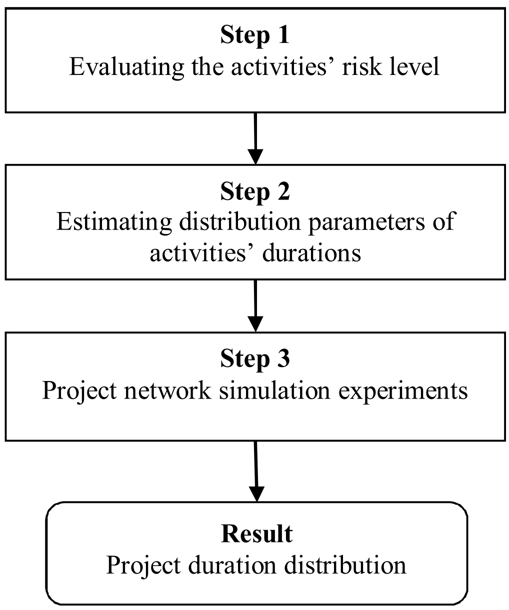

Figure 1. The proposed procedure of project risk assessment

2.2 Evaluating the activities' risk level

The precedence relationships between schedule activities, i.e. construction processes, are modeled by a unigraph directed, acyclic, in activity-on-node representation with single start and end nodes.

The frequency and impact of risk factors on a particular construction process depend on the project-specific, contractor-specific and location-specific conditions.

Table 1 lists ten construction project conditions considered to be of the greatest impact on risk and deviation in activities duration, identified on the basis of a survey among chartered engineers employed by construction companies in Poland.

Table 1. Construction project conditions affecting project risk level

No.	Condition
1	Season of the year
2	Human resources: skill and availability (concerns also subcontractors)
3	Quality and completeness of design documents
4	Quality of project and construction management systems
5	Labour conditions
6	Financial standing of project participants, project's finance conditions
7	Quality of the supply system
8	Site layout, site location
9	Project environment (economic, political, legal, geographic, labour market, suppliers etc.)
10	Equipment – quality and availability

The state of each condition was assumed to be scored using a five-point scale 0, 0.25, 0.5, 0.75, 1, where score 0 stands for ideal conditions, 0,5 – average conditions, and 1 – most adverse conditions. In the process of assigning scores, knowledge and experience of experts should be used. Group decision making involves aggregation of diverse individual preferences to obtain a single collective preference. To achieve consensus of the expert judgements, the authors propose the Delphi method.

The aggregated score for a project condition state is calculated according to the following formula:

$$PC = \sum_{j=1}^{n} pc_j \cdot w_j \,, \tag{1}$$

where: pc_j = evaluation of condition j state, w_j = weight of condition j, n = number of evaluated conditions (here, n=10).

The weights of particular project conditions should reflect their impact on extension of activities' duration (risk level). They can be found by means of Analytical Hierarchy Process. Let us consider a group of K experts involved in a decision making process. They compare, pairwisely, n criteria (project conditions) with respect to the project risk level. Each expert provides a set of $m = n(n-1)/2$ comparison judgments – assigns a numerical value of an importance ratio – using a fundamental scale: 1/9,1/7, 1/5, 1/3, 1, 3, 5, 7, 9. The scale may be extended by some intermediate values: 1/8, 1/6, 1/4, 1/2, 2, 4, 6, 8 if necessary.

As a result of the pairwise comparison that uses the above crisp ratios, a set of K matrices is created $A_k = \{a_{ijk}\}$, $i = 1, 2, ..., n-1$, $j = 2, 3, ..., n$, $j >$ i, $k = 1, 2, ..., K$, where a_{ijk} stands for a relative preference of criterion i to j, as assessed by the expert k.

In the classical AHP method, Saaty proposed the geometric mean method of aggregating ratio judgments (Saaty & Vergas 2007). This is to assure satisfying the Pareto optimality axiom: the variant preferred by each expert or decision maker should be preferred by the whole group (Van Den Honert & Lootsma 1996).

Scoring the state of each project condition and determining each condition's weight for each particular construction process is not necessary, as construction processes can be divided into groups that are similarly affected by certain risk factors. For instance, in the case of housing projects, six activities groups were identified by authors to represent all the types of activities in project schedule. These groups are Mobilization, Foundations, Structural works, Internal and External finishing, and Services.

2.3 Estimating distributions parameters of activities' durations

As a result of disturbances caused by risk factors, the duration of a activity j is a random variable. Its actual distribution is unknown. If there is only a limited number of sample data, the continuous triangular distribution (with lower limit a_j, mode m_i and upper limit b_i) is often used for a proxy of actual distribution (Johnson 1997). Similarly to PERT, the lower and upper limits can be evaluated properly as optimistic and pessimistic (or 5% and 10% fractiles) estimates of activity j duration. Instead, they could be derived from the planner's database of past experience, if such was available.

The duration's mode m_i can be calculated on the basis of median duration estimate based on a unit production time. As unit production times are established for average states of project conditions, the distribution function formulated this way would reflect the variability of activity duration only in the case of PC=0.5.

To construct a project schedule, one needs to assume fixed activities' duration estimates t_j. The risk connected with these decisions can be described using following formula:

$$r^{PC}\left(t_j\right) = \int_{t_j}^{b_j^{PC}} \left(x - t_j\right) \cdot f_j^{PC}(x)\, dx, \tag{2}$$

where:

$r^{PC}\left(t_i\right)$ = risk associated with expressing the duration of activity j, as a fixed value t_j, when the state of project conditions is assessed as PC; it is the expected value of extension of duration over the estimate t_j,

$f_j^{PC}(x)$ = activity j duration's distribution function when the state of project conditions is assessed as PC (with parameters a_j^{PC}, m_j^{PC}, b_j^{PC}).

The analytical formula to calculate the approximate risk value that bases on the assumption of a triangular distribution takes the following form:

$$r^{PC}\left(t_j\right) = \\ = \begin{cases} \dfrac{\frac{1}{3}t_j^3 - a_j^{PC} t_j^2 - \left(m_j^{PC}\right)^2 t_j + 2a_j^{PC} m_j^{PC} t_j + \frac{2}{3}\left(m_j^{PC}\right)^3 - a_j^{PC}\left(m_j^{PC}\right)^2}{\left(b_j^{PC} - a_j^{PC}\right)\cdot\left(m_j^{PC} - a_j^{PC}\right)} + \\ + \dfrac{-\left(m_j^{PC}\right)^2 t_j + 2b_j^{PC} m_j^{PC} t_j + \frac{2}{3}\left(m_j^{PC}\right)^3 - b_j^{PC}\left(m_j^{PC}\right)^2 - \left(b_j^{PC}\right)^2 t_j + \frac{1}{3}\left(b_j^{PC}\right)^3}{\left(b_j^{PC} - a_j^{PC}\right)\cdot\left(b_j^{PC} - m_j^{PC}\right)}, \\ t_i \in \left[a_j^{PC}, m_j^{PC}\right] \\ \dfrac{-t_j^3 + 3b_j^{PC} t_j^2 - \left(b_j^{PC}\right)^2 t_j + \left(b_j^{PC}\right)^3}{3\left(b_j^{PC} - a_j^{PC}\right)\cdot\left(b_j^{PC} - m_j^{PC}\right)}, \; t_j \in \left[m_j^{PC}, b_j^{PC}\right] \end{cases} \tag{3}$$

Figure 2 presents the results of using this formula: the risk curve of fixed activity duration estimate for a activity of the following parameters of triangular distribution function: $a_j^{0,5} = 0$, $m_j^{0,5} = 0{,}3$, $b_j^{0,5} = 1$ and PC=0.5.

To find the parameters of the distribution function for other states of project conditions ($PC \neq 0.5$), the authors propose using the least squares technique and fitting the risk curve under the following assumptions:

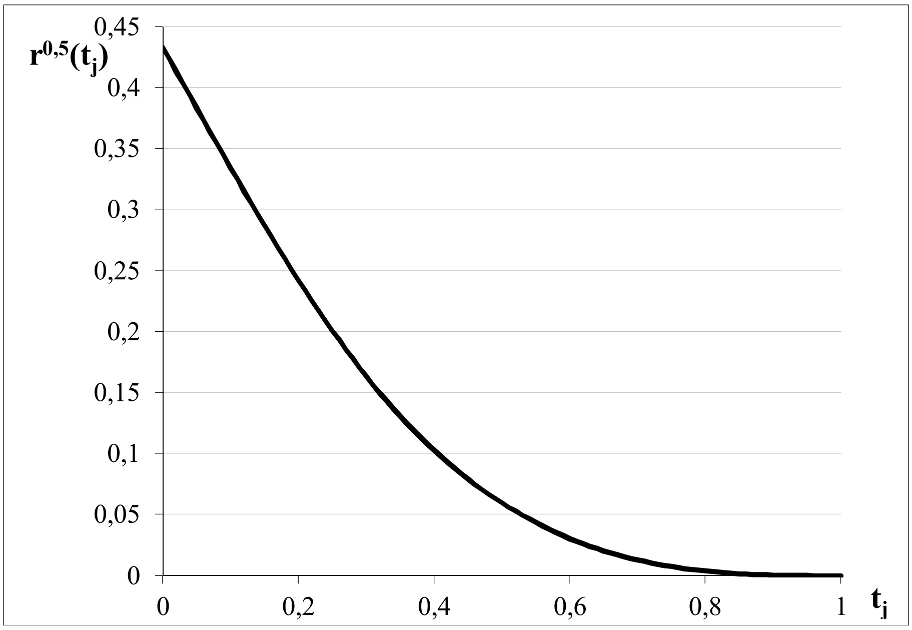

Figure 2. Risk curve of fixed activity duration estimate (example)

1. The risk associated with fixed duration estimate t_j of activity j is linearly dependent on the state of the project conditions:
   $$r^{PC}(t_j) = r^{0,5}(t_j) \cdot \frac{PC}{0,5}, \quad \forall t_j \in \left[a_j^{PC}, b_j^{PC}\right].$$
2. If PC>0.5 then lower limit and the mode of the distribution function can be increased.
3. If PC<0.5 then the upper limit and the mode of the distribution function can be reduced.

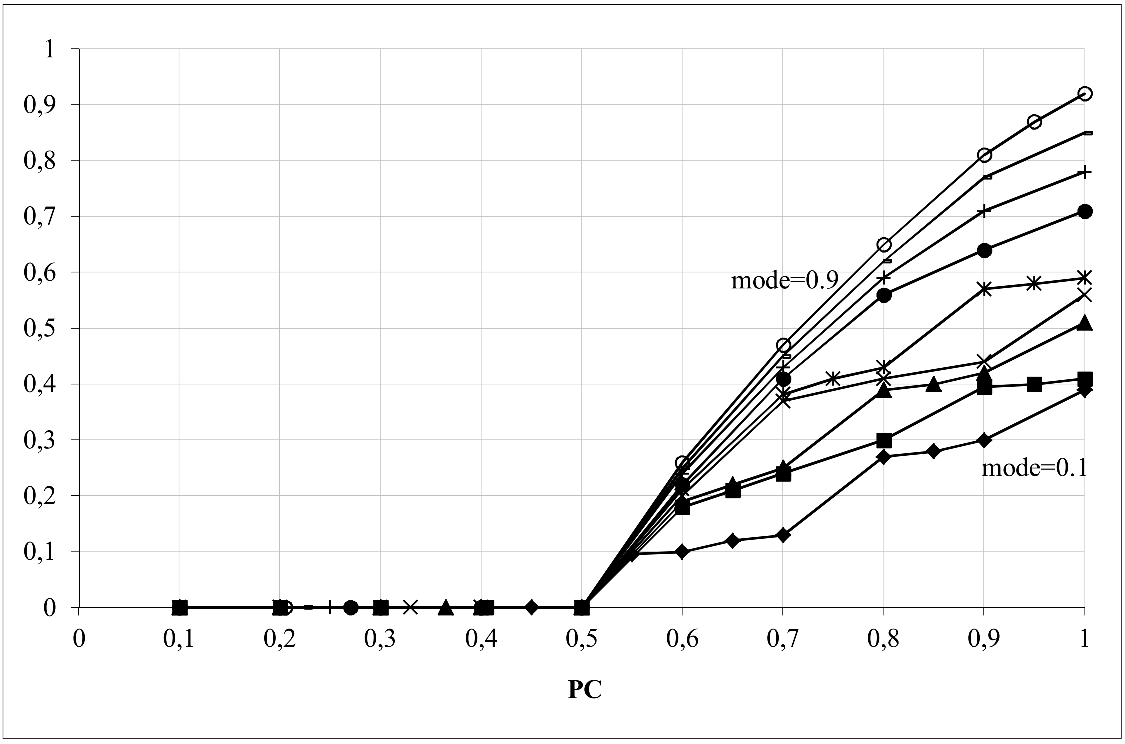

Figure 3. Effect of the state of project conditions on the a_{sj}^{PC} parameter

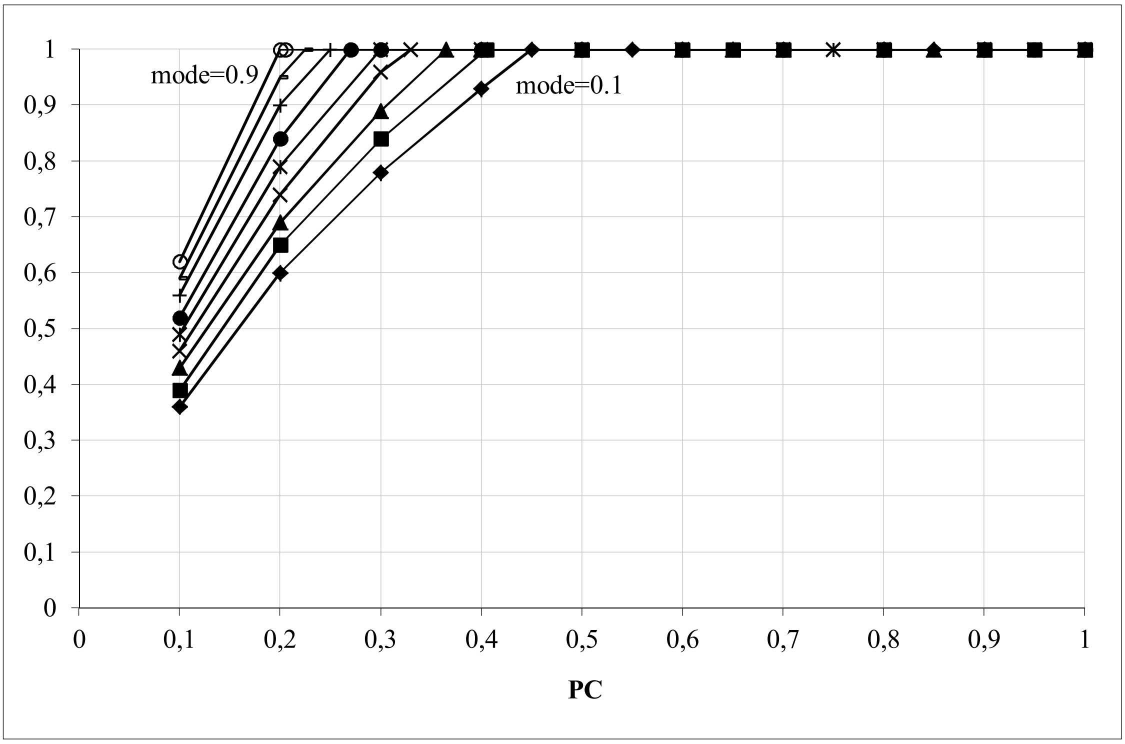

Figure 4. Effect of the state of project conditions on the b_{sj}^{PC} parameter

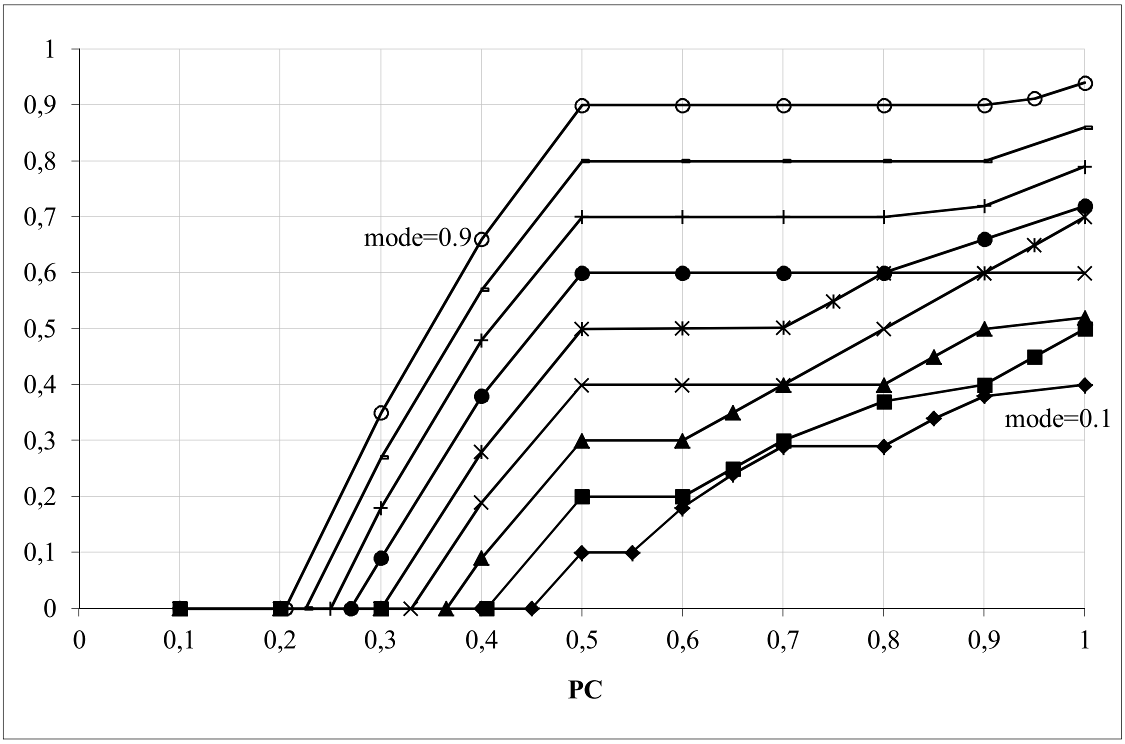

Figure 4. Effect of the state of project conditions on the m_{sj}^{PC} parameter

The sum of the squares of the errors was minimised for limited number of t_j values. Figures 3–5 present the relationship between the state of project conditions *PC* and the parameters of the standardized triangular distribution function for the activity j with original standardized parameters a_{sj}=0, m_{sj}=mode, b_{sj}=1 for PC=0.5; coefficient of determination R^2 takes values 0.77–0.96.

Parameters of the distribution function can be determined using graphs on Figures 3–5 and the following equations:

$$a_j^{PC} = a_j^{0,5} + a_{sj}^{PC} \cdot \left(b_j^{0,5} - a_j^{0,5}\right), \tag{4}$$

$$m_j^{PC} = a_j^{0,5} + m_{sj}^{PC} \cdot \left(b_j^{0,5} - a_j^{0,5}\right), \tag{5}$$

$$b_j^{PC} = a_j^{0,5} + b_{sj}^{PC} \cdot \left(b_j^{0,5} - a_j^{0,5}\right). \tag{6}$$

2.4 Project network simulation experiments

There are a number of methods that allow the planner to consider the effect of random occurrences on the project performance and to assess the chances of meeting the deadlines defined by the contract. The first attempt to allow for risks in project planning was made by the inventors of *PERT* (Program Evaluation and Review Technique).

The assumptions of *PERT* made it possible to reduce the complexity of network model analyses but, at the same time, affected the accuracy of time estimates of individual project events and the project as a whole (Biruk & Jaskowski 2010). Therefore, project networks are often analysed by means of the *Monte Carlo* simulation.

The Monte Carlo method simulates the project network many times, each time randomly choosing a value for activities' duration from their probability distribution. The outcome is a probability distribution of the project duration, evaluated on the basis of project durations calculated in consecutive iterations of the network. Monte Carlo simulation may be applied to quantify the confidence in the target project completion date or total project duration. The project manager is able to report the probability of completing the project at any particular date, which allows him to set a schedule reserve for the project.

The simulation experiments can be conducted using standard project management software, such as Microsoft Project or Primavera, along with Monte Carlo simulation add-ins, such as @Risk or Risk + (Kwak & Ingall 2007).

To illustrate the impact of project conditions on its duration, the following example is introduced. Figure 5 presents a simple construction project network model. The estimates of process durations (random variables of triangular distribution) are presented in Table 2. The Monte Carlo simulation was conducted using Minuteman GPSS World software.

The cumulative distributions of project duration for three PC scores (0.5 – for average and 0.7 – expected conditions also 0.4 – for project risk level after planned mitigation actions) and are shown in Figure 6. Let us assume that the project manager determines the reliability of the contractual project due date at the level of 0.6. The planned risk mitigation actions allow to reduce the project duration in this simple example from about 20 to 18 days (c.a. 10 %).

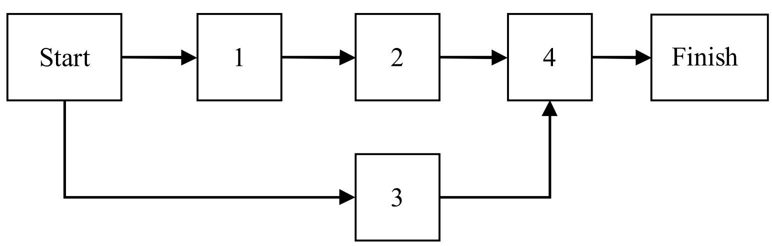

Figure 5. Precedence relationships among processes of the example

Table 2. Estimates of processes durations (example) [days]

Activity	*PC* = 0.4			*PC* = 0.5			*PC* = 0.7		
j	a_j	b_j	m_j	a_j	b_j	m_j	a_j	b_j	m_j
Start	0	0	0	0	0	0	0	0	0
1	5	10	5.95	5	10	7	6.85	10	7
2	4	8.99	5	4	9	5	5.20	9	5.50
3	8	14	9.68	8	14	11	10.29	14	11
4	3	7	4.12	3	7	5	4.53	7	5
Finish	0	0	0	0	0	0	0	0	0

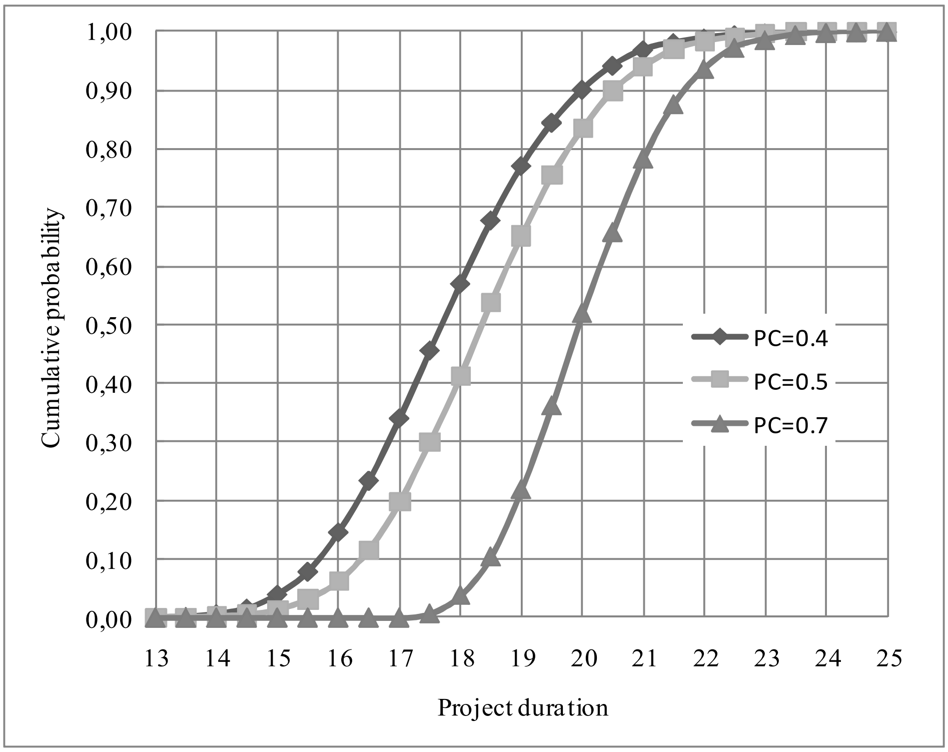

Figure 6. Cumulative distribution function of project duration realized for different conditions scores

3 CONCLUSIONS

The paper presents the framework for construction project risk assessment. The approach enables the planner to estimate the probability distribution of project duration on the basis of the project conditions' evaluation and Monte Carlo simulation technique. The input needed for the analysis should be stored in a contractor database (i.a. the weights of particular project conditions for groups of processes, upper and lower limits per unit, unit production times). A considerable advantage of this approach is seen in the possibility of automated assessment of the impact of risk mitigation actions on the duration of the project.

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DISTRIBUTIONS OF NUMBERS OF CONNECTIVITY COMPONENTS IN RECURSIVELY DEFINED GRAPHS WITH UNRELIABLE ARCS

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ABSTRACT

In this paper a problem of accuracy and approximate calculations of connectivity characteristics in recursively defined random graphs is considered. This problem is solved using low and upper bounds for numbers of connectivity components in graphs and limit theorems of probability theory: law of large numbers and central limit theorem.

1. INTRODUCTION

In this paper a problem of an accuracy and approximate calculations of connectivity characteristics in recursively defined random graphs is considered. This problem is analyzed in papers [1] – [6] and many other ones. But when a graph becomes more complicated a complexity of this solution increases significantly. So it is necessary to introduce additional characteristics of connectivity like numbers of connectivity components. It allows to widen a set of analyzed random graphs essentially.

Analogously to [6, Figure 4] connectivity probability and mean number of connectivity components for parallel aggregation of chains with identical arcs is calculated accurately. But accuracy formulas do not allow to consider manifold practically interesting random graphs. So first step to obtain estimates of the connectivity is to analyze completely connective random graph (where each pair of nodes is connected by single arc). Sufficient conditions of tendency of connectivity probability of this graphs to one are obtained.

Then we transit from accuracy formulas to upper and low bounds. Analogously to [7] upper and low bounds of numbers of connectivity components are constructed for recursively defined graphs which are obtained by a gluing of defined graphs in few nodes. The gluing in single node creates graphs of treelike structure with a bridge or radial-circle generating graphs. But it is not enough and a step to the gluing in a few nodes is made.

In this case upper and low bounds of numbers of connectivity components are obtained by numbers of failed arcs and some deterministic summands. Applying to these bounds law of large numbers and central limit theorem it is possible to remove deterministic summands and to obtain variants of limit theorems. These results are used to parallel aggregations of chains with equal lengths.

2. CONNECTIVITY PROBABILITY IN PARALLEL AGGREGATIONS OF CHAINS OF IDENTICAL ARCS

Consider parallel aggregation of m chains with lengths $n_1 > 0,...,n_m > 0$. Each chain consists of independently working arcs with the failure probability $q = 1 - p$, $0 < p < 1$. Our problem is to calculate the probability Q of the event C , that this aggregation is not connective.

Define the event A that there is a chain with more than one failed arc and the event B, that there is single failed arc in each chain. It is obvious that the events A, B are inconsistent and $A \subset C$, $B \subset C$, $C \cap \overline{A} = B$, consequently, $C = A \cup B$, $Q = P(C) = P(A) + P(B)$, where

$$P(A) = 1 - \prod_{i=1}^{m} \left(p^{n_i} + n_i p^{n_i - 1} q \right), \; P(B) = q^m \prod_{i=1}^{m} n_i p^{n_i - 1} \; . \tag{1}$$

If the chain i consists of arcs which work with the probability p_i and fail with the probability $q_i = 1 - p_i$, then the formulas (1) transforms as follows

$$P(A) = 1 - \prod_{i=1}^{m} \left(p_i^{n_i} + n_i p_i^{n_i - 1} q_i \right), \; P(B) = q^m \prod_{i=1}^{m} n_i p_i^{n_i - 1} q_i \; . \tag{2}$$

If the graph $G = G_1 \to G_2$ is constructed by a gluing of the graphs G_1, G_2 in a single node then the connectivity probability of the graph G equals a product of the connectivity probabilities of the graphs G_1, G_2.

3. NETWORKS WITH LARGE NUMBERS OF NODES AND ARCS

E.A. Nurminsky (oral information) using numerical experiments formulated a hypothesis that if a number of graph nodes and a number of arcs is large also then this graph connectivity probability is close with one. In this section a model of a graph which satisfies this hypothesis is constructed and its sufficient conditions are formulated.

Consider non oriented connective graph G_n with the arcs set W_n and the nodes set $U_n = \{u_1, ..., u_n\}$. Suppose that each pair of nodes may be connected no more than by a single arc. Denote $\varphi_n(i, j)$ a number of nodes $u_k \in U_n$ so that the arcs $w_{ik} = (u_i, u_k) \in W_n$, $w_{kj} = (u_k, u_j) \in W_n$ and put

$$\varphi_n = \min_{1 \le i < j \le n} \varphi_n(i, j), \; \psi_n = \min_{1 \le i < j \le n} \psi_n(i, j),$$

$$\psi_n(i, j) = \min\left(p(w_{ik}) p(w_{kj}) : u_k \in U_n, k \ne i, k \ne j \right).$$

Theorem 1. Suppose that the graph G_n arcs work independently with probabilities $p(w)$, $w \in W_n$ and

$$-\varphi_n \psi_n + 2 \ln n \to -\infty, \; n \to \infty.$$

Then the connectivity probability of the graph G_n satisfies the relation

$$\lim_{n \to \infty} P(G_n) = 1 \; . \tag{3}$$

Proof. Denote $P_n(u_i, u_j)$ the probability that the nodes $u_i, u_j \in U_n$ are connected in the graph G_n. It is obvious that $\overline{P}_n(u_i, u_j) = 1 - P_n(u_i, u_j)$ does not exceed failure probability of all ways (w_{ik}, w_{kj}), which pass through some nodes $u_k \in U_n$, $k \ne i$. Consequently from the inequality $x \ge 1 - \exp(-x)$, $x > 0$, we obtain that

$$\overline{P}_n(u_i, u_j) \le (1 - \psi_n)^{\varphi_n} \le \exp(-\varphi_n \psi_n).$$

As the number of the graph G_n arcs does not exceed $n(n-1)/2$ then

$$0 \le 1 - P(G_n) \le \sum_{1 \le i < j \le n} \overline{P}_n(u_i, u_j) \le \frac{n(n-1)}{2} \exp(-\varphi_n \psi_n) \le \frac{1}{2} \exp(-\varphi_n \psi_n + 2\ln n).$$

From this inequality we obtain the limit relation (3).

Corollary1. If the graphs G_n are completely connective, $n \ge 1$, and for some $c > 0$ the inequalities

$$p\left(w \ge \sqrt{\frac{(2+c)\ln n}{n}} \right), \ w \in W_n,$$

are true then the formula (3) takes place.

Corollary2. If in the graphs G_n, $n \ge 1$, the conditions $\varphi_n - \ln n \to \infty$, $n \to \infty$, are true and for some $p > 0$ the inequalities $p(w) \ge p$, $w \in W_n$ take place then the formula (3) is true also.

4. MEAN NUMBER OF CONNECTIVITY COMPONENTS IN PARALLEL AGGREGATION OF CHAINS

Calculate now the mean number S of connectivity components in parallel aggregation of m chains with $n_1 = ... = n_m = n$ arcs.

Theorem 2. The following formula is true:

$$S = mnq + (1 - p^n)^m + (1 - m) + mp^n \ . \qquad (4)$$

Proof. Define auxiliary expressions $P_n(k_1, ..., k_m)$ - the probability of k_i failures in the chain $i, k_i = 0, ..., n$, $i = 1, ..., m$, $p_n(k) = C_n^k q^k p^{n-k}$, $k_i = 0, ..., n$,

$$S_1 = \sum_{k_1 > 0, ..., k_m > 0} (k_1 + ... + k_m - m + 2) P_n(k_1, ..., k_m) =$$

$$= \sum_{k_1 > 0, ..., k_m > 0} (k_1 + ... + k_m - m + 2) p_n(k_1) \cdot ... \cdot p_n(k_m) =$$

$$= mnq(1 - p^n)^{m-1} + (2 - m)(1 - p^n)^m,$$

$$S_2 = \sum_{r=1}^{m} C_m^r p^{nr} \left[(m - r) nq (1 - p^n)^{m-r-1} + (1 - m + r)(1 - p^n)^{m-r} \right], \ S = S_1 + S_2.$$

Here S_1 - is mean number of connectivity components if there are failures in all chains, S_2 is mean number of connectivity components if there is positive number of chains without failures. Denote

$$S_2' = \sum_{r=1}^{m} C_m^r p^{nr} (m - r) nq (1 - p^n)^{m-r-1} = (t = m - r) =$$

$$= (1 - p^n)^{-1} nq \sum_{t=1}^{m-1} C_m^t p^{n(m-t)} t (1 - p^n)^t =$$

$$= (1 - p^n)^{-1} nq \sum_{t=0}^{m-1} C_m^t p^{n(m-t)} t (1 - p^n)^t = (1 - p^n)^{-1} nq \left[m(1 - p^n) - m(1 - p^n)^{m-1} \right] =$$

$$= mnq \left[1 - (1 - p^n)^{m-1} \right],$$

$$S_2'' = \sum_{r=1}^{m} C_m^r p^{nr} (1 - m + r)(1 - p^n)^{m-r} = \left(\sum_{r=0}^{m} - \sum_{r=0}^{0} \right) C_m^r p^{nr} (1 - m + r)(1 - p^n)^{m-r} =$$

$$=(1-m)\left(1-\left(1-p^n\right)^m\right)+\sum_{r=0}^{m}C_m^r p^{nr} r\left(1-p^n\right)^{m-r}=(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n .$$

Consequently

$$S_2=S_2'+S_2''=mnq\left[1-\left(1-p^n\right)^{m-1}\right]+(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n ,$$

and so

$$S=mnq\left(1-p^n\right)^{m-1}+(2-m)\left(1-p^n\right)^m+mnq\left[1-\left(1-p^n\right)^{m-1}\right]+$$

$$+(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n=mnq+(2-m)\left(1-p^n\right)^m+(1-m)\left(1-\left(1-p^n\right)^m\right)+mp^n=$$

$$=mnq+\left(1-p^n\right)^m+(1-m)+mp^n .$$

5. DISNRIBUTION OF NUMBER OF CONNECTIVITY COMPONENTS IN RECURSIVELY DEFINED GRAPHS

Recursively defined class of graphs. Consider recursively defined class $\mathcal{A}$ of graphs with identical arcs. Suppose that $A-$ is enumerable set of arcs called a system of generating arcs. Each graph $g\in A$ characterizes by numbers $n(g)=1$, $m_i(g)=0$, $i\geq 1$. The class $\mathcal{A}$ is defined by rolls: $A\subset\mathcal{A}$, if, $g_1\subset\mathcal{A}$, $g_2\subset\mathcal{A}$ and the sets of these graphs arcs do not intersect, then the aggregation $g_1\overset{(i)}{\cdot}g_2$ constructed by a gluing of the graphs g_1,g_2 in $i\geq 1$ nodes belongs to the class $\mathcal{A}$.also and

$$n\left(g_1\overset{(i)}{\cdot}g_2\right)=n(g_1)+n(g_2),\quad m_j\left(g_1\overset{(i)}{\cdot}g_2\right)=m_j(g_1)+m_j(g_2)+\delta_{ij},\quad 1\leq i,\quad 1\leq j .$$

Here δ_{ij} - is Kroneker symbol, $n(g)$- is a number of arcs and $m_i(g)$ - is a number of "$\overset{(i)}{\cdot}$"connections in a graph $g\in A$.

Example 1. The class of parallel-sequential graphs is an example of recursively defined class $\mathcal{A}$ which is widely used in reliability theory [1].

Inequalities for numbers of connectivity components in random realizations of graphs. Assume that arcs of a graph $g\in A$ work independently with the probability $p, 0<p<1$ and fail with the probability $q=1-p$. For each realization of the graph $g'\in A$ arcs it is possible to define random number $l(g')$ of failed arcs and random number $k(g')$ of connectivity components. Assume that edges of failed arcs belong to this graph realization. Designate $m(g)=\sum_{i\geq 1}(i-1)m_i(g)$

.

Lemma 1. For each random realization g' of the graph $g\in A$ the following inequalities take place:

$$l(g')-2m(g)+1\leq k(g')\leq l(g')+1 . \qquad (5)$$

Proof. Using recursive definition of the class $\mathcal{A}$ it is easy to prove that almost surely the following formulas are true: for realizations g_1',g_2' of graphs $g_1\in A$, $g_2\in A$ and for a realization g' of a graph $g\in A$

$$k(g_1')+k(g_2')-2i+1\le k\left(g_1' \overset{(i)}{\cdot} g_2'\right)\le k(g_1')+k(g_2')-1,$$
$$k(g')=l(g')+1,\ l\left(g_1' \overset{(i)}{\cdot} g_2'\right)=l(g_1')+l(g_2'),\ i\ge 1. \tag{6}$$

Indeed for $g\in A$ the inequalities (5) are true as $k(g')=l(g')+1$. Assume that these inequalities take place for random realizations g_1', g_2' of the graphs $g_1\in A$, $g_2\in A$. Then

$$k\left(g_1' \overset{(i)}{\cdot} g_2'\right)\le k(g_1')+k(g_2')-1\le l(g_1')+1+l(g_2')+1-1=l\left(g_1' \overset{(i)}{\cdot} g_2'\right)+1,$$

$$k\left(g_1' \overset{(i)}{\cdot} g_2'\right)\ge k(g_1')+k(g_2')-2i+1\ge l(g_1')-2m(g_1)+1+l(g_2')-2m(g_2)+1-2i+1=$$

$$=l\left(g_1' \overset{(i)}{\cdot} g_2'\right)-m\left(g_1 \overset{(i)}{\cdot} g_2\right)+1.$$

Limit theorems for numbers of connectivity components in recursively defined graphs. Remark that random quantity $l(g')$ may be represented as a sum $\sum_{i=1}^{n(g)}\eta_i$ of independent random variables η_i, $P(\eta_i=1)=q$, $P(\eta_i=0)=p$, $i=1,...,l(g)$.

Theorem 3. Suppose that $m(g)/n(g)\to 0$, $n(g)\to\infty$. Then almost surely

$$\frac{k(g')}{qn(g)}\to 1\ ,n(g)\to\infty\ . \tag{7}$$

Proof. Rewrite the inequalities (4) as follows

$$\frac{l(g')}{qn(g)}+\frac{1-2m(g)}{qn(g)}\le\frac{k(g')}{qn(g)}\le\frac{l(g')}{qn(g)}+\frac{1}{qn(g)}. \tag{8}$$

Theorem 3 statement is a corollary of the inequalities (7) and enforced law of large numbers [8, chapter IV, §3].

Theorem 4. Suppose that $m(g)/\sqrt{n(g)}\to 0$, $n(g)\to\infty$, then random variable $\left(k(g')-qn(g)\right)/\sqrt{pqn(g)}$ distribution tends to normal distribution with zero mean and single variation.

Proof. Rewrite the inequality (5) as follows

$$\frac{l(g')-qn(g)}{\sqrt{pqn(g)}}+\frac{1-2m(g)}{\sqrt{pqn(g)}}\le\frac{k(g')-qn(g)}{\sqrt{pqn(g)}}\le\frac{l(g')-qn(g)}{\sqrt{pqn(g)}}+\frac{1}{\sqrt{pqn(g)}}\ , \tag{9}$$

Then Theorem 4 is a corollary of the inequalities (9) and integral Muavre-Laplas theorem [8, chapter I, §6].

Limit theorems for numbers of connectivity components in parallel aggregation of chains. Consider important partial case of the graph g_m which is an aggregation of m parallel chains with the length n. Using Theorems 3, 4 it is possible to prove that for $n\to\infty$ random sequence

$k(g'_m)/qnm$ almost surely tends to 1, $n\to\infty$ for m which may depend on n arbitrarily. More over if $m/n\to 0, n\to\infty$, then distribution of random variable $\left(k(g')-qnm\right)/\sqrt{pqnm}$ for $n\to\infty$ tends to normal distribution with zero mean and single variation. But there is a question connected with a behavior of this sequence when $m\to\infty$ if for example there is $N<\infty$ so that $1<n<N<\infty$. This problem may be solved as follows.

It is easy to prove the recurrent formula

$$k(g'_{m+1})=k(g'_m)+\left(\gamma_{m+1}-1+\chi(\gamma_{m+1}=0)\right)-\chi(\gamma_{m+1}=0)\chi(\gamma_1>0,...,\gamma_m>0), m\ge 1, \qquad (10)$$

and the initial condition

$$k(g'_1)=\gamma_1+1. \qquad (11)$$

Here γ_m is a number of failed arcs in m - th chain, $\chi(.)$ is an indicator function of an event ".". Using the formulas (10), (11) it is easy to obtain that

$$k(g'_m)=1+\gamma_1+\sum_{i=2}^{m}\left(\gamma_i-1+\chi(\gamma_i=0)\right)-\sum_{i=2}^{m}\chi(\gamma_i=0)\chi(\gamma_1>0,...,\gamma_{i-1}>0),\ m\ge 1.$$

Consequently we have that

$$k(g'_m)=2+\sum_{i=1}^{m}\left(\gamma_i-1+\chi(\gamma_i=0)\right)-\sum_{i=1}^{m}\chi(\gamma_i=0)\chi(\gamma_1>0,...,\gamma_{i-1}>0), m\ge 1. \qquad (12)$$

and

$$1+\sum_{i=1}^{m}\left(\gamma_i-1+\chi(\gamma_i=0)\right)\le k(g'_m)\le 2+\sum_{i=1}^{m}\left(\gamma_i-1+\chi(\gamma_i=0)\right). \qquad (13)$$

Calculate now a mean and a variation of the random variable $\left(\gamma_i-1+\chi(\gamma_i=0)\right)$:

$$M\left(\gamma_i-1+\chi(\gamma_i=0)\right)=nq-1+p^n, \qquad (14)$$

$$M\left(\gamma_i-1+\chi(\gamma_i=0)\right)^2=M\left[\gamma_i^2+1+\chi(\gamma_i=0)-2\gamma_i+2\gamma_i\chi(\gamma_i=0)-2\chi(\gamma_i=0)\right]=$$

$$=npq+n^2q^2+1+p^n-2nq-2p^n=npq+n^2q^2+1-2nq-p^n,$$

$$D\left(\gamma_i-1+\chi(\gamma_i=0)\right)=npq+n^2q^2+1-2nq-p^n-\left(nq-1+p^n\right)^2=$$

$$=npq+n^2q^2+1-2nq-p^n-n^2q^2-1-p^{2n}+2nq+2p^n-2nqp^n=npq-p^{2n}+p^n-2nqp^n. \qquad (15)$$

From the formulas (13) - (15) and enforced law of large numbers we obtain that almost surely $k(g'_m)/m\to nq-1+p^n,\ m\to\infty$. And from central limit theorem the distribution of random variable

$$\frac{k(g'_m)-m\left(nq-1+p^n\right)}{\sqrt{m\left(npq-p^{2n}+p^n-2nqp^n\right)}}$$

tends to normal distribution with zero mean and single variation.

5. CONCLUSION

In this paper a static model of a graph with unreliable arcs is considered and a connectivity probability and a distribution of connectivity components are considered. But all obtained results may be spread onto a graph in which an arc w has failure intensity λ_w и and renewal intensity μ_w

so that $\lambda_w / \left(\lambda_w + \mu_w \right) = p$. In this case limit connectivity probability of the graph G and limit distribution of connectivity components are analogous to the same characteristics of static model.

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DETERMINATION OF MEAN TIME TO FAILURE OF A NETWORK CONSISTING OF IDENTICAL NON-REPAIRABLE ELEMENTS

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ABSTRACT

It is suggested the analytical model permitting to get expression for determination of mean time to failure of a network consisting of identical non-repairable elements that fail independently of one another and have exponential distribution of time to failure. To determine values of obtained expressions it is necessary to determine probability of network failure in failure of defined quantity of its elements. This factor may be determined exactly with analysis of all possible combinations of failed elements or approximately with Monte-Carlo method.

1. INTRODUCTION

In assessment of reliability of networks consisting of non-repairable elements, determination of mean time to failure is of great interest.

In suggested work it is considered Markov process, states of which are characterized with quantity of failed elements and the state of network. The analysis of this process permits obtaining of analytical expression for mean time to failure of networks consisting of identical non-repairable elements. It is supposed that network elements fail independently one on another and have exponential distribution of time to failure.

In considered examples we will use connectivity of a network as criterion of its operability, however, obtained expressions are true for other criterion of its operability.

2. RELIABILITY MODEL OF NETWORK CONSISTING OF IDENTICAL NON-REPAIPABLE ELEMENTS

We supposed that network nods are absolutely reliable, and edges are identical in reliability, fail independently of one another and have exponential distribution of time to failure.

We will use probability of network failure due to ***i*** elements failure as main network parameter permitting us to determine values of network reliability factor under consideration. We will denote this factor as - $\mathbf{Z_i}$.

Value of $\mathbf{Z_i}$ is equal to ratio quantity of non-operable states of the network in case failure of ***i*** elements ($\mathbf{Y_i}$) to the total quantity of possible combinations of ***i*** elements of ***n***, where ***n*** is the quantity of network elements.

$$Z_i = \frac{Y_i}{\binom{n}{i}} \tag{1}$$

Let us determine values $\mathbf{Y_i}$ for network presented in Figure 1. Since considered network is biconnected, therefore one edge moving off cannot break its connectivity. Hence $\mathbf{Y_1=0}$. For definition of $\mathbf{Y_2}$ and $\mathbf{Y_3}$ we consider all possible states of the network when 2 and 3 edges are moved off respectively. It is possible to define that 2 cutsetss of capacity 2, and 14 cutsetss of capacity 3 are within the considered network, hence $\mathbf{Y_2=2}$, $\mathbf{Y_3=14}$. Any combination of i edges at $\boldsymbol{i > 3}$ is a cutsets, hence $Y_i = \binom{n}{i}$ for $\boldsymbol{i>3}$.

Values $\mathbf{Z_i}$ for this network: $\mathbf{Z_1=0, Z_2=2/21, Z_3=14/35, Z_4=Z_5=Z_6=Z_7=1}$. More detailed analysis of this network is described in article Tkachev (2011).

For small $\boldsymbol{n}$ the values of $\mathbf{Y_i}$ can be determined by means of enumeration of all possible network states. For great values of $\boldsymbol{n}$ it is necessary to use Monte-Carlo method.

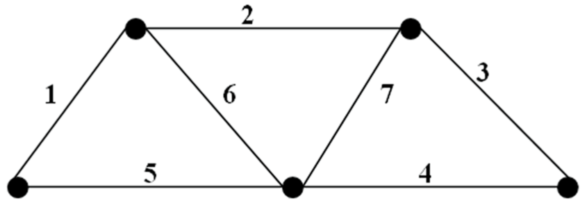

Figure1: Example of a network

Let us denote with $\mathbf{Z^*_i}$ the probability of network failure in failure of $\boldsymbol{i}$ element, if the network was operable for $\boldsymbol{i-1}$ failed elements. Correlation of values $\mathbf{Z^*_i}$ и $\mathbf{Z_i}$ was established in article Tkachev (1983).

Let us consider Markov chain describing change of network states at the moments of its elements failure.

If for state of $\boldsymbol{i-1}$ failed elements the network is operable, then for failure of $\boldsymbol{i}$-element it transforms with probability $\mathbf{Z^*_i}$ to non-operable state, or with probability $\mathbf{1- Z^*_i}$ it will remain in operable state.

If in presence of $\boldsymbol{i-1}$ failed elements the network was non- operable, than for the failure of $\boldsymbol{i}$-element it will remain with probability of $\boldsymbol{1}$ in non-operable state.

Transition diagram of considered process is shown in Figure 2. States $\boldsymbol{i'}$ correspond to network operable states, and states $\boldsymbol{i''}$ correspond to non-operable states of network for failure of $\boldsymbol{i}$ elements.

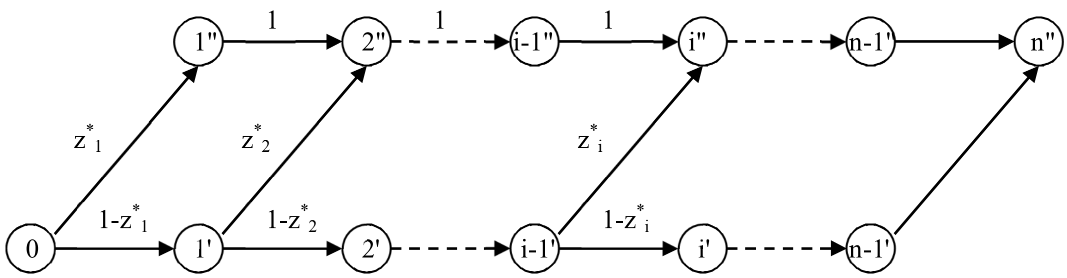

Figure 2: Network state transition diagram

Let us denote $\boldsymbol{P_{i'}}$ – probability of operable state, and $\boldsymbol{P_{i''}}$ –probability of non-operable state of network in the case of failure of $\boldsymbol{i}$ elements. From definition of $\boldsymbol{Z_i}$ it follows

$$P_{i'} = 1 - Z_i \tag{2}$$

$$P_{i''} = Z_i \tag{3}$$

In accordance with diagram (Figure 2), the network state transition can be written as

$$P_{i'} = P_{(i-1)'}(1 - Z^*{}_i) \tag{4}$$

wherefrom

$$Z^*_i = \frac{P_{(i-1)'} - P_{(i)'}}{P_{(i-1)'}} \tag{5}$$

Substituting in (5) values $P_{i'}$ и $P_{i''}$ from (2) and (3) we obtain

$$Z^*_i = \frac{Z_i - Z_{i-1}}{1 - Z_{i-1}} \tag{6}$$

Besides, from expression (4) it follows

$$P_{i'} = \prod_{j=1}^{i} 1 - Z^*_j = 1 - Z_i \tag{7}$$

For determination of mean time to failure we will consider continuous Markov process, describing system behavior in time. Process states are preset with quantity of failed elements and states of network. State transition diagram of this process is shown in Figure 3.

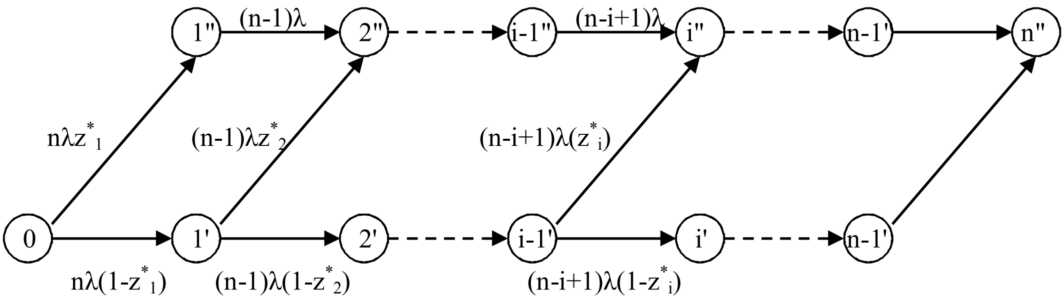

Figure 3: The Markov process of network state transition.

Let us denote elements failure rate as λ. Let in some moment of time to be i failed elements and the network to be operable. In infinitely small time interval Δt it can occur one of the following events:

- network will remain in operable state.

Probability of this event **(1-(n-i)λ Δt);**

- One more element will fail and the network will transit to non-operable state.

Probability of this event **(1-(n-i)λ Z^*_{i+1} Δt);**

- One more element will fail but the network will remain in operable state.

Probability of this event **(1-(n-i)λ $(1-Z^*_{i+1})$ Δt).**

To simplify analytic calculations we will substitute a set of non-operable states with one absorbing state Figure 4.

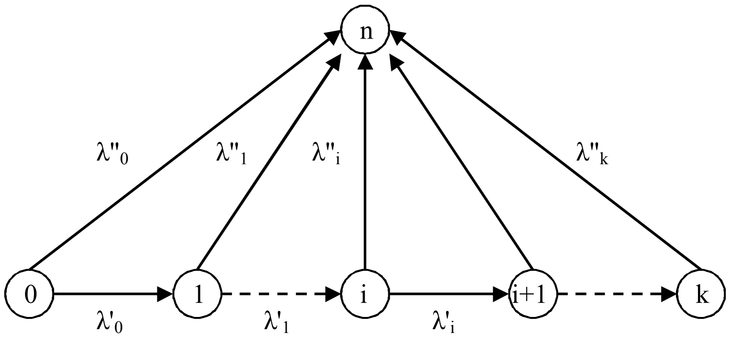

Figure 4: The Markov process of network state transition with absorbing state

Probabilities of process being in different states in arbitrary moment of time **$P_i(t)$** can be found by means of solving the following differential equation system:

$$\begin{aligned}
\frac{\partial P_0(t)}{\partial t} &= -\lambda_0^{'} P_0(t) - \lambda_0^{''} P_0(t) = -\lambda_0 P_0(t) \\
\frac{\partial P_i(t)}{\partial t} &= \lambda_{i-1}^{'} P_{i-1}(t) - \lambda_i^{'} P_i(t) - \lambda_i^{''} P_i(t) = \lambda_{i-1}^{'} P_{i-1}(t) - \lambda_i P_i(t) \\
\frac{\partial P_n(t)}{\partial t} &= \sum_{i=0}^{k} \lambda_i^{''} P_i(t)
\end{aligned} \tag{8}$$

Here

$\lambda_i=(n-i)\lambda$

$\lambda'_i=(1- Z^*_{i+1})\, \lambda_i$

$\lambda''_i=Z^*_{i+1}\, \lambda_i$

k – is the maximum quantity of elements, after the failure of which the network can be operable.

Let us solve differential equation system (8) using Laplace transformation under initial conditions $P_0(0)=1$, $P_i(0)=0 \;\; \forall i \neq 0$.

Let us designate

$$F(s) = \int_0^{\infty} P(t) e^{-st} \partial t$$

Then differential equation system (8) is deduced into algebraic equation system relatively to F(s).

$$\begin{aligned}
sF_0(s) &= 1 - \lambda_0 F_0(s) \\
sF_i(s) &= \lambda_{i-1}^{'} F_{i-1}(s) - \lambda_i F_i(s) \\
sF_n(s) &= \sum_{i=0}^{k} \lambda_i^{''} F_i(s)
\end{aligned} \tag{9}$$

Wherefrom we have

$$\begin{aligned}
F_0(s) &= \frac{1}{s+\lambda_0} \\
F_i(s) &= \frac{\lambda_{i-1}^{'}}{s+\lambda_i} F_{i-1}(s) \\
F_n(s) &= \frac{1}{s} \sum_{i=0}^{k} \lambda_i^{''} F_i(s)
\end{aligned} \tag{10}$$

This solution permits obtaining of expression for mean time to failure. According to definition Kozlov&Ushakov (1975).

$$T = \sum_{i=0}^{k} F_i(s) \Big| s = 0 \tag{11}$$

Expanding (10) we obtain

$$F_0(0) = \frac{1}{\lambda_0} = \frac{1}{n\lambda} \tag{12}$$

$$F_i(0) = \frac{\lambda'_{i-1}}{\lambda_i} F_{i-1}(0) = \frac{\prod_{j=1}^{i}(1 - Z_j^*)}{(n-i)\lambda} \tag{13}$$

Using (7) the expression (13) may be simplified

$$F_i(0) = \frac{\prod_{j=1}^{i}(1 - Z_j^*)}{(n-i)\lambda} = \frac{1 - Z_i}{(n-i)\lambda} \tag{14}$$

Then from (11) it follows

$$T = \frac{1}{\lambda}\sum_{i=0}^{k}\frac{(1 - Z_i)}{n - i} \tag{15}$$

Expression (15) permits determine the value of mean time to failure for known values $\boldsymbol{Z_i}$ and $\boldsymbol{\lambda}$.

Let us determine value $\boldsymbol{T}$ for network, shown in Figure 1, for λ=0,01 (1/hour)

$$T = 100(\frac{1}{7} + \frac{1}{6} + (\frac{19}{21})\frac{1}{5} + (\frac{21}{35})\frac{1}{4}) = 100 * \frac{269}{420} \approx 64{,}047(hour)$$

3. COMPARISON WITH KNOWN RESULTS

To check obtained expression let us determine mean time to failure of a system, for which there are known analytical assessments Figure 5.

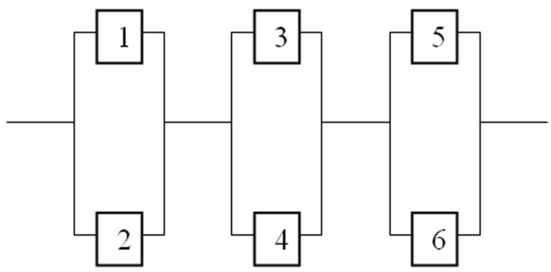

Figure 5: Series-parallel system

In Gnedenko&Belyayev&Solovyov (1965) there can be found following expressions. Probability of failure-safe operation of system consisting of 2 parallel identical elements.

$$P_{(t)} = 1 - (1 - e^{-\lambda t})^2 = 2e^{-\lambda t} - e^{-2\lambda t}$$

Probability of failure-safe operation of system consisting of 3 parallel identical, series-connected subsystems.

$$P_{c(t)} = (P_{(t)})^3$$

$$P_{c(t)} = (2e^{-\lambda t} - e^{-2\lambda t})^3 = 8e^{-3\lambda t} - 12e^{-4\lambda t} + 6e^{-5\lambda t} - e^{-6\lambda t}$$

Mean time to failure is equal to

$$T = \int_0^{\infty} P_{c(t)} \partial t = \frac{1}{\lambda}(\frac{8}{3} - \frac{12}{4} + \frac{6}{5} - \frac{1}{6}) = \frac{1}{\lambda}(\frac{42}{60}) = 0{,}7\frac{1}{\lambda}$$

For determination of mean time to failure with use of expression (15), it is necessary to determine values Z_i. For considered system they are equal to: Z_1=0,0; Z_2=3/15; Z_3=12/20; Z_4 = Z_5 = Z_6=1.

Substituting these values into expression (15), we obtain:

$$T = \frac{1}{\lambda}(\frac{1}{6} + \frac{1}{5} + (\frac{12}{15})\frac{1}{4} + (\frac{8}{20})\frac{1}{3}) = \frac{1}{\lambda}(\frac{42}{60}) = 0{,}7\frac{1}{\lambda}$$

This example also shows that suggested method can be used for determination of mean time to failure of complicated series-parallel and parallel-series systems consisting of identical non-repairable elements.

4. EXAMPLE OF NETWORKS RELIABILITY ANALYSIS

Let us denote the number of nodes by ***m*** and the number of edges by ***n***. Let us consider network with parameters m=20, n=24, Figure 6. Values **Zi** of this network can be determined by means of enumeration of all possible network states. Calculation results are shown in table 5.

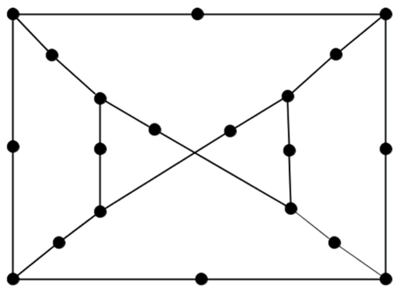

Figure 6: Network with parameters m=20, n=24.

Table 5. Values $\mathbf{Y_i}$ and $\mathbf{Z_i}$ for network on Figure 6.

i	1	2	3	4	5	i>=6
Y_i	0	12	328	4082	29960	
Z_i	0	0,043478	0,162055	0,384252	0,704875	1

Suggested method can be used for other criteria of a network operability. For example: the network is operable, if number of connected nodes **>= m-k**.

Obtained results can also be used in this case, but it is necessary to make some corresponding corrections in determination algorithm of values $\mathbf{Z_i}$**.** In table 6 there are listed values $\mathbf{Z_i}$ for different values of **k** for network shown in Figure 6. Determination of values $\mathbf{Z_i}$ for k>0 was carried out with Monte-Carlo method. Number of tests amounted 10^6.

Table 6. Values of $\mathbf{Z_i}$ for different values of **k** for network on Figure 6.

i / k	0	2	4	6	8	10
1	0,0000000	0,00000	0,00000	0,00000	0,00000	0,00000
2	0,0434783	0,00000	0,00000	0,00000	0,00000	0,00000
3	0,1620553	0,01581	0,00000	0,00000	0,00000	0,00000
4	0,3841521	0,07980	0,01498	0,00435	0,00296	0,00000
5	0,7048748	0,25247	0,08813	0,03329	0,01644	0,00000
6	1,0000000	0,56194	0,27850	0,13070	0,05976	0,00324
7	1,0000000	0,86648	0,56942	0,33252	0,17912	0,04445
8	1,0000000	1,00000	0,82642	0,58123	0,36764	0,16783
9	1,0000000	1,00000	0,96544	0,79849	0,57894	0,34029
10	1,0000000	1,00000	1,00000	0,93211	0,76173	0,52593
11	1,0000000	1,00000	1,00000	0,98807	0,89262	0,69336
12	1,0000000	1,00000	1,00000	1,00000	0,96470	0,82566
13	1,0000000	1,00000	1,00000	1,00000	0,99360	0,91743
14	1,0000000	1,00000	1,00000	1,00000	1,00000	0,97046
15	1,0000000	1,00000	1,00000	1,00000	1,00000	0,99403
α	0,214850615	0,29501	0,354027	0,411813	0,476755	0,571528

$$\alpha = \sum_{i=0}^{k} \frac{(1-Z_i)}{n-i}$$

$$T = \alpha * \lambda$$

It should be noticed that obtained results can be used in the case when nods failed, though edges are absolutely reliable. Node failure can be modeled by removal of all edges coming from this node.

5. CONCLUSION

It was obtained the analytical expression for determination of mean time to failure of networks consisting of identical non-repairable elements that fail independently of one another and have exponential distribution of time to failure. And as the nods, so the edges can be assumed as absolutely reliable.

Suggested method can also be used for reliability assessment of complicated series-parallel systems.

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CFBLTQ: A CLOSED FEED BACK LOOP TYPE QUEUING SYSTEM; MODELING AND ANALYSIS

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ABSTRACT

This paper presents an innovative approach to solve probability distributions of a closed feed back loop type queuing system with general service time distribution. This model is applied to a multi-processors system where some of its nodes are performed a repair procedure during a node's malfunction condition. Our model is appropriate for a multiprocessor system that employs a common bus or for a multi-node system in computer network. A meticulous analysis of the system's model has been conducted and numerical results have been obtained to scrutinize the proposed model.

1 INTRODUCTION

The queuing system is widely classified into an open-type system and a closed-type system model which are described by Baskett et al. (1975). The open-type system model customers arrive from outside and depart to the outside of the system while in the closed system, the customers operate internally where no customers arrive from outside or depart to outside of the system. Numerous research works have been extensively dedicated to investigate the open system model (classical model) which is widely used in computer systems and computer networks. However, the closed system model has not been paid much attention in spite of its paramount importance to computer systems. Some research works have devoted to find an optimal solution to the closed queuing behaviour at a group of systems that form networks (See Benson & Gregory 1961, Jeffrey & Buzen 1973, Kakubava 2010, Denning & Buzen1978, Lavenberg & Reiser1980, Reiser & Lavenberg 1980), or at a particular system such as cyclic systems (See Koenigsberg, 1958, Gordon et al. 1967, Lipsky 1985, Lavenberg 1989). The reliability and flexibility of closed queuing systems with repairable elements has been investigated by Chinho et al. (1994) & Kakubava (2010), both of these works are considered two types of service operations where in Chinho (1994) examined the flexibility and the capacity of the repair station, while Kakubava (2010) scrutinized the closed queuing system for replacements and renewals.

This paper has made very punctilious efforts to formulate the closed system's behaviour for maintenance operation at a maintenance repair center in the course of malfunctions which might develop in the system during repairing procedures. A failed element is removed from the system while keeping the system operates normally. Our proposed a closed queuing system consists of only one type of service operation with a closed feed back loop type queuing system *CFBLTQ* model. We study the reliability and the limitation of the system in case of failed elements increased, also we will investigate of how far faults tolerance that the system can offer.

2 MODEL OF A CLOSED LOOP TYPE QUEUING SYSTEM

The proposed system consists of multi elements or processors which are autonomously operated. When any of the system's elements malfunction is reported, this element is required for repair operation (or service) at the service repair center (server). The repaired element is subsequently put to work in the system again.

This kind of element's failure and repair procedure is called Closed Feed Back Loop Type *CFBLTQ* queuing system, an example of this model is illustrated in Figure 1. The *CFBLTQ* model has fault tolerance, with respect to a single or a multi-element failure. To formulate the system's model, we define λ as elements' failure rate per unit time and μ as the rate of completions' repairing per unit time when the system is busy. Meanwhile, the exponential distributions for both arrival time and service time have been considered. The system has also self configuration feature that can tolerant temporary failures while distributing tasks which have been assigned to the failure element to other active elements. The system can tolerant up to m of N elements $(m \in N)$ while the system operation will be in a normal operation condition if the number of faulty elements are less than or equal to m. The fault tolerant is defined by probability of working elements in the system which the probability of a minimum number of elements m in the system while keeping the system operates normally. The following sub-section will present a mathematical procedure of how to obtain the probability of working elements.

As mentioned above, the proposed system's model is applicable for numerous systems' applications in computer systems and we will focus on a maintenance operation at a maintenance repair station.

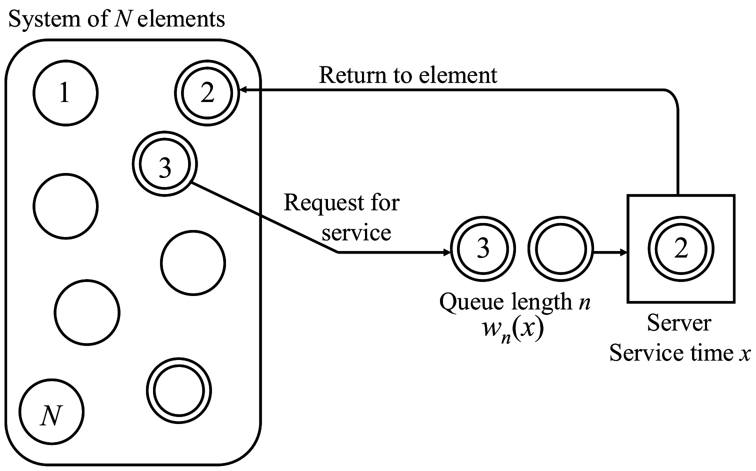


Figure 1. An example model of a closed feed back loop type queuing system model.

3 AN ANALYSIS OF THE CLOSED QUEUING SYSTEM

A systematic approach is given in this section for proper analysis of *CFBLTQ* model, an example of this model is shown in Figure 1. The *CFBLTQ* model is described below and the properties of the model is given by

(1) We consider the system is in a steady state, and the queue is first in and first service (FIFS) model's discipline.

(2) Let the number of elements in the system be N (N >1). The request arrivals for service due to elements' malfunctions follow an exponential distribution with elements' failure rate of λ.

(3) Let the service time distribution be a general distribution. Suppose the probability that a service is started between arbitrary time t and time $t+\Delta$ is equal to $\mu(x)\Delta+o(\Delta^2)$ on service time x, and the density function $f(x)$ of service time distribution is given by

$$f(x) = \mu(x) \cdot e^{-\int_0^x \mu(y)dy} \tag{1}$$

The full derivation of above equation is given in the Appendix of this paper.

(4) Let the probability density function of the service time x with n queue length be $w_n(x)$. The state probabilities p_n are given by

$$p_n = \int_0^{\infty} w_{n-1}(x)\,dx \qquad (n = 1,2,\cdots,N) \tag{2}$$

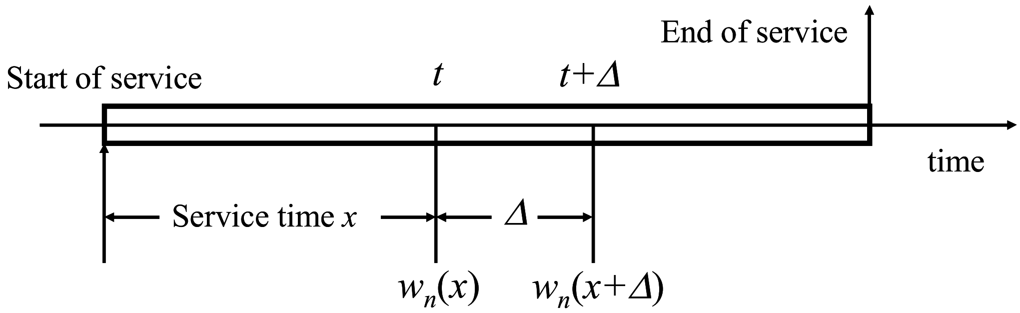


Figure 2. The relationship among n states $w_n(x)$ at arbitrary time t and $t+\Delta$, when the service continues.

From above notations and since the service is continued, as shown in Figure 2, the relationship between $w_n(x)$ in arbitrary time t and $w_n(x+\Delta)$ in time $(t+\Delta)$ is given as follows

$$w_0(x+\Delta) = \{1-(N-1-n)\cdot\lambda\Delta\}\cdot\{1-\mu(x)\Delta\}\cdot w_0(x) + o(\Delta^2) \tag{3}$$

$$\begin{aligned} w_n(x+\Delta) &= \{1-(N-1-n)\cdot\lambda\Delta\}\cdot\{1-\mu(x)\Delta\}\cdot w_n(x) \\ &\quad + (N-n)\cdot\lambda\Delta\cdot w_{n-1}(x) + o(\Delta^2) \qquad (n = 1,2,\cdots,N-1) \end{aligned} \tag{4}$$

For $\Delta \to 0$, we can differentiate the above equations with respect to x to obtain

$$\frac{d}{dx} w_0(x) + \{(N-1)\cdot\lambda + \mu(x)\}\cdot w_0(x) = 0 \tag{5}$$

$$\frac{d}{dx} w_n(x) + \{(N-1-n)\cdot\lambda + \mu(x)\}\cdot w_n(x) = (N-n)\cdot\lambda\cdot w_{n-1}(x) \quad (n = 1,2,\cdots,N-1) \tag{6}$$

In order to solve the above differential equations, the differential Equations $w_n(x)$ of (5) and (6) become

$$w_0(x) = C_0 \cdot e^{-(N-1)\cdot\lambda x - \int_0^x \mu(x)dx} \tag{7}$$

$$\begin{aligned} w_n(x) = \{C_n &+ (-1)^1 \frac{(N-n)}{1!}\cdot C_{n-1}\cdot e^{-\lambda x} + (-1)^2 \frac{(N-n+1)(N-n)}{2!}\cdot C_{n-2}\cdot e^{-2\cdot\lambda x} \\ &+ \cdots + (-1)^n \frac{(N-1)\cdots(N-n)}{n!}\cdot C_0\cdot e^{-n\cdot\lambda x}\}\cdot e^{-(N-n-1)\cdot\lambda x - \int_0^x \mu(y)dy} \\ &\qquad (n = 1,2,\cdots,N-1) \end{aligned} \tag{8}$$

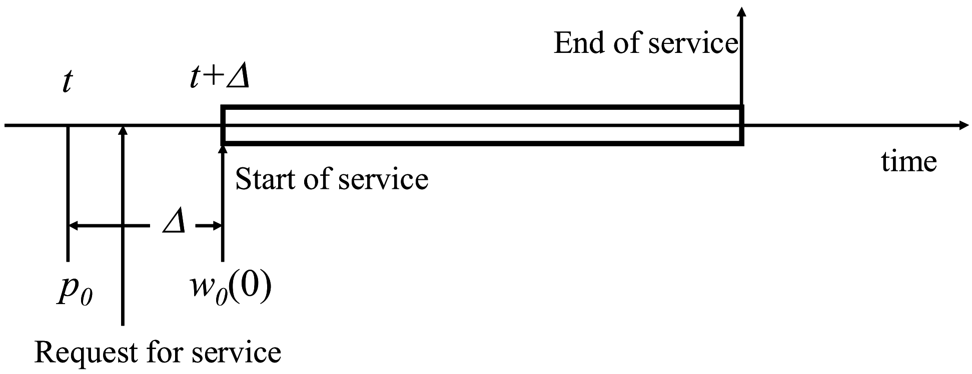
where C_n $(n = 0, 1, 2, \ldots, N-1)$ is a constant value given by the boundary conditions at the start point or at the end point of service. Moreover, from (2), the state probabilities are

$$p_1 = \int_0^{\infty} w_0(x)dx = C_0 \cdot \frac{1-f^*\{(N-1)\lambda\}}{(N-1)\lambda} \tag{9}$$

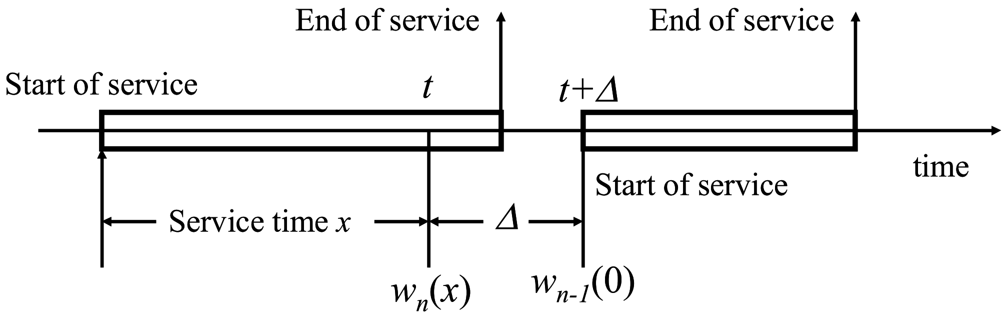
$$p_n = \int_0^\infty w_{n-1}(x)dx = C_{n-1} \cdot \frac{1-f^*\{(N-n)\lambda\}}{(N-n)\lambda} + (-1)^1 \frac{(N-n+1)!}{1!} \cdot C_{n-2} \cdot \frac{1-f^*\{(N-n+1)\lambda\}}{(N-n+1)\lambda} + \cdots$$
$$+ (-1)^{n-1} \frac{(N-1)\cdots(N-n+1)}{(n-1)!} \cdot C_0 \cdot \frac{1-f^*\{(N-1)\lambda\}}{(N-1)\lambda}$$
$$+ (-1)^{n-1} \frac{(N-1)\cdots(N-n+1)}{(n-1)!} \cdot C_0 \cdot \frac{1-f^*\{(N-1)\lambda\}}{(N-1)\lambda} \qquad (n = 2, 3, \cdots, N) \tag{10}$$

where $f^*(i\lambda)$ is the Laplace transform of the function $f(x)$ that was given in (1) and is given by

$$f^*(i\lambda) = \int_0^\infty f(x) \cdot e^{-i \cdot \lambda x} dx \qquad (i = 0,1,2,\cdots, N-1) \tag{11}$$



(a)



(b)

Figure 3. The states exchange between arbitrary time t and $t+\Delta$ at the start point and/or at the end point of the service. (a) when n=0 at arbitrary time t (b) when n>0 at arbitrary time t

On the other hand, the boundary conditions at the start point or at the end point of the service, as shown in Fig.3, are given by the following formulas

$$p_0 = \{1 - N \cdot \lambda\Delta\} \cdot p_0 + \int_0^\infty \mu(x)\Delta \cdot w_0(x)\, dx + o(\Delta^2) \tag{12}$$

$$w_0(0)\Delta = \int_0^\infty \mu(x)\Delta \cdot w_1(x)\, dx + N \cdot \lambda\Delta \cdot p_0 + o(\Delta^2) \tag{13}$$

$$w_{n-1}(0)\Delta = \int_0^\infty \mu(x)\Delta \cdot w_n(x)\, dx + o(\Delta^2) \qquad (n = 2, 3, \cdots, N-1) \tag{14}$$

If $\Delta \to 0$, the above equations become as follows

$$N \cdot \lambda \cdot p_0 = \int_0^\infty \mu(x) \cdot w_0(x)\, dx \tag{15}$$

$$w_0(0) = \int_0^\infty \mu(x) \cdot w_1(x)\, dx + N \cdot \lambda \cdot p_0 \tag{16}$$

$$w_{n-1}(0) = \int_0^\infty \mu(x) \cdot w_n(x)\, dx \qquad (n = 2,3,\cdots,N-1) \tag{17}$$

By substituting the values of w_0 and w_n in Equations (7) and (8), respectively, into (15), (16) and (17) to obtain the constant values C_0, $C_1$and C_n which are given as follows

$$C_0 = \frac{N \cdot \lambda \cdot p_0}{f^*\{(N-1)\lambda\}} \tag{18}$$

$$C_1 = C_0 \cdot \frac{1 + (N-2) \cdot f^*\{(N-1)\lambda\}}{f^*\{(N-2)\lambda\}} \tag{19}$$

$$\begin{aligned} C_n = & \frac{1}{f^*\{(N-n-1)\lambda\}} \cdot [\{1 + \frac{N-n}{1} \cdot f^*((N-n)\lambda)\} \cdot C_{n-1} \\ & + (-1)^1 \frac{(N-n+1)}{1!}\{1 + \frac{N-n}{2} \cdot f^*((N-n+1)\lambda)\} \cdot C_{n-2} + \cdots \\ & + (-1)^{n-1} \frac{(N-1)\cdots(N-n+1)}{(n-1)!} \cdot \{1 + \frac{N-n}{n} \cdot f^*((N-1)\lambda)\} \cdot C_0] \\ & \qquad (n = 2, 3, \cdots, N-1) \end{aligned} \tag{20}$$

Insert the above constant values into the Equations (9) into (10), then we have the state probabilities p_n (n=0, 1, 2, . . ., N-1). The empty state probability p_0 is given by

$$\sum_{n=0}^{N} p_n = 1 \tag{21}$$

To calculate the average waiting time, we need to define another parameter which is average service time T_S, this parameter is given by

$$T_S = \int_0^\infty x \cdot f(x)\, dx \tag{22}$$

If the state probabilities p_n (n=0, 1, 2, . . ., N) are solved by above (9) to (10) and (19) to (22) then the average queue length L_q and the average waiting time W_q are given respectively by

$$L_q = \sum_{n=1}^{N} (n-1)\, p_n = \sum_{n=0}^{N-1} n\, p_{n+1} \tag{23}$$

$$\begin{aligned} W_q = & \sum_{n=0}^{N-1} \int_0^\infty w_n(x) \left\{ nT_S + \int_0^\infty y \frac{f(x+y)}{1-F(x)} dy \right\} dx = L_q \cdot T_S + \sum_{n=0}^{N-1} \{C_n \cdot \frac{f^*((N-n-1)\lambda)}{((N-n-1)\lambda)^2} \\ & + \sum_{i=1}^{n} (-1)^i C_{n-i} \cdot \frac{(N-n+i-1)\cdots(N-n)}{i!} \cdot \frac{f^*((N-n+i-1)\lambda)}{((N-n+i-1)\lambda)^2}\} \end{aligned} \tag{24}$$

Suppose P_w is working probability which it means that the probability of a minimum number of elements m in the system while keeping the system operates normally and it's given by

$$P_w = p_0 + p_1 + \cdots + p_{N-m} \qquad (N \geq m > 0) \tag{25}$$

The above (25) means that the system is able to operate normally with minimum elements in operations. Therefore, our model has fault tolerance and has the ability to respond properly to an unexpected failure, and we can realize that the system's operation works normally even the system has some faulty elements.

4 NUMERICAL RESULTS AND DISCUSSIONS

In previous section, we assumed in our previous calculations that the service time was general service distribution, however, if we consider that the service time is Erlang type k distributed then Equation (11) will be

$$f^*(i\lambda) = \int_0^\infty f(x)e^{-i\lambda x}dx = \int_0^\infty \frac{(k\mu)^k}{(k-1)!}x^{k-1}e^{-k\mu x}e^{-i\lambda x}dx = \left(\frac{k\mu}{i\lambda + k\mu}\right)^k \quad (26)$$

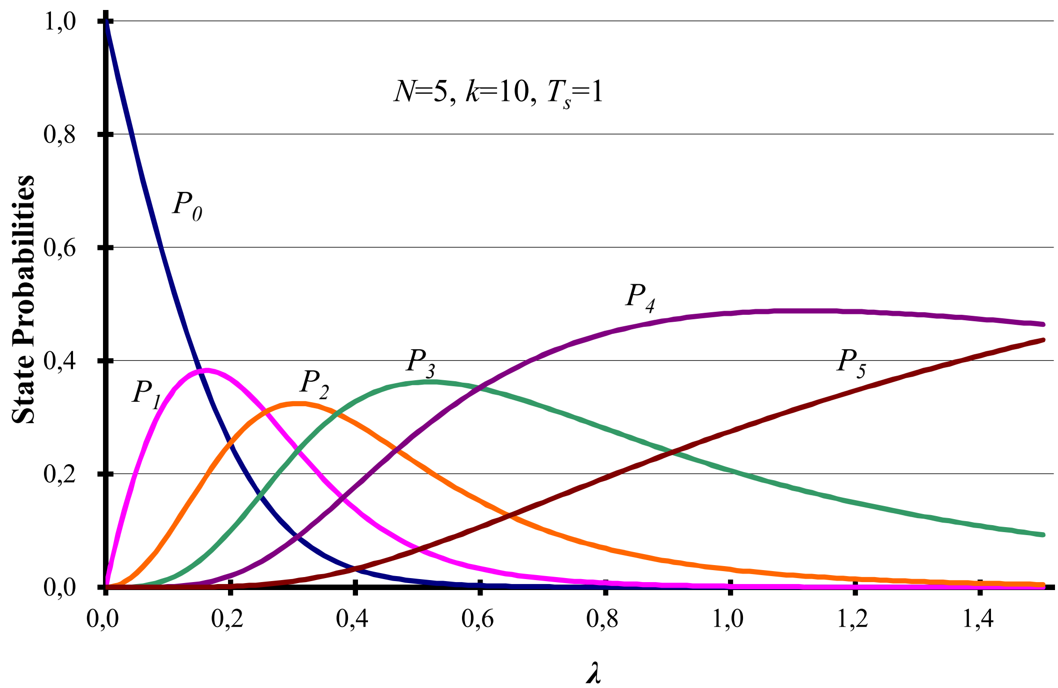


Figure 4. The state probabilities p_n versus the arrival rate λ on condition of N=5, k=10, T_S=1 (unit time)

Equation (33) is Laplace transformation of $f(x)$, where μ is any arbitrary positive service rate. We analyze the initial system performance by evaluating (9) to (21) and (22) for N=5, T_s=1 with different k phase and is plotted in Figure 4. As shown in figures, the state probability of all elements in operations p_0, where no defective elements in the system or n=0 is reported, is sharply decreased while the state probability p_n for (n=N) is increased as arrival rate λ expanded. However, for each state probabilities p_n ($0< n <N$) has a maximum value at a specific rate of λ. This maximum value becomes higher as k increased. As we mentioned above, the number of faulty elements should be less than or equal to m ($m \in N$), for keeping the system operates normally.

Figure 5 shows the probability of the system operates normally P_w versus arrival rate of the defective elements with various k phases. However, if the system has not fault tolerance capability then any defective element in the system will cause a system's termination. For example, if the system has 5 elements then the probability $P_w = p_0$ =0.269 at λ=0.2 with k=1. However, in case of the system has fault tolerance capability, suppose that the system has 3 of 5 elements are out of services, then the probability P_w=0.798 at λ=0.2 with k=1. Therefore, our model has fault tolerance and has the ability to respond reasonably to an unexpected failure, and we can realize that system's operation works normally even the system has some faulty elements.

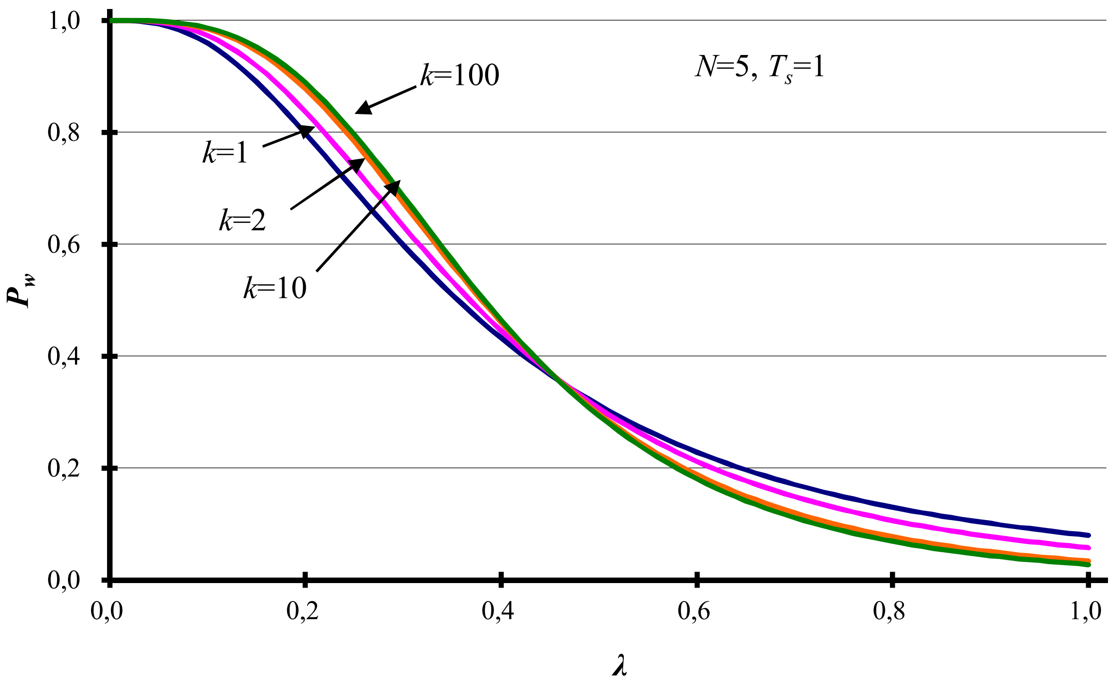


Figure 5. The probability P_w of the 3 out of 5 system versus the arrival rate λ on conditions of N=5, T_S=1 (unit time).

5 CONCLUSIONS

A study of a closed system's behavior is very important because is widely used in recent computer systems and in factories. In this paper, we presented an analytical method for a closed feed back loop type queuing *CFBLTQ* model, which is appropriated for failure and repair processes in the maintenance station. Numerical examples were given to gain a better understanding of the system's behavior. The system model has fault tolerance capabilities by considering the system has a self configuration feature that can tolerant temporary failures while operation's tasks which have been assigned to failure elements are distributed to other active elements.

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APPENDIX

In this Appendix, we will derive (1) which is the density function *f(x)* of service time distribution (See Yoshioka 1988, Yoshioka 2004). Suppose the probability that service is completed on arbitrary service time *x* to *x+Δ* is the conditional probability and is defined by

$$\frac{F(x+\Delta)-F(x)}{1-F(x)} = \frac{f(x)}{1-F(x)}\cdot\Delta + o(\Delta^2) = \mu(x)\cdot\Delta + o(\Delta^2) \tag{A1}$$

where $F(x)$ and $f(x)$ are the probability distribution function and the probability density function, respectively. From (A1), we can obtain the following function

$$\frac{f(x)}{1-F(x)} = \mu(x) \tag{A2}$$

By taking limit integration for both sides of (A2) with respect to t, where $(0 \le t \le x)$

$$\int_0^x \frac{f(t)}{1-F(t)}dt = \log|1-F(x)| = \int_0^x \mu(t)dt \tag{A3}$$

The function $F(x)$ in (A3) can be expressed as

$$F(x) = 1 - e^{-\int_0^x \mu(t)dt} \tag{A4}$$

Finally, the probability density function $f(x)$ can be obtained, as well

$$f(x) = \frac{d}{dx}F(x) = \mu(x)\cdot e^{-\int_0^x \mu(t)dt} \tag{A5}$$

COOPERATIVE EFFECTS IN COMPLETE GRAPH WITH LOW RELIABLE ARCS

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ABSTRACT

An analysis of the limit $\lim\limits_{n\to\infty} P_n = A$ of connectivity probability (CP) P_n of complete graph with n nodes and independent arcs which have working probability n^{-a} is made. It is proved that for $0 < a < 1$ we have the equality $A = 1$ and for $1 < a$ the equality $A = 0$.

1. INTRODUCTION

An analysis of the limit $\lim\limits_{n\to\infty} P_n = A$ of connectivity probability (CP) P_n of complete graph with n nodes and independent arcs which have working probability n^{-a} is made. In the complete graph each pair of nodes is connected by single arc. It is proved that for $0 < a < 1$ we have the equality $A = 1$ and for $1 < a$ the equality $A = 0$.

Analogously with [1] this zero-one low may be interpreted as a transition from chaos to order in a structure with all possible connections between nodes. The parameter a may be called order parameter with critical meaning $a = 1$. Such model may be applied to an analysis of connection structure in the internet for example in social networks. Another field of applications may be a modeling of self organizing systems.

A calculation of CP for graphs with unreliable arcs is considered in a lot of monographs [2] – [5] which become classical. This list may be added by articles on upper and low bounds of CP [6] – [11], on transfer matrices [12], [13], on an application of groups of disjoint events [14] for accuracy CP calculation of two nodes in a graph, on accelerated algorithms [15] of accuracy CP calculation and on an application of Monte-Carlo simulations with some combinatory formulas for CP estimates [16]. There is a large number of other articles and monographs devoted to this very important problem of applied mathematics and applied probability.

But a consideration of complete graph with large number of nodes demand to construct special upper and low bounds of the probability P_n and its asymptotic analysis for $n \to \infty$. These approaches look like proofs of limit theorems in combinatory probability theory [17].

A formulation of considered problem appears from oral communication of E.A. Nurminsky. It is based on numerical experiment showed a fact of a transition from "chaos to order" in a graph with large number of connections between nodes. Our researches are accompanied by a lot of sufficiently long Monte-Carlo simulations which helped to define main properties of random graphs with large number of arcs and nodes.

2. FORMULATION OF MAIN RESULTS

Consider complete graph G_n with nodes $1,...,n$ and with independently working arcs. Denote p, $0 < p < 1$, working probability of an arc and put $P_n(i, j)$ CP of the nodes i, j in the graph G_n,

$\overline{P}_n(i,j)=1-P_n(i,j)$. It is obvious that $P_n(i,j)=P_n(k,s)$ for all pairs of nodes $i\neq j$, $k\neq s$. Denote P_n the CP of random realization of the whole graph, $\overline{P}_n=1-P_n$.

Theorem 1. Suppose that $p=p_n=n^{-a}$, $a\geq 0$, then

$$\lim_{n\to\infty}P_n=1,\ 0<a<1\ . \tag{1}$$

$$\lim_{n\to\infty}P_n=0,\ a>1\ . \tag{2}$$

Denote $Q_n(b)$ the probability that in random realization of the graph G_n there is more than $[n^b]$ connectivity components, $0<b\leq 1$. Here $[r]$ is integer part of real number *r*.

Theorem 2. Assume that $p=p_n=n^{-a}$, $a>0$, $0<b\leq 1$. Then

$$\lim_{n\to\infty}Q_n(b)=1,\ 1+b<a\leq 2,\ . \tag{3}$$

$$\lim_{n\to\infty}Q_n(1)=1,\ a>2\ . \tag{4}$$

Remark 1. The condition $p=n^{-a}$ may be replaced in Theorems 1, 2 by more general condition $p=\min(1,cn^{-a})$, where c is arbitrary positive number.

Remark 2. The condition of the graphs G_n, $n>1$, completeness may be replaced by the suggestion that there is C, $1/2<C\leq 1$, so that each node of the graph G_n is connected with more than $[Cn]-1$ other nodes.

3. PROOFS OF MAIN RESULTS

Theorem 1. Suppose that $0<a<1$ then it is possible to use obvious statement that the probability of the graph G_n random realization disconnection equals with the probability that there are k, $0<k\leq[n/2]$, nodes which are not connected with rest $n-k$ nodes of the graph G_n. Consequently the inequality

$$\overline{P}_n\leq T=\sum_{0<k\leq[n/2]}C_n^k q^{k(n-k)}\ , \tag{5}$$

is true with $q=1-p$. Remark that the functions C_n^k, $q^{k(n-k)}$ of the discrete argument k for $0<k\leq[n/2]$ do not decrease.

Choose $\gamma>0$ and integer K from the conditions

$$0<\gamma<1-a\ ,\ 0<1-K\gamma<\gamma\ . \tag{6}$$

and so

$$K\gamma>a\ . \tag{7}$$

Choose N so that for $n>N$ the inequality $n^{K\gamma}<n/2$ is true and put then $n>N$. Represent the sum T as follows

$$T=\sum_{i=1}^{K}T_i+T_0\,,\ T_i=\sum_{[n^{(i-1)\gamma}]\leq k<[n^{i\gamma}]}C_n^k q^{k(n-k)},\ 1\leq i\leq K\,,\ T_0=\sum_{[n^{K\gamma}]\leq k\leq[n/2]}C_n^k q^{k(n-k)}. \tag{8}$$

As the functions C_n^k, $k(n-k)$, $0<k\leq[n/2]$, do not decrease then

$$T_i \le n^{i\gamma} q^{\left[n^{(i-1)\gamma}\right]\left(n-n^{(i-1)\gamma}\right)} C_n^{\left[n^{i\gamma}\right]}, \quad T_0 \le \left[\frac{n}{2}\right] q^{\left[n^{K\gamma}\right]\left(n-n^{K\gamma}\right)} C_n^{[n/2]}.$$

From the formula $q = 1 - n^{-a}$ and monotone increasing of the sequence $\left(1-n^{-1}\right)^n$, $n \ge 1$, to the limit $\exp(-1)$ it is easy to obtain the inequalities

$$T_i \le n^{i\gamma} \exp\left(-n^{-a}\left[n^{(i-1)\gamma}\right]\left(n-n^{(i-1)\gamma}\right)\right) C_n^{\left[n^{i\gamma}\right]}, \tag{9}$$

$$T_0 \le \frac{n}{2} \exp\left(-n^{-a}\left[n^{K\gamma}\right]\left(n-n^{K\gamma}\right)\right) C_n^{[n/2]}. \tag{10}$$

From the Sterling formula [18, chapter II, paragraph 9, formula (9.15)] we obtain for $0 < \delta < 1$:

$$C_n^{\left[n^\delta\right]} = \frac{n!}{\left[n^\delta\right]!\left(n-\left[n^\delta\right]\right)!} \le$$

$$\le \frac{\exp(1/12n)\, n^n \exp(-n)\sqrt{2\pi n}}{\left[n^\delta\right]^{\left[n^\delta\right]} \exp\left(-\left[n^\delta\right]\right)\sqrt{2\pi\left[n^\delta\right]}\left(n-\left[n^\delta\right]\right)^{n-\left[n^\delta\right]} \exp\left(-\left(n-\left[n^\delta\right]\right)\right)\sqrt{2\pi\left(n-\left[n^\delta\right]\right)}}, \tag{11}$$

Denote $R_\delta = \left(1-2^{-\delta}\right)^{2^\delta}$ then for $n > 1$

$$\left[n^\delta\right]^{\left[n^\delta\right]} \ge \left(n^\delta - 1\right)^{n^\delta - 1} \ge n^{\delta\left(n^\delta - 1\right)} R_\delta, \tag{12}$$

$$\left(n-\left[n^\delta\right]\right)^{n-\left[n^\delta\right]} \ge \left(n-n^\delta\right)^{n-n^\delta} = n^{n-n^\delta}\left(1-n^{\delta-1}\right)^{n-n^\delta} \ge n^{n-n^\delta} R_{1-\delta}^{n^\delta}. \tag{13}$$

From the formulas (11) - (13) for $0 < \delta < 1$ we obtain

$$C_n^{\left[n^\delta\right]} \le \frac{\exp(1/12n)\, n^{n^\delta(1-\delta)+\delta}}{\sqrt{2\pi\left(n^\delta-1\right)\left(1-n^{\delta-1}\right)} R_\delta R_{1-\delta}^{n^\delta}}. \tag{14}$$

Analogously we have

$$C_n^{[n/2]} = \frac{n!}{([n/2])!\left(n-[n/2]\right)!} \le \frac{\exp(1/12n)\, n^n \sqrt{2\pi n}}{[n/2]^{[n/2]}\sqrt{2\pi[n/2]}\left(n-[n/2]\right)^{n-[n/2]}\sqrt{2\pi\left(n-[n/2]\right)}}. \tag{15}$$

As $(n-1)/2 \le [n/2] \le n/2$ so

$$[n/2]^{[n/2]} \ge \left((n-1)/2\right)^{(n-1)/2} \ge \frac{n^{n/2-1/2}}{2^{n/2-1/2}}(2/3)^{3/2}, \tag{16}$$

$$\left(n-[n/2]\right)^{n-[n/2]} \ge \left(n-n/2\right)^{n-n/2} \ge [n/2]^{[n/2]} \ge \frac{n^{n/2-1/2}}{2^{n/2-1/2}}(2/3)^{3/2}, \ n > 2. \tag{17}$$

From the formulas (15) - (17) we obtain

$$C_n^{[n/2]} \le \exp\left(\frac{1}{12n}\right) \cdot 27 \cdot 2^{n-7/2}\sqrt{\frac{n}{\pi(1-2/n)}}. \tag{18}$$

Consequently from the formulas (9), (14) and from the condition (6) and from the existence of the number $f < \infty$ so that

$$R_{i\gamma} > f^{-1} > 0, \quad R_{1-i\gamma} > f^{-1} > 0, \ 1 \le i \le K,$$

we have

$$T_i \le n^{i\gamma} \exp\left(-\left[n^{(i-1)\gamma}\right]\left(n-n^{(i-1)\gamma}\right)n^{-a}\right)\frac{\exp(1/12n) f^{1+n^{i\gamma}} n^{n^{i\gamma}+i\gamma}}{\sqrt{2\pi\left(n^{i\gamma}-1\right)\left(1-n^{i\gamma-1}\right)}} \to 0,\ \ n\to\infty. \tag{19}$$

Analogously the formulas (10), (18) and the conditions (6), (7) lead to

$$T_0 \le 27\exp\left(-\left[n^{K\gamma}\right]n^{-a}\left(n-n^{K\gamma}\right)+\frac{1}{12n}\right)2^{n-9/2}\sqrt{\frac{n^3}{\pi(1-2/n)}} \to 0\ ,n\to\infty. \tag{20}$$

Unite the formulas (8), (19), (20) we obtain that $T\to 0$, $n\to\infty$. Consequently from the formula (5) we have (1).

Assume now that $a>1$. If all arcs connected with the node 1 do not work then the nodes 1; 2 are disconnected and so

$$\overline{P}_n \ge \overline{P}_n(1,2) \ge (1-p)^{n-1} = \left(1-n^{-a}\right)^{n-1} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{\frac{n-1}{n^a}}. \tag{21}$$

As $\left(1-n^{-a}\right)^{n^a}\to\exp(-1)$, $n\to\infty$, and $a>1$, then $\overline{P}_n\to 1$, $n\to\infty$. The formula (2) is proved.

Theorem 2. It is obvious that $Q_n(b)$ is not smaller than the probability that the nodes $1,2,...\left[n^b\right]$ are isolated in random realization of the graph G_n. That is

$$Q_n(b) \ge \left(1-n^{-a}\right)^{n\left[n^b\right]} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{n^{1-a}\left[n^b\right]}. \tag{22}$$

Suppose that $1+b<a<2$, then from the formula $\left(1-n^{-a}\right)^{n^a}\to\exp(-1)$, $n\to\infty$, the condition $a>1+b$ and the formula (22) we obtain the inequality (3).

Assume that $a>2$ then

$$Q_n(1) \ge \left(1-n^{-a}\right)^{n^2} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{n^{2-a}}. \tag{23}$$

Consequently from the condition $a>2$ the formula $\left(1-n^{-a}\right)^{n^a}\to\exp(-1)$, $n\to\infty$, and the formula (23) we obtain the equality (4).

Remarks 1, 2. Remark 1 proof almost word by word repeats Theorems 1, 2 proofs. To prove Remark 2 it is enough to replace the inequality (5) by

$$\overline{P}_n \le T \le \sum_{0<k\le[n/2]} C_n^k q^{k([Cn]-1-n/2)},$$

the inequality (21) by

$$\overline{P}_n \ge \overline{P}_n(1,2) \ge (1-p)^{[Cn]-1} = \left(1-n^{-a}\right)^{Cn-1} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{\frac{Cn-1}{n^a}},$$

the inequality (22) by

$$Q_n(b) \ge \left(1-n^{-a}\right)^{Cn^{b+1}} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{Cn^{b+1-a}},$$

the inequality (23) by

$$Q_n(1) \ge \left(1-n^{-a}\right)^{Cn^2} = \left(\left(1-n^{-a}\right)^{n^a}\right)^{Cn^{2-a}}.$$

4. CONCLUSION

This paper is written using complicated numerical calculations. It is obvious that further for a realization of these calculations it is necessary to use supercomputers.

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Good memory of Academician
Boris Vladimirovich Gnedenko

DUBBED THE REPLACEMENT SYSTEM WITH CONTROL

Smagin V.A.

1. Model B.V.Gnedenko. The system consists of two elements, one of which is in working condition, and another - in the unloaded reserve. After the failure of the main item, it goes to recovery, back-up substitutes for the failed primary element. If refuses to work element, and the backup does not have time to recover, it is a system failure. If a back-up time to recover during the time of the main element, it becomes redundant and the process then repeats. A mathematical model of system uptime is represented by two equations [1]:

$$\begin{aligned} P_2(t) &= P(t) + \int_0^t W(t-z)a(z)dz, \\ W(t) &= P(t) + \int_0^t G(z)W(t-z)a(z)dz. \end{aligned} \tag{1}$$

In (3) $P_2(t), P(t) -$ probability of failure of the system and a basic element, $a(t) -$ probability density of time to failure of the element, $W(t) -$ conditional probability of failure of the system, provided that the initial time a back-up involved in the work, and the principal immediately began to recover. Control of the state elements continuous and perfect.
Using Laplace transform, we write (1) as:

$$\begin{aligned} \overset{*}{P}_2(s) &= \overset{*}{P}(s) + \overset{*}{W}(s)\overset{*}{a}(s), \\ \overset{*}{W}(s) &= \overset{*}{P}(s) + \overset{*}{W}(s)\overset{*}{b}(s), \end{aligned} \tag{2}$$

Where $\overset{*}{b}(s) = \int_0^\infty e^{-sz}G(z)a(z)dz$. The system of equations (2) has the form:

$$\overset{*}{P}_2(s) = \frac{(1-\overset{*}{a}(s))(1-\overset{*}{b}(s)+\overset{*}{a}(s))}{s(1-\overset{*}{b}(s))}. \tag{3}$$

From (3) we find, for example, mean time to failure of the system:

$$T_2 = \frac{T(1-\overset{*}{b}(0)+\overset{*}{a}(0))}{(1-\overset{*}{b}(0))}. \tag{4}$$

In particular, the exponential distribution of time to failure $P(t) = e^{-\lambda t}$ and recovery time $G(t) = 1 - e^{-\mu t}$ we get:

$$T_2 = \frac{2\lambda + \mu}{\lambda^2}. \tag{5}$$

2. Accounting for non-ideality of control system elements. In [2] proposed to take into account no ideality of the control system to be restored by introducing the probability at this value is multiplied by "resource recovery", namely:

$$G_q(t) = 1 - e^{-q\int_0^t \mu(z)dz}, \; R_q(t) = 1 - G_q(t),$$

where $R_q(t) = e^{-q\int_0^t \mu(z)dz}$ - probability of unrecovered elements of time, provided that his failure wasdetected with a probability q.

Value of the integral $\vartheta(t) = q\int_0^t \mu(z)dz$ by analogy with the "resource security" professor N.M. Sedyakin $r(t) = \int_0^t \lambda(z)dz$, where $\lambda(t)$ – failure intensity of an element is called a "resource recovery". Taking into account the probability of introducing it in the future will be denoted as the probability $G_q(t), R_q(t)$, and the value of the resource $-\vartheta_q(t)$.

In the above expressions (1) - (5) with a probability q must be directly related $G(t)$, $\overset{*}{b}(s)$, so finally we can write:

$$\overset{*}{P}_{2q}(s) = \frac{(1-\overset{*}{a}(s))(1-\overset{*}{b}_q(s)+\overset{*}{a}(s))}{s(1-\overset{*}{b}_q(s))}, \tag{6}$$

$$T_{2q} = \frac{T(1-\overset{*}{b}_q(0)+\overset{*}{a}(0))}{(1-\overset{*}{b}_q(0))}. \tag{7}$$

In the special case for exponential distributions, will have the formula:

$$T_{2q} = \frac{2\lambda+\mu\cdot q}{\lambda^2}. \tag{8}$$

Thus, the average uptime of the system duplicated in this case is directly proportional to the reliability of monitoring the state of the failed element.

Example 1. Let $\lambda = 0{,}01\, ч^{-1}$, $\mu = 0{,}1\, ч^{-1}$. Then $T_{2q} = 200\, ч$ at $q = 0$, $T_{2q} = 700\, ч$ at $q = 0{,}5$, $T_{2q} = 1200\, ч$ at $q = 1$.

Example 2. Let the law of distribution of time to failure is normal. Density distribution is $a(t) = \frac{1}{\sqrt{2\pi}3}e^{-\frac{(t-25)^2}{2\cdot 3^2}}$. Distribution law for the recovery time Weibull. The distribution function of recovery time is $G(t) = 1-e^{-0{,}2\cdot t^2}$. Then $\overset{*}{b}_q(0) = \int_0^\infty (1-e^{-q\cdot 0{,}2\cdot z^2})\cdot\frac{1}{\sqrt{2\pi}3}e^{-\frac{(z-25)^2}{2\cdot 3^2}}dz$. So, $T = 25\, ч., \Theta = 6{,}267\, ч$. Figure 1 shows plots $\overset{*}{b}_q(q), T_{2q}(q)$, plotted for different values of probability q. They imply that an increase in the reliability of control system elements to refuse the value of conditional probability $\overset{*}{b}_q(q)$ increases. With the increase of this probability as the probability q, value of the mean time to failure of the duplicated $T_{2q}(q)$ increases, and increase is nonlinear. This demonstrates the importance of the value of reliability control in the duplicated system.

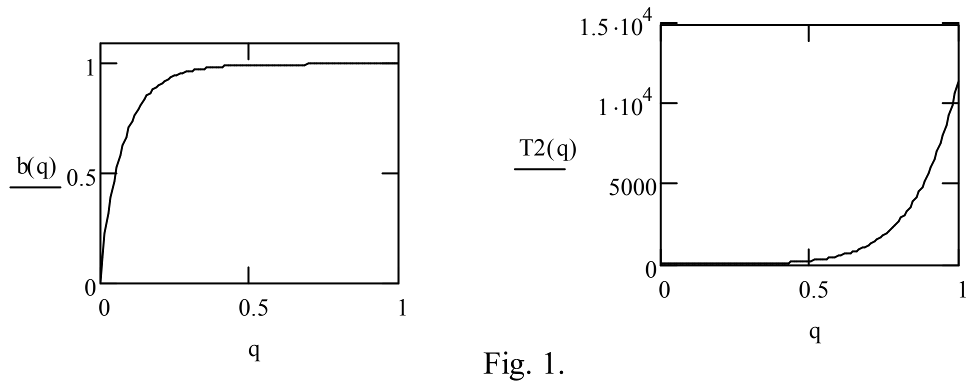

Fig. 1.

3. Willingness duplicated system with control. Obtain equations for the study of readiness duplicated system with arbitrary distributions directly, as was done to determine the probability (1), rather difficult.

Simply use the expression (6) and find a picture of him Laplace distribution density of time to failure duplicated system, using the formula:

$$\overset{*}{P}_{2q}(s) = \frac{1 - \overset{*}{a}_{2q}(s)}{s}, \tag{9}$$

When $\overset{*}{a}_{2q}(s) -$ image of the desired density. Performing the necessary transformations, we obtain:

$$\overset{*}{a}_{2q}(s) = \frac{\overset{*}{a}(s)(\overset{*}{a}(s) - \overset{*}{b}_q(s))}{1 - \overset{*}{b}_q(s)}. \tag{10}$$

Next, use the formula for the image of the function of readiness in the form:

$$\overset{*}{K}_{\Gamma q}(s) = \frac{1 - \overset{*}{a}_{2q}(s)}{s(1 - \overset{*}{a}_{2q}(s)\overset{*}{g}_{2q}(s))}, \tag{11}$$

in which $\overset{*}{g}_{2q}(s) -$ the image probability density recovery duplicated system after its failure.

Pay attention to the fact that this density can take different forms depending on the discipline system recovery, namely the recovery of both elements can be performed by one or two brigades. Consider this further in the analysis of readiness.

Expression(11) after substituting in it (10) reduces to:

$$\overset{*}{K}_{\Gamma q}(s) = \frac{1 - \overset{*}{b}_q(s) - \overset{*}{a}(s)(\overset{*}{a}(s) - \overset{*}{b}_q(s))}{s(1 - \overset{*}{b}_q(s) - \overset{*}{a}(s)(\overset{*}{a}(s) - \overset{*}{b}_q(s))\overset{*}{g}_{2q}(s))}. \tag{12}$$

Recall that $K_\Gamma = K_\Gamma(\infty) = \lim_{s \to 0} s \overset{*}{K}_\Gamma(s)$. Performing the limit, we find the coefficient of readiness:

$$K_{\Gamma q} = \frac{T(2 - \overset{*}{b}_q(0))}{T(2 - \overset{*}{b}_q(0)) + \Theta_{2q}(1 - \overset{*}{b}_q(0))}, \tag{13}$$

$\Theta_{2q} = \int_0^\infty g_{2q}(z)dz -$ average recovery time of one or two brigades.

Example 3. Assume that the distributions of time to failure and recovery of elements exponential. Failure rate and recovery elements are equal λ, μ. If System Restore is one team, then by (13) we obtain:

$$K_{\Gamma q} = \frac{q\mu(2\lambda + q\mu)}{q^2\mu^2 + 2\lambda q\mu + 2\lambda^2}. \tag{14}$$

If system restore is performed by two teams, then the coefficient of readiness will be:

$$K_{\Gamma q} = \frac{2q\mu(2\lambda + q\mu)}{2q^2\mu^2 + 4\lambda q\mu + 3\lambda^2}. \tag{15}$$

Correctness (14) and (15) can be checked by applying a system of differential equations.

Example 4. Determine the availability of the system if the probability density of time to failure of the element $a(t) = \frac{1}{\sqrt{2\pi}3} e^{-\frac{(t-25)^2}{2\cdot 3^2}}$, and the distribution function of his recovery time $G_q(t) = 1 - e^{-q\cdot 0,2\cdot t^2}$.

Then $\overset{*}{b}_q(0) = \int_0^\infty (1 - e^{-q\cdot 0,2\cdot z^2}) \cdot \frac{1}{\sqrt{2\pi}3} e^{-\frac{(z-25)^2}{2\cdot 3^2}} dz$.

Maintenance of two failed elements is one brigade. The average recovery time of both elements is equal to: $\Theta_{2q}(q) = \int_0^\infty g_{2q}(z)dz,\ g_{2q}(t) = \int_0^t g_q(z) g_q(t-z)dz.\ g_q(t) = 0,4qze^{-0,2z^2}$.

$$\Theta_{2q}(q) = \frac{10}{q} \cdot \Gamma(\frac{3}{2}).$$

$$\overset{*}{b}_q(q) = \int_0^\infty (1 - e^{-q\cdot 0,2\cdot z^2}) \cdot \frac{1}{\sqrt{2\pi}3} e^{-\frac{(z-25)^2}{2\cdot 3^2}} dz.$$

Substituting these values into the formula

$$K_{\Gamma q}(q) = \frac{T(2 - \overset{*}{b}_q(0))}{T(2 - \overset{*}{b}_q(0)) + \Theta_{2q}(1 - \overset{*}{b}_q(0))},$$

obtain the dependence of the system from the parameter control. It is shown in Picture 2. It can be seen that even for relatively small values of control availability becomes sufficiently close to unity.

When servicing two teams of recovery, this effect will increase.

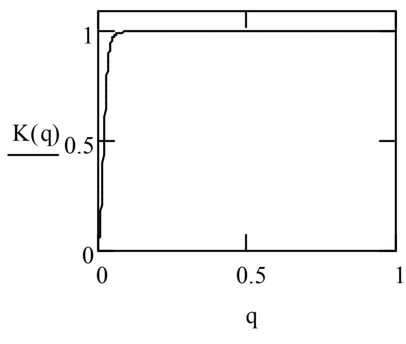

Fig. 2.

Conclusion. An expression for the function of readiness duplicated system in the Laplace transform and its significance is the steady state for arbitrary distributions of time to failure and recovery of constituent elements.

In these expressions have introduced the parameter reliability monitoring the state of the elements after their refusal. The value of this option allows you to take into account the duration of the recovery elements after their refusal.

Because of this, a generalization of the result obtained B.V.Gnedenko, ready for the duplicated system with control of the state elements.

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AVAILABILITY AND RELIABILITY MEASURES FOR MULTISTATE SYSTEM BY USING MARKOV REWARD MODEL

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ABSTRACT

This paper describes some models and measures of reliability for multistate systems. The expected cumulative reward for the continuous time Markov reward models are used for deriving the structure function for a multistate system where the system and its components can have different performance levels ranging from perfect functioning to complete failure. The suggested approach presents with respect to the non-homogeneous and homogeneous Markov reward model of two stochastic process for computation of these availability and reliability measures. A particular case for three levels is analyzed numerically by assuming Weibull and exponential distributions for failure and repair times.

Keywords: Markov reward model, demand, multistate system, availability and reliability measures.

1. INTRODUCTION

Traditional binary-state reliability models allow for a system and its components only two possible states: perfect functioning (up) and complete failure (down). However, a system can have a finite number of performance rates. And, many real-world systems are composed of components that in their turn can have different performance levels and for which one cannot formulate an "all or nothing" type of failure criterion. Failures of some system elements lead, in these cases, only to performance degradation. Such systems are called multi-state systems (MSS) [11]. Traditional reliability theory, which is based on a binary approach, has recently been extended by allowing components and systems to have an arbitrary finite number of states.

According to the generic multi-state system model [8], any system element $j \in \{1, 2, \dots, n\}$ can have k different states corresponding to the performance rates, represented by the set $g_j = \{g_{j1}, g_{j2}, \dots, g_{jk}\}$, where g_{ji} is the performance rate of element j in the state $i, i \in \{1, 2, \dots, k\}$. The performance rate $G_j(t)$ of element j at any instant $t \geq 0$ is a discrete-state continuous-time stochastic process that takes its values from $g_j : G_j(t) \in g_j$. The system structure function $G(t) = \phi(G_1(t), \dots, G_n(t))$ produces the stochastic process corresponding to the output performance of the entire MSS. In practice, a desired level of system performance (demand) also can be represented by a discrete-state continuous-time stochastic process $W(t)$. The relation between the MSS output performance and the demand represented by two corresponding stochastic processes should be studied in order to define reliability measures for the entire MSS. For reliability assessment, MSS output performance and the desired performance level (demand) are often assumed to be independent stochastic processes. In practice, the most commonly used MSS reliability measures are probability of failure-free operation during time interval $[0, t]$ or MSS reliability function $R(t)$, MSS availability, mean time to MSS failure, mean accumulated performance deficiency for a fixed time interval $[0, t]$, and so on.

Many technical systems are subjected during their lifetime to aging and degradation. After any failure, maintenance is performed by a repair team. Maintenance and repair problems have been widely investigated in the literature. [1], [4], [16] survey and summarize theoretical developments and practical applications of maintenance models. Aging is usually considered as a process which results in an age-related increase of the failure rate. The most common shapes of failure rates have been observed in [12], [18]. An interesting approach was introduced in [7], where it was shown that aging is not always manifested by the increasing failure rate.

After each corrective maintenance action or repair, the aging system's failure rate $\lambda(t)$ can be expressed as:

$$\lambda(t) = q\,\lambda(0) + (1-q)\,\lambda^*(t),$$

where q is an improvement factor that characterizes the quality of the overhauls $(0 \le q \le 1)$ and $\lambda^*(t)$ is the aging system's failure rate before repair [20]. If $q = 1,$ it means that the maintenance action is perfect (system becomes "as good as new" after repair). If $q = 0,$ it means that the failed system is returned back to a working state by minimal repair (system stays "as bad as old" after repair), in which failure rate of the system is nearly the same as before. The minimal repair is usually appropriate for multi-state systems. In such situation, the failure pattern can be described by non-homogeneous Poisson process (NHPP). Incorporating the time-varying failure intensity into existing Markov model was suggested in [17] for reliability modeling of hardware/software systems. More details and interesting examples one can find in [19]. Based on this, the extended approach is suggested, which incorporates the time-varying failure intensity of aging component into Markov reward model that is using for general reliability measures evaluation of non-aging MSS [7]. Such unified model will be called as a non-homogeneous Markov reward model.

This paper considers measures of availability and reliability for a multi-state system where the system and its components can have different performance levels ranging from perfect functioning to complete failure. In section 2 a general approach is presented for the computation of main MSS reliability measures. This approach is based on the application of the Markov reward model. The main MSS reliability measures can be found by corresponding reward matrix definitions for this model and then by using a standard procedure for finding expected accumulated rewards during a time interval $[0, t]$ as a solution of a system of differential equations. In section 3 a general approach is presented for computing reliability measures for aging MSS under corrective maintenance with minimal repair. This approach is based on non-homogeneous Markov reward model, where specific reward matrix is determined for finding any reliability measure. This chapter is based on [9], [11], and presents a model representing demand as a continuous-time Markov chain with three logic levels. In section 4 we introduce illustrative example in order to illustrate the approaches.

2. MARKOV REWARD MODEL FOR MULTI-STATE SYSTEM

2.1. Generalized MSS Reliability Measures

The MSS behavior is characterized by its evolution in the space of states. The entire set of possible system states can be divided into two disjoint subsets corresponding to acceptable and unacceptable system functioning. MSS entrance into the subset of unacceptable states constitutes a failure. The MSS reliability can be defined as its ability to remain in the acceptable states during the operation period. The system state acceptability depends on the relation between the MSS output performance and the desired level of this performance (demand $W(t)$) that is determined outside the system. Often the demand $W(t)$ is also a random process that can take discrete values from the set

$w = \{w_1, \ldots, w_M\}$. The desired relation between the system performance and the demand at any time instant t can be expressed by the acceptability function $\Phi(G(t), W(t))$. In many practical cases, the MSS performance should be equal to or exceed the demand. So, in such cases, the acceptability function takes the following form:

$$\Phi(G(t), W(t)) = G(t) - W(t) \tag{1}$$

and the criterion of state acceptability can be expressed as: $\Phi(G(t), W(t)) \geq 0$.
A general expression defining MSS reliability measures can be written in the following form:

$$R = E\{F[\Phi(G(t), W(t))]\}, \tag{2}$$

where $E =$ expectation symbol, $F =$ functional that determines corresponding type of reliability measure, and $\Phi =$ acceptability function. Many important MSS reliability measures can be derived from the expression (2) depending on the functional F that may be determined in different ways. For example, it may be a probability $\Pr\{\Phi(G(t), W(t)) \geq 0\}$ throughout a specified time interval $[0, t]$ and the acceptability function (1) will be nonnegative. In this case, this probability characterizes MSS availability. It may be also an expectation of an appropriate function up to the time of the MSS's initial entrance into the set of unacceptable states, where $\Phi(G(t), W(t)) < 0$ is the number of such entrances within time interval $[0, t]$ and so on. For a power system where the available generating capacity at time instant t is $G(t)$ and the corresponding load demand is $W(t)$, if the acceptability function is defined as:

$$\Phi(G(t), W(t)) = \begin{cases} W(t) - G(t), & \text{if } W(t) > G(t) \\ 0, & \text{if } W(t) \leq G(t) \end{cases}$$

A function,

$$F[\Phi(G(t), W(t))] = \int_0^T \Phi(G(t), W(t))\, dt,$$

will characterize an accumulated performance deficiency during time interval [0, T].

2.2. Markov Reward Model: General Description

The general Markov reward model was introduced in [6]. It considers the continuous-time Markov chain $\{X(t) \mid t \geq 0\}$ with a set of states $\{1, \ldots, k\}$ and a transition intensity matrix $A = [a_{ij}], i, j = 1, \ldots, k$. It is assumed that while the process is in any state i during any time unit, some money r_{ii} should be paid. It is also assumed that if there is a transition from state i to state j the amount r_{ij} will be paid. The amounts r_{ii} and r_{ij} are called rewards. Rewards can be negative while representing a loss or penalty. Such a reward process associated with its states or/and transitions is called a Markov process with rewards. For such processes, in addition to the transition intensity matrix, a reward matrix $r = [r_{ij}], i, j = 1, \ldots, k$ should be determined. The main problem is to find the total expected reward accumulated up to time instant t under specified initial conditions.

Let $V_i(t)$ denotes the total expected reward accumulated up to time t at state i. The following system of differential equations must be solved under the initial conditions: $V_i(0) = 0,\ i = 1, \ldots, k$ in order to find the total expected reward.

$$\frac{dV_i(t)}{dt} = r_{ii} + \sum_{\substack{j=1,\\ j\neq i}}^{k} a_{ij} r_{ij} + \sum_{j=1}^{k} a_{ij} V_j(t), \quad i = 1, \ldots, k \tag{3}$$

Markov reward models are widely used in financial calculations and operations research [5]. General Markov reward models for system dependability and performability analysis one can find in [2], [14], and [10]. Here the new approach is presented where the main MSS reliability measures can be found by determination of the corresponding reward matrix. Such an idea was primarily introduced for a binary-state system and constant demand in [15]. In this chapter, the approach is extended for multi-state systems and variable demand.

2.3. Rewards Determination for MSS Reliability Computation

MSS instantaneous (point) availability $A(t)$ is the probability that the MSS at instant $t > 0$ is in one of the acceptable states: $A(t) = \Pr\{\Phi(G(t), W(t)) \geq 0\}$.

The MSS average availability $\overline{A}(t)$ is defined in [13] as a mean fraction of time when the system resides in the set of acceptable states during the time interval [0, t], $\overline{A}(t) = \frac{1}{t}\int_0^t A(t)\,dt$.

In order to assess $\overline{A}(t)$ for MSS the rewards in matrix r for the MSS model should be determined in the following manner:

- The rewards associated with all acceptable states should be defined as one.
- The rewards associated with all unacceptable states should be zeroed as well as all rewards associated with all transitions.

The mean reward $V_i(t)$ accumulated during interval $[0, t]$ will define a time that MSS will be in the set of acceptable states in the case when the state i is the initial state. This reward should be found as a solution of the system (3). After solving (3) and finding $V_i(t)$, MSS average availability can be obtained for every initial state $i = 1, \ldots, k$, $\overline{A}_i(t) = (V_i(t))/t$.

Usually, the initial state is assumed as the best state.

Mean number $N_f(t)$ of MSS failures during time interval [0, t] measure can be treated as the mean number of MSS entrances to the set of unacceptable states during time interval [0, t]. For its computation the rewards associated with each transition from the set of acceptable states to the set of unacceptable states should be defined as one. All other rewards should be zeroed. In this case mean accumulated reward $V_i(t)$ will define the mean number of entrances in the unacceptable area during time interval [0, t]: $N_f(t) = V_i(t)$.

Mean time to failure (MTTF) is the mean time up to the instant when the MSS enters the subset of unacceptable states for the first time. For its computation the combined performance-demand model should be transformed; all transitions that return MSS from unacceptable states should be forbidden, because for this case all unacceptable states should be treated as absorbing states. In order to assess MTTF for MSS the rewards in matrix r for the transformed performance-demand model should be determined in the following manner:

- The rewards associated with all acceptable states should be defined as one.
- The rewards associated with unacceptable (absorbing) states should be zeroed as well as rewards associated with transitions.

In this case mean accumulated reward $V_i(t)$ will define the mean time accumulated up to the first entrance into the subset of unacceptable states or MTTF.

Probability of MSS failure during time interval [0, t]: The model should be transformed as in the previous case; all unacceptable states should be treated as absorbing states, and therefore all transitions that return MSS from unacceptable states should be forbidden. Rewards associated with all transitions to the absorbing states should be defined as one. All other rewards should be zeroed. Mean accumulated reward $V_i(t)$ will define for this case the probability of MSS failure during time interval [0, t] if the state i is the initial state. Therefore, the MSS reliability function can be obtained as: $R_i(t) = 1 - V_i(t)$, where $i = 1, \ldots, k$.

3. NON-HOMOGENEOUS MARKOV REWARD MODEL FOR AGING MULTI-STATE SYSTEM UNDER MINIMAL REPAIR

3.1. Model Description

The MSS output performance $G(t)$ at any instant $t \geq 0$ is a continuous-time Markov chain that takes its values from the set $g = \{g_1, \ldots, g_k\}, G(t) \in g$, where g_i is the MSS output performance in state $i, i = 1, \ldots, k$. For Markov MSS transition rates (intensities) a_{ij} between states i and j are defined by the corresponding system failure λ_{ij} and repair μ_{ij} rates. The minimal repair is a corrective maintenance action that brings the aging equipment to the conditions it was in just before the failure occurrence. Aging MSS subject to minimal repairs experiences reliability deterioration with the operating time, i.e., there is a tendency toward more frequent failures. In such situations, the failure pattern can be described by a Poisson process whose intensity function monotonically increases with t. A Poisson process with a non-constant intensity is called non-homogeneous, since it does not have stationary increments [4]. It was shown (see, for example, [20]) that NHPP model can be integrated into the Markov model with time-varying transition intensities $a_{ij}(t) = \lambda_{ij}(t)$. Therefore, for aging MSS transition intensities corresponding to failures of aging components will be functions of time $a_{ij}(t)$.

3.2. Non-Homogeneous Markov Reward Model

For non-homogeneous Markov model a system's state at time t can be described by a continuous-time Markov chain with a set of states $\{1, \ldots, k\}$ and a transition intensity matrix $A(t) = [a_{ij}(t)], \quad i, j = 1, \ldots, k,$ where each transition intensity may be a function of time t. For such model, in addition to the transition intensity matrix, a reward matrix $r = [r_{ij}]$, $i, j = 1, \dots, k$ should be determined [2].

Let $V_i(t)$ be the expected total reward accumulated up to time t given the initial state of the process at time instant $t = 0$ is state i. Howard differential equations [14] with time-varying transition intensities $a_{ij}(t)$ should be solved under specified initial conditions in order to find the total expected rewards:

$$\frac{dV_i(t)}{dt} = r_{ii} + \sum_{\substack{j=1,\\ j\neq i}}^{k} a_{ij}(t) r_{ij} + \sum_{j=1}^{k} a_{ij}(t) V_j(t), \quad i = 1, \dots, k \tag{4}$$

In the most common case, MSS begins to accumulate rewards after time instant $t = 0$, therefore, the initial conditions are:

$$V_i(0) = 0, \quad i = 1, \dots, k \tag{5}$$

If for example the state k with the highest performance level is defined as the initial state, the value $V_k(t)$ should be found as a solution of the system (4).

It was shown in [7] and [11] that many important reliability measures for non-aging MSS can be found by determination of rewards in a corresponding reward matrix. Here this approach is extended for aging MSS under minimal repair. And, notice that the approach is applied only for minimal repair.

3.3. Rewards Determination for Computation of Different Reliability Measures for Aging MSS

The reliability measures can be determined by the same manner as it was indicated in section 2.3.

4. ILLUSTRATIVE EXAMPLE

Consider the air-conditioning system used in a hospital. The system consists of three identical air conditioners which are connected in parallel. Demand is a continuous-time Markov chain with three levels: peak, middle, and low. The state-space diagram for this system is presented in figure (1).

There are 12 states. States from 1 to 4 associated with the low demand period, states from 5 to 8 associated with the middle demand period, and states from 9 to 12 associated with the peak demand period.

States 12, 8, and 4 indicate all components work, the system performance is $g_{12} = g_8 = g_4 = 3$. States 11, 7, and 3 indicate two components work and the third component failed, the system performance is $g_{11} = g_7 = g_3 = 2$. States 10, 6, and 2 indicate that one component only works, the system performance is $g_{10} = g_6 = g_2 = 1$. States 9, 5, and 1 indicate full system failure, the system performance is $g_9 = g_5 = g_1 = 0$. If in the peak-demand period the required demand level is w = 3, in the middle-demand period the required demand level is w = 2, and in the low-demand period the required demand level is w = 1, then there are six acceptable states: 12, 8, 4, 7, 3, and 2. States: 11, 10, 6, 9, 5, and 1 are unacceptable.

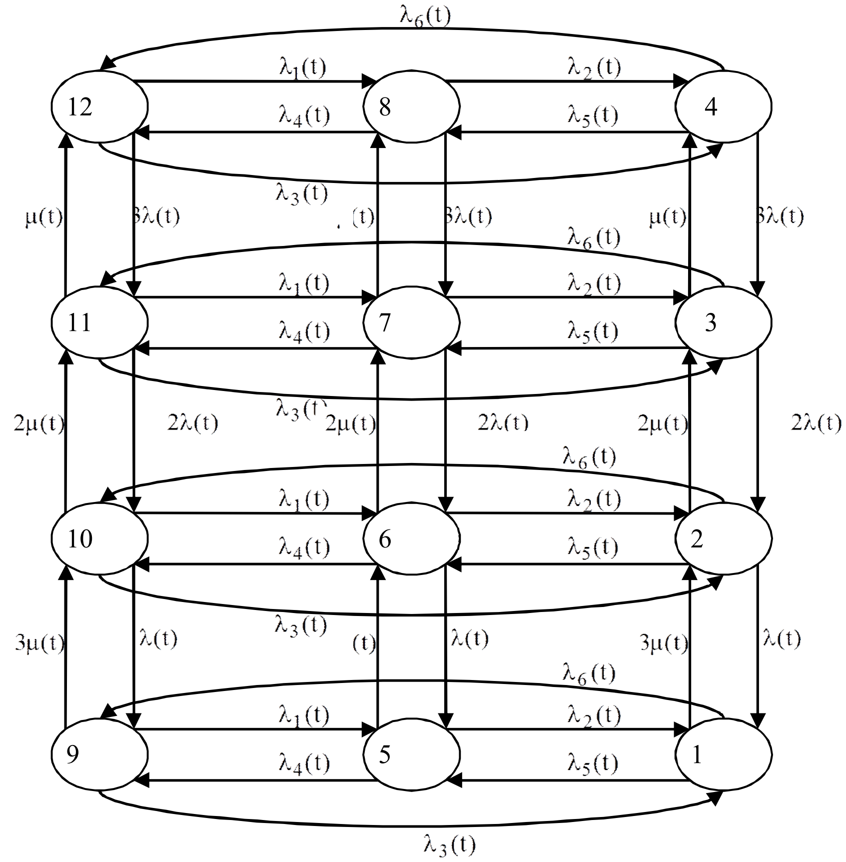

Figure (1): The state-space diagram for a system with three identical air conditioners

The transitions from state 12 to state 11, from state 8 to state7, and from state 4 to state 3 are associated with the failure of one of the three conditioners and have an intensity of $3\lambda(t)$. The transitions from state 11 to state 10, from state 7 to state 6, and from state 3 to state 2 are associated with the failure of the second conditioner and have intensity of $2\lambda(t)$. The transitions from state 10 to state 9, from state 6 to state 5, and from state 2 to state 1 are associated with the failure of the third conditioner and have intensity of $\lambda(t)$.

The transitions from state 1 to state 2, from state 5 to state 6, and from state 9 to state 10 are associated with repair of one of the three failed conditioners and have intensity of $3\mu(t)$. The transitions from state 2 to state 3, from state 6 to state 7, and from state 10 to state 11 are associated with repair of one of the two failed conditioners and have intensity of $2\mu(t)$. The transitions from state 3 to state 4, from state 7 to state 8, and from state 11 to state 12 are associated with repair of the failed conditioner and have intensity of $\mu(t)$.

The transitions from state 12 to state 8, from state 11 to state 7, from state 10 to state 6, and from state 9 to state 5 are associated with a variable demand and have intensity of $\lambda_1(t)$. The transitions from state 8 to state 4, from state 7 to state 3, from state 6 to state 2, and from state 5 to state 1 are associated with a variable demand and have intensity of $\lambda_2(t)$. The transitions from state 12 to state 4, from state 11 to state 3, from state 10 to state 2, and from state 9 to state 1 are associated with a variable demand and have intensity of $\lambda_3(t)$. The transitions from state 8 to state

12, from state 7 to state 11, from state 6 to state 10, and from state 5 to state 9 are associated with a variable demand and have intensity of $\lambda_4(t)$. The transitions from state 4 to state 8, from state 3 to state 7, from state 2 to state 6, and from state 1 to state 5 are associated with a variable demand and have intensity of $\lambda_5(t)$. The transitions from state 4 to state 12, from state 3 to state 11, from state 2 to state 10, and from state 1 to state 9 are associated with a variable demand and have intensity of $\lambda_6(t)$.

In order to find the MSS average availability $\overline{A}(t)$ we should present the reward matrix r_A in the following form:

$$r_A = [r_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

In this matrix, rewards associated with all acceptable states are defined as one and rewards associated with all unacceptable states are zeroed as well as all rewards associated with all transitions.

The system of differential equations (7) can be written in order to find the expected total rewards $V_i(t), i = 1, \ldots, 12.$ The initial conditions are: $V_i(0) = 0, i = 1, \ldots, 12.$
After solving this system and finding $V_i(t),$ MSS average availability can be obtained as follows: $\overline{A}(t) = V_{12}(t)/t,$ where the 12-th state is the initial state.

$$
\begin{cases}
\dfrac{dV_1(t)}{dt} = -C_1(t)V_1(t) + 3\,\mu(t)V_2(t) + \lambda_5(t)V_5(t) + \lambda_6(t)V_9(t) \\
\dfrac{dV_2(t)}{dt} = 1 - C_2(t)V_2(t) + \lambda(t)V_1(t) + 2\,\mu(t)V_3(t) + \lambda_5(t)V_6(t) \\
\qquad + \lambda_6(t)V_{10}(t) \\
\dfrac{dV_3(t)}{dt} = 1 - C_3(t)V_3(t) + 2\lambda(t)V_2(t) + \mu(t)V_4(t) + \lambda_5(t)V_7(t) \\
\qquad + \lambda_6(t)V_{11}(t) \\
\dfrac{dV_4(t)}{dt} = 1 - C_4(t)V_4(t) + 3\lambda(t)V_3(t) + \lambda_5(t)V_8(t) + \lambda_6(t)V_{12}(t) \\
\dfrac{dV_5(t)}{dt} = -C_5(t)V_5(t) + 3\,\mu(t)V_6(t) + \lambda_2(t)V_1(t) + \lambda_4(t)V_9(t) \\
\dfrac{dV_6(t)}{dt} = -C_6(t)V_6(t) + \lambda(t)V_5(t) + 2\,\mu(t)V_7(t) + \lambda_2(t)V_2(t) \\
\qquad + \lambda_4(t)V_{10}(t) \\
\dfrac{dV_7(t)}{dt} = 1 - C_7(t)V_7(t) + 2\lambda(t)V_6(t) + \mu(t)V_8(t) + \lambda_2(t)V_3(t) \\
\qquad + \lambda_4(t)V_{11}(t) \\
\dfrac{dV_8(t)}{dt} = 1 - C_8(t)V_8(t) + 3\lambda(t)V_7(t) + \lambda_2(t)V_4(t) + \lambda_4(t)V_{12}(t) \\
\dfrac{dV_9(t)}{dt} = -C_9(t)V_9(t) + 3\,\mu(t)V_{10}(t) + \lambda_1(t)V_5(t) + \lambda_3(t)V_1(t) \\
\dfrac{dV_{10}(t)}{dt} = -C_{10}(t)V_{10}(t) + \lambda(t)V_9(t) + 2\,\mu(t)V_{11}(t) + \lambda_1(t)V_6(t) \\
\qquad + \lambda_3(t)V_2(t) \\
\dfrac{dV_{11}(t)}{dt} = -C_{11}(t)V_{11}(t) + 2\lambda(t)V_{10}(t) + \mu(t)V_{12}(t) + \lambda_1(t)V_7(t) \\
\qquad + \lambda_3(t)V_3(t) \\
\dfrac{dV_{12}(t)}{dt} = 1 - C_{12}(t)V_{12}(t) + 3\lambda(t)V_{11}(t) + \lambda_1(t)V_8(t) + \lambda_3(t)V_4(t)
\end{cases}
\tag{7}
$$

where,

$$
\begin{cases}
C_1(t) = 3\,\mu(t) + \lambda_5(t) + \lambda_6(t) \\
C_2(t) = \lambda(t) + 2\,\mu(t) + \lambda_5(t) + \lambda_6(t) \\
C_3(t) = 2\lambda(t) + \mu(t) + \lambda_5(t) + \lambda_6(t) \\
C_4(t) = 3\lambda(t) + \lambda_5(t) + \lambda_6(t) \\
C_5(t) = 3\,\mu(t) + \lambda_2(t) + \lambda_4(t) \\
C_6(t) = 2\,\mu(t) + \lambda(t) + \lambda_2(t) + \lambda_4(t) \\
C_7(t) = \mu(t) + 2\lambda(t) + \lambda_2(t) + \lambda_4(t) \\
C_8(t) = 3\lambda(t) + \lambda_2(t) + \lambda_4(t) \\
C_9(t) = 3\,\mu(t) + \lambda_1(t) + \lambda_3(t) \\
C_{10}(t) = \lambda(t) + 2\,\mu(t) + \lambda_1(t) + \lambda_3(t) \\
C_{11}(t) = 2\lambda(t) + \mu(t) + \lambda_1(t) + \lambda_3(t) \\
C_{12}(t) = 3\lambda(t) + \lambda_1(t) + \lambda_3(t)
\end{cases}
\tag{8}
$$

In order to find the mean total number of system failures $N_f(t)$ we should present the reward matrix r_N in the form (9). In this matrix the rewards associated with each transition from the set of acceptable states to the set of unacceptable states should be defined as one. All other rewards should be zeroed.

$$r_N = [r_{ij}] = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix} \tag{9}$$

The following system of differential equations (10) can be written in order to find the expected total rewards $V_i(t), i = 1, \ldots, 12$.

$$\begin{cases} \dfrac{dV_1(t)}{dt} = -C_1(t)V_1(t) + 3\mu(t)V_2(t) + \lambda_5(t)V_5(t) + \lambda_6(t)V_9(t) \\ \dfrac{dV_2(t)}{dt} = \lambda(t) + \lambda_5(t) + \lambda_6(t) - C_2(t)V_2(t) + \lambda(t)V_1(t) \\ \qquad + 2\mu(t)V_3(t) + \lambda_5(t)V_6(t) + \lambda_6(t)V_{10}(t) \\ \dfrac{dV_3(t)}{dt} = \lambda_6(t) - C_3(t)V_3(t) + 2\lambda(t)V_2(t) + \mu(t)V_4(t) \\ \qquad + \lambda_5(t)V_7(t) + \lambda_6(t)V_{11}(t) \\ \dfrac{dV_4(t)}{dt} = -C_4(t)V_4(t) + 3\lambda(t)V_3(t) + \lambda_5(t)V_8(t) + \lambda_6(t)V_{12}(t) \\ \dfrac{dV_5(t)}{dt} = -C_5(t)V_5(t) + 3\mu(t)V_6(t) + \lambda_2(t)V_1(t) + \lambda_4(t)V_9(t) \\ \dfrac{dV_6(t)}{dt} = -C_6(t)V_6(t) + \lambda(t)V_5(t) + 2\mu(t)V_7(t) + \lambda_2(t)V_2(t) \\ \qquad + \lambda_4(t)V_{10}(t) \\ \dfrac{dV_7(t)}{dt} = 2\lambda(t) + \lambda_4(t) - C_7(t)V_7(t) + 2\lambda(t)V_6(t) + \mu(t)V_8(t) \\ \qquad + \lambda_2(t)V_3(t) + \lambda_4(t)V_{11}(t) \\ \dfrac{dV_8(t)}{dt} = -C_8(t)V_8(t) + 3\lambda(t)V_7(t) + \lambda_2(t)V_4(t) + \lambda_4(t)V_{12}(t) \\ \dfrac{dV_9(t)}{dt} = -C_9(t)V_9(t) + 3\mu(t)V_{10}(t) + \lambda_1(t)V_5(t) + \lambda_3(t)V_1(t) \\ \dfrac{dV_{10}(t)}{dt} = -C_{10}(t)V_{10}(t) + \lambda(t)V_9(t) + 2\mu(t)V_{11}(t) + \lambda_1(t)V_6(t) \\ \qquad + \lambda_3(t)V_2(t) \\ \dfrac{dV_{11}(t)}{dt} = -C_{11}(t)V_{11}(t) + 2\lambda(t)V_{10}(t) + \mu(t)V_{12}(t) + \lambda_1(t)V_7(t) \\ \qquad + \lambda_3(t)V_3(t) \\ \dfrac{dV_{12}(t)}{dt} = 3\lambda(t) - C_{12}(t)V_{12}(t) + 3\lambda(t)V_{11}(t) + \lambda_1(t)V_8(t) \\ \qquad + \lambda_3(t)V_4(t) \end{cases} \tag{10}$$

Here $C_1, \ldots, C_{12}$ are calculated via formulas (8).

The initial conditions are: $V_i(0) = 0, i = 1, \ldots, 12.$ After solving this system and finding $V_i(t),$ the mean total number of system failures $N_f(t)$ can be obtained as follows: $N_f(t) = V_{12}(t),$ where the 12-th state is the initial state.

In order to calculate the mean time to failure (MTTF), the initial model should be transformed; all transitions that return MSS from unacceptable states should be forbidden and all unacceptable states should be treated as absorbing states. The transformed model is shown in figure (2).

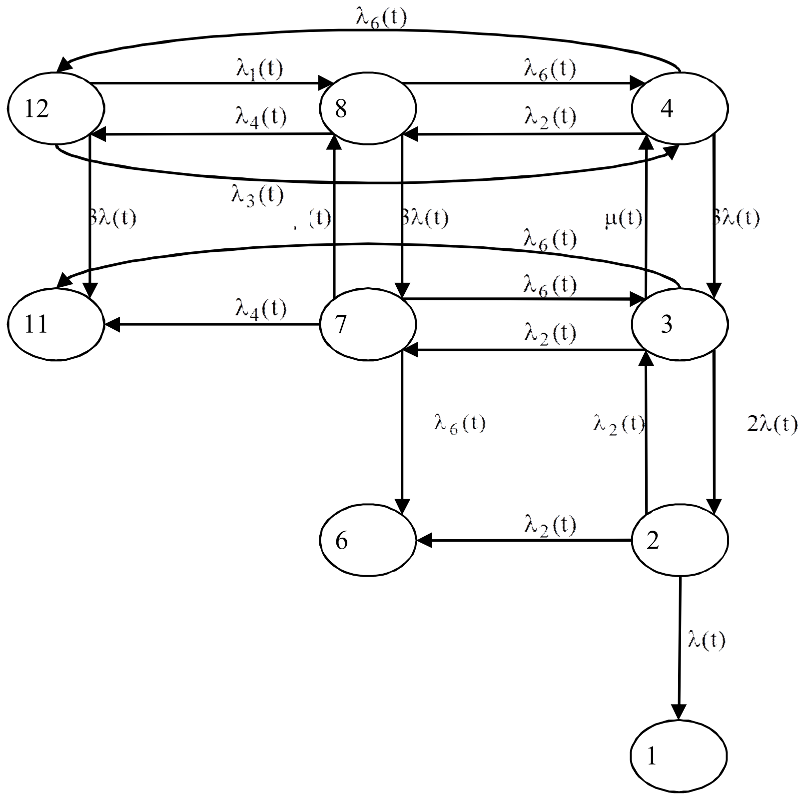

Figure (2): The state-space diagram for the transformed system with three identical air conditioners with absorbing states

In order to assess MTTF for MSS, the rewards in matrix r for the transformed model should be determined in the following manner. The rewards associated with all acceptable states should be defined as one and the rewards associated with unacceptable (absorbing) states should be zeroed as well as all rewards associated with transitions.

The reward matrix r for this system is as follows:

$$r = [r_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

The following system of differential equations can be written in order to find the expected total rewards $V_i(t), i = 1, 2, 3, 4, 6, 7, 8, 11, 12.$

$$\begin{cases} \frac{dV_1(t)}{dt} = 0 \\ \frac{dV_2(t)}{dt} = 1 - C_2(t)V_2(t) + \lambda(t)V_1(t) + 2\mu(t)V_3(t) + \lambda_5(t)V_6(t) \\ \frac{dV_3(t)}{dt} = 1 - C_3(t)V_3(t) + 2\lambda(t)V_2(t) + \mu(t)V_4(t) + \lambda_5(t)V_7(t) \\ \qquad + \lambda_6(t)V_{11}(t) \\ \frac{dV_4(t)}{dt} = 1 - C_4(t)V_4(t) + 3\lambda(t)V_3(t) + \lambda_5(t)V_8(t) + \lambda_6(t)V_{12}(t) \\ \frac{dV_6(t)}{dt} = 0 \\ \frac{dV_7(t)}{dt} = 1 - C_7(t)V_7(t) + \lambda_2(t)V_3(t) + \mu(t)V_8(t) + 2\lambda(t)V_6(t) \\ \qquad + \lambda_4(t)V_{11}(t) \\ \frac{dV_8(t)}{dt} = 1 - C_8(t)V_8(t) + 3\lambda(t)V_7(t) + \lambda_2(t)V_4(t) + \lambda_4(t)V_{12}(t) \\ \frac{dV_{11}(t)}{dt} = 0 \\ \frac{dV_{12}(t)}{dt} = 1 - C_{12}(t)V_{12}(t) + \lambda_1(t)V_8(t) + \lambda_3(t)V_4(t) + 3\lambda(t)V_{11}(t) \end{cases} \tag{12}$$

where,

$$\begin{cases} C_2(t) = \lambda(t) + 2\mu(t) + \lambda_5(t) \\ C_3(t) = 2\lambda(t) + \mu(t) + \lambda_5(t) + \lambda_6(t) \\ C_4(t) = 3\lambda(t) + \lambda_5(t) + \lambda_6(t) \\ C_7(t) = \lambda_2(t) + \mu(t) + 2\lambda(t) + \lambda_4(t) \\ C_8(t) = \lambda_2(t) + \lambda_4(t) + 3\lambda(t) \\ C_{12}(t) = \lambda_1(t) + \lambda_3(t) + 3\lambda(t) \end{cases} \tag{13}$$

The initial conditions are: $V_i(0) = 0, i = 1, 2, 3, 4, 6, 7, 8, 11, 12.$

After solving this system and finding $V_i(t)$, the MTTF for MSS can be obtained as $V_{12}(t)$, where the 12-th state is the initial state.

To calculate the probability of MSS failure during time interval [0, t] the model should be transformed as in the previous case: all unacceptable states should be treated as absorbing states and, therefore, all transitions that return MSS from unacceptable states should be forbidden. Rewards associated with all transitions to the absorbing state should be defined as one. All other rewards should be zeroed.

The reward matrix r for this system is as follows:

$$r = [r_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{14}$$

Mean accumulated reward $V_i(t)$ will define the probability $Q(t)$ of MSS failure during time interval [0, t].

The following system of differential equations can be written in order to find the expected total rewards $V_i(t), i = 1, 2, 3, 4, 6, 7, 8, 11, 12.$

$$\begin{cases} \dfrac{dV_1(t)}{dt} = 0 \\ \dfrac{dV_2(t)}{dt} = \lambda(t) + \lambda_5(t) - C_2(t)V_2(t) + \lambda(t)V_1(t) + 2\mu(t)V_3(t) \\ \qquad + \lambda_5(t)V_6(t) \\ \dfrac{dV_3(t)}{dt} = \lambda_6(t) - C_3(t)V_3(t) + 2\lambda(t)V_2(t) + \mu(t)V_4(t) \\ \qquad + \lambda_5(t)V_7(t) + \lambda_6(t)V_{11}(t) \\ \dfrac{dV_4(t)}{dt} = -C_4(t)V_4(t) + 3\lambda(t)V_3(t) + \lambda_5(t)V_8(t) + \lambda_6(t)V_{12}(t) \\ \dfrac{dV_6(t)}{dt} = 0 \\ \dfrac{dV_7(t)}{dt} = 2\lambda(t) + \lambda_4(t) - C_7(t)V_7(t) + \lambda_2(t)V_3(t) + \mu(t)V_8(t) \\ \qquad + 2\lambda(t)V_6(t) + \lambda_4(t)V_{11}(t) \\ \dfrac{dV_8(t)}{dt} = -C_8(t)V_8(t) + 3\lambda(t)V_7(t) + \lambda_2(t)V_4(t) + \lambda_4(t)V_{12}(t) \\ \dfrac{dV_{11}(t)}{dt} = 0 \\ \dfrac{dV_{12}(t)}{dt} = 3\lambda(t) - C_{12}(t)V_{12}(t) + \lambda_1(t)V_8(t) + \lambda_3(t)V_4(t) \\ \qquad + 3\lambda(t)V_{11}(t) \end{cases} \tag{15}$$

Here $C_i, i = 1, 2, 3, 4, 6, 7, 8, 11, 12$ are calculated via formulas (13). The initial conditions are:

$$V_i(0) = 0, i = 1, 2, 3, 4, 6, 7, 8, 11, 12.$$

After solving this system and finding $V_i(t)$, MSS reliability function can be obtained as $R(t) = 1 - V_{12}(t)$, where the 12-th state is the initial state.

Now, we consider two types of the parameters as follows:

(i) The air conditioners failure and repair rates are time-varying

As a particular case, we assume that the working time and the repair time of each conditioner are both Weibully distributed. We can then write:

$$\lambda_1(t) = 1.1\, t^{0.1} \qquad \lambda_5(t) = 1.7\, t^{0.7}$$
$$\lambda_2(t) = 1.2\, t^{0.2} \qquad \lambda_6(t) = 1.8\, t^{0.8}$$
$$\lambda_3(t) = 1.4\, t^{0.4} \qquad \lambda(t) = 1.5\, t^{0.5}$$
$$\lambda_4(t) = 1.6\, t^{0.6} \qquad \mu(t) = 1.9\, t^{0.9}$$

Using MAPLE program, the MSS average availability $\overline{A}(t)$ against time is illustrated in figure (3) with numerical solutions based on Runge-Kutta method.

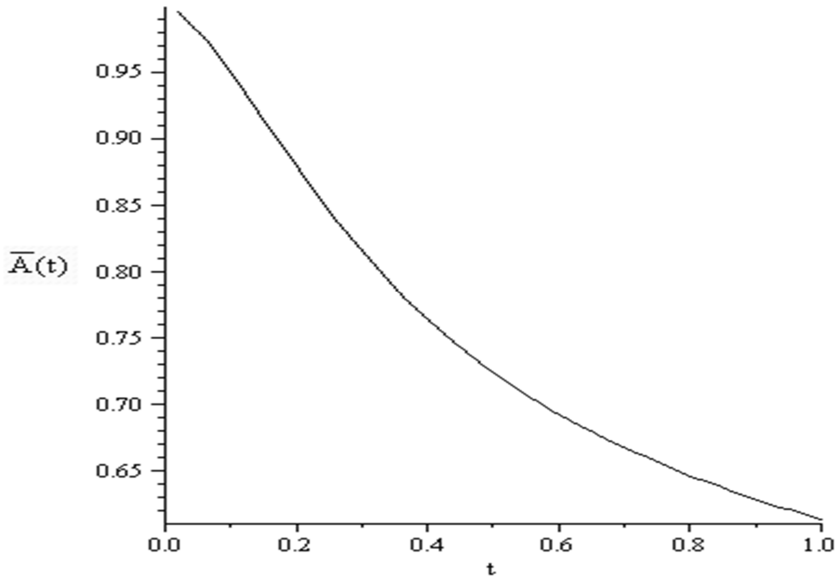

Figure (3): The average availability $\overline{A}(t)$ versus the time t (case i)

Similarly, the mean total number of system failures $N_f(t)$, the MTTF for MSS, and the MSS reliability function $R(t)$ against time are illustrated in figures (4), (5), and (6), respectively.

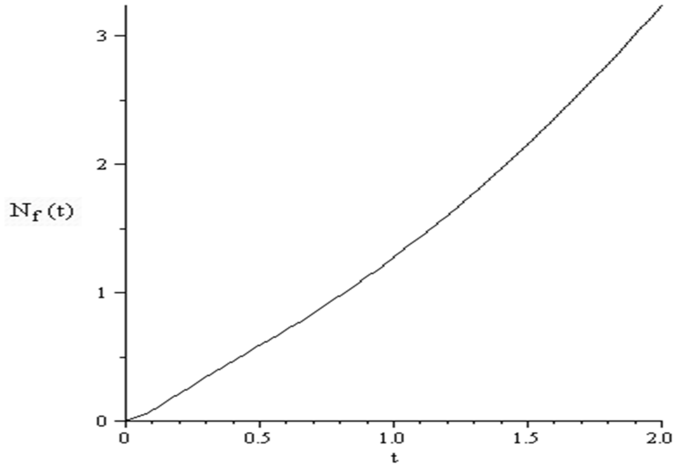

Figure (4): The mean total number of system failures $N_f(t)$ versus the time t (case i)

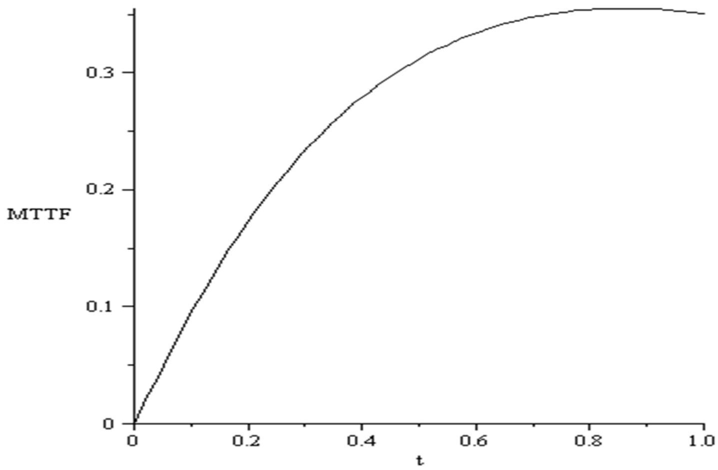

Figure (5): The MTTF for MSS versus the time t (case i)

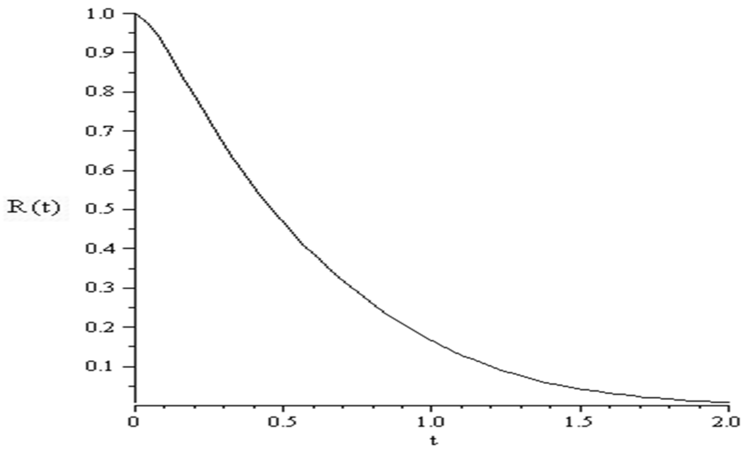

Figure (6): The MSS reliability function $R(t)$ versus the time t (case i)

(ii) The air conditioners failure and repair rates are constant:

As a particular case, we assume that the working time and the repair time of each conditioner are both exponentially distributed. We can then write:

$$\lambda_1(t) = \lambda_1 = 0.5 \qquad \lambda_5(t) = \lambda_5 = 0.25$$
$$\lambda_2(t) = \lambda_2 = 0.25 \qquad \lambda_6(t) = \lambda_6 = 0.2$$
$$\lambda_3(t) = \lambda_3 = 0.25 \qquad \lambda(t) = \lambda = 0.3$$
$$\lambda_4(t) = \lambda_4 = 0.2 \qquad \mu(t) = \mu = 0.6$$

Using MAPLE program, the MSS average availability $\overline{A}(t)$ against time is illustrated in figure (7) with solutions based on Laplace transform method.

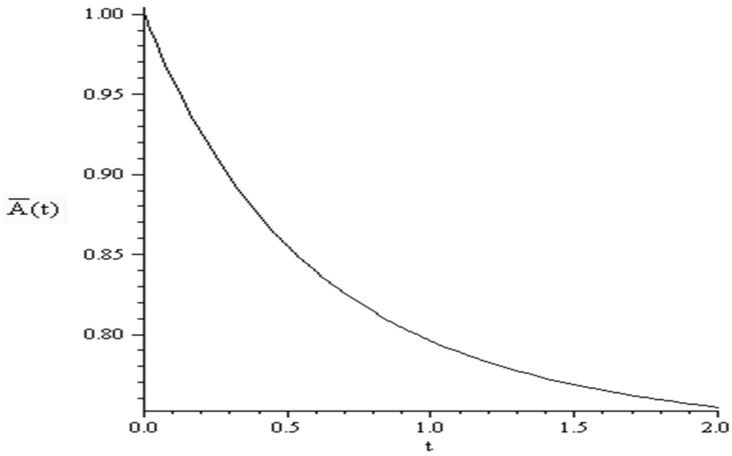

Figure (7): The average availability $\overline{A}(t)$ versus the time t (case ii)

Similarly, the mean total number of system failures $N_f(t)$, the MTTF for MSS, and the MSS reliability function $R(t)$ against time are illustrated in figures (8), (9), and (10), respectively.

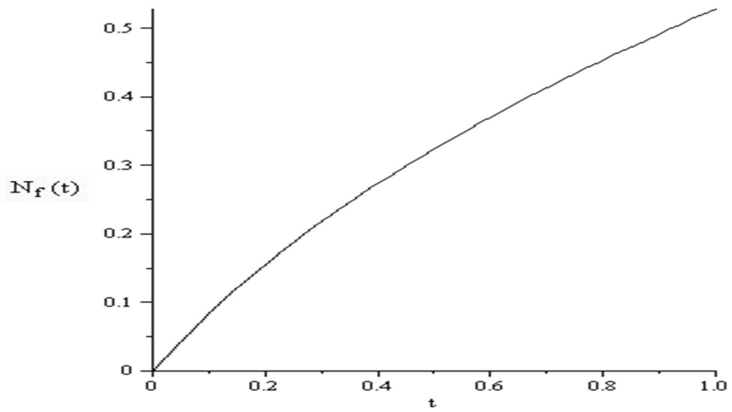

Figure (8): The mean total number of system failures $N_f(t)$ versus the time t (case ii)

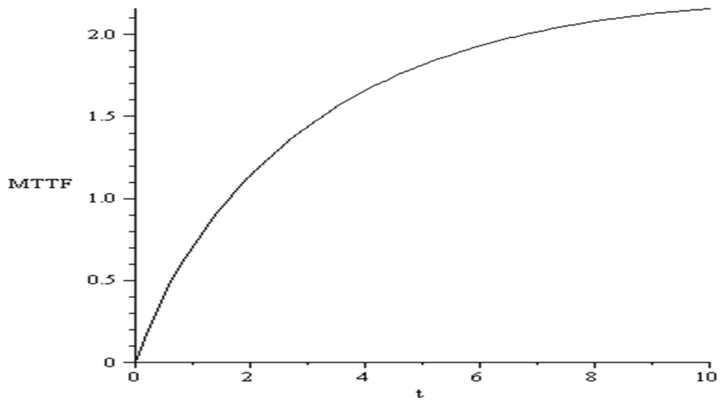

Figure (9): The MTTF for MSS versus the time t (case ii)

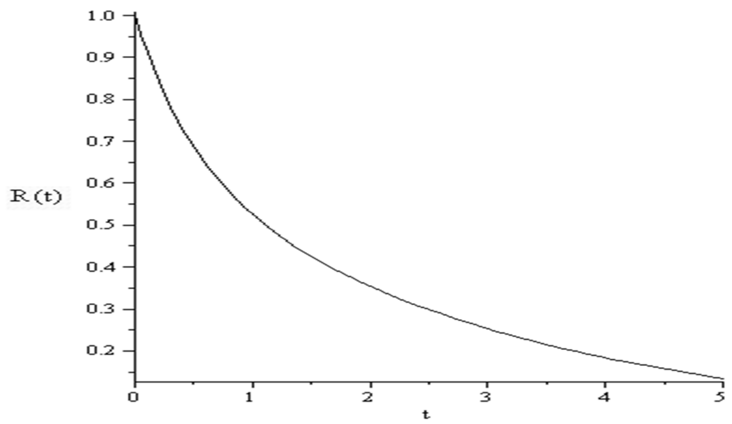

Figure (10): The MSS reliability function $R(t)$ versus the time t (case ii)

time to MSS failure, mean accumulated performance deficiency for a fixed

5. CONCLUSIONS

1. Extension of continuous-time Markov chain to Markov reward models make them even more useful.

2. A Markov reward models was developed as the basis for the generalized computation of availability and reliability measures.

3. The method has been suggested for the computation of MSS reliability and availability measures based on a different reward matrix determination for the Markov reward model.

4. A Markov reward models is well formalized and suitable for practical application in reliability engineering.

5. The numerical results are presented in order to illustrate the suggested model.

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MATHEMATICAL MODELS OF DECISION SUPPORT SYSTEM FOR THE HEAD OF THE FIREFIGHTING DEPARTMENT ON RAILWAYS

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Rail transport is an important basic sector of economy of Ukraine, provides its internal and external transport and economic connections and transportation needs of the population.

Currently, the country's railways transport a large quantity of various goods, including nine form of dangerous goods.

We know that traffic accidents on the railway, according to the causes of origin, are divided into technological, natural, social, political and military character and, according to territorial distribution, disruption of traffic and volumes of technical and financial resources necessary to eliminate them, they can be national, regional and local or object level.

These transport events may lead to loss of life or create a dangerous situation on particular area or particular object which may lead to loss of health, lead to the destruction of buildings, equipment and vehicles, violations of the manufacturing or transport process and harm the environment.

Especially dangerous are accidents, accompanied by explosion of tanks with liquefied hydrocarbon gases and flammable liquids and throwing of flammable liquids and potent poison. Much of the danger comes from solid combustible substances during their transportation and storage [1].

Fighting fires on the railway is marked difficulty in organizing the operations of fire departments and related units of the railway, because of a large number of goods that have a variety of fire and explosive properties, the need to power down the contact network, the complexity of assessing the situation on the fire, the concentration necessary capabilities, etc. [2].

From the above, for the effective management of forces and means of fire must have an effective system of fire departments and relevant units of railroads, to organize a scientific basis for Task Force work to eliminate traffic accidents and fires at the operational headquarters, which is impossible without widespread use of modern information technology including decision support systems (DSS).

Application of DSS allows the informational, technological, analytical and organizational support iterative interactive process of analysis of the situation as a result of accidents, training and evaluation of solutions managers eliminate traffic accidents and fire suppression and selection of the final decision to eliminate such events.

To implement the DSSfor our opinion it is necessary to create mathematical models to assess the situation and the process of developing recommendations for their elimination, and the mathematical model for evaluating the effectiveness of firefighting units [3].

Creating models to assess the situation which arose as a result of traffic accidents accompanied by fire, in our view, should do with the productive systems that provide a view of human experience evaluating the situation in dangerous situations in the form of rules and allow the results to develop a search algorithm and are effective tool for developing expert systems [4].

Based on an analysis of work in addressing issues of liquidation of traffic accidents on the railway and general systems theory, set-theoretic model that reflects the cause-effect relationships of processes that create extraordinary situation, is given by [5]:

$$S=\{\ CS(t_1),\ \ FS(t_2)\ \},$$

where S - the set of emergencies; CS (t) i FS (t) - set the current and final states of emergency, respectively; t = t + t, t - duration of the emergency, t1 - the beginning of an emergency.

Current state of emergency determined by tuple:

$$CS\ (t_1) = \{E, \Gamma, X, Z, \Omega, \Xi, \Delta, T\}.$$

The set E defines a set of character attributes burning hazardous substances fire signs of the impact of hazardous substances at the emergency and the adjacent tank includes a set of Γ; warning signs of cargo tanks are set on X, the set Z consists of the signs of the impact of combustion leakage of substance on the emergency and the adjacent tank; signs of the impact of other (neighboring) emergency tank fires are set Ω, the set Ξ consists of flame characteristics influence another (neighboring) fires in emergency cases or a neighboring tank features of the environment and weather conditions contained in the set Δ, the set T consists of a period of time.

The set of final states FS (t2) emergency includes a set of SE, ME, i HE, which determine the results of such situations. SE set contains the final states of emergencies, bomb threats, when the tank is missing, the set consists of the ME final states of emergency when there is some danger of explosion, and the HE set includes final states with an explicit threat of explosion of the tank.

$$FS(t_2) = \{HE, ME, SE\}.$$

To create an information model of emergency, which is accompanied by fire hazardous cargo, the method of production systems, which is now widely used in artificial intelligence theory to retrieval algorithms and modeling problem solving person. Productive system manages the process of solving problems on the model and consists of a set of production rules, working memory and control cycle "recognition – action".

The generalized production rules to determine the result of an accident is determined by the expression:

$$\{e_i^{fs}\} = [[\{cs_i(t_1)\} = \{\varepsilon_a\} \wedge \{\gamma_b\} \wedge \{\chi_c\} \wedge \{\xi_d\} \wedge \{\omega_k\} \wedge \{\zeta_m\} \wedge \{\delta_q\}] \rightarrow \{fs_l(t_2)\}], \quad (1)$$

Where

$$e_i^{fs} \in S, \varepsilon_a \in E, \gamma_b \in \Gamma, \chi_c \in X, \xi_d \in \Xi, \omega_k \in \Omega, \zeta_m \in Z, \delta_q \in \Delta, fs_i(t_2) \in FS.$$

Assessment of the situation and making recommendations on elimination of emergency situations that are accompanied by fire-hazardous cargo, is a complex multistep process. Given that the problem being solved and the heads of an emergency fire suppression in fire suppression is poorly formalized, their solutions should be undertaken using models, which formalizes the knowledge of experts to solve such problems.

The development of such models currently being carried out by means of information technologies using expert techniques and theories of artificial intelligence techniques that allow professionals to take into account experience and easier to work with DSS not prohramuyuchoho specialist who has knowledge of the organization dealing with breakdowns, accompanied by fire of dangerous goods and involved in the process of making necessary decisions.

In order to create information models and recommendations to eliminate fire on the analysis of guidance documents that define the order of liquidation of accidents with dangerous goods during their transportation by rail, also found that elements of the decisions the leaders of an emergency and a fire that define for fire suppression, cooling and emergency protection of rolling stock and infrastructure, evacuation, equipment and rolling stock, and take into account the experience of experts in making decisions regarding operations of fire departments in emergency situations, which are accompanied by fire-dangerous goods.

Accordingly, the expression for the information model recommendations regarding emergency is defined components:

$$Ra = \{Ex, Col, Ev, Liq, Ac, Det, Ob\}$$

Ex set includes recommendations for actions aimed at putting an emergency rolling stock and prevent formation of explosive concentrations of fuel mixture; Col set includes recommendations for cooling rolling stock; recommendations for measures to eliminate leakage of dangerous substances includes set Liq; Ev has set guidelines that define the arrangements for the evacuation of people, equipment and rolling stock; recommendations for the General measures: isolation of the danger zone, in her handling of personnel, compliance with fire safety measures, provide first aid to victims, etc. contained in the set of Ac, Det set includes recommendations for the parameters of explosion and fire areas and the number of forces and capabilities necessary for fire, cooling and protection of rolling stock and facilities; recommendations for analysis of infrastructure, environment are in set Ob.

Situation assessment and recommendation of the software DSS to eliminate accidents accompanied by fire, carried out in several stages [6].

In the first phase recommendations determined by state emergency rolling stock, infrastructure and rolling stock that are in dangerous areas of the accident, and fire suppression methods in such hardware.

The first procedure φ_2' of this stage is to identify the hazardous event occurring in emergency hardware. Procedure φ_1' by using the generalized production rules (1).

The procedure φ_2' allows to identify the infrastructure and rolling stock of railway transport, which are located in dangerous areas of the accident.

Calculating the size of zones of influence of hazards and accidents and detection of rolling stock in these areas is the methodology and algorithms are created in developing automatic working place for leader fighting a fire at the facilities and rolling stock of railway transport.

The procedure φ_2', based on the above factors is given by:

$$\varphi_2' = \{ \varphi_1', Det', Det'', T, Ob, Ev' \}. \tag{2}$$

Elements Det^{I} subsets are recommendations from the calculation of accident hazards of liquefied hydrocarbon gases. Guidelines for determining the size of accidents hazards with combustible (flammable) liquids are in the subset of Det^{II}. Ev^{I} subset includes recommendations for evacuation.

Given (2) production rules for such a procedure defined by the expression:

$$\varphi_2' = [[\{ec_{2i}'\} = \{e_i^{fs}\} \wedge \{det_r'(t_2)\} \wedge \{det_0''(t_2)\}] \rightarrow$$
$$\rightarrow [\{so_{2\rho}\} = \{ob_{\rho_1}^{P}\} \wedge \{ob_{\rho_2}^{q}\} \wedge \{ev_\eta'\}]],$$

where $ec_{2i}' \in EC$, $EC \subset S$ - subset of the set of emergencies S, which takes into account the consequences of an accident e_i^{fs}; $det_r'(t_2) \in Det'$; $det_0''(t_2) \in Det''$ - calculated from a formula determining the size of hazardous zones accident at the time t_2 ; $so_{2\rho} \in SO$, $SO \subset S$ - subset of the the set of emergencies S, considering the state of rolling stock and facilities that are in dangerous areas of the accident: $ob_{\rho_1}^{P}$ - rolling stock items that are in areas of excessive pressure of the front shock wave from the explosion facilities and rolling stock, suffering from flames; ev_η'

$\in Ev'$ - guidelines for notification and evacuation of the objects that are in dangerous areas of the accident.

After detection of such facilities and rolling stock, using the procedure φ_1' identified hazardous events have been found damaged rolling stock of dangerous goods and set its impact on the surrounding objects and railway rolling stock.

Iterative process of application procedures φ_1' and φ_2' ends when they have not found an emergency rolling with the dangerous goods in zones hazards of accidents, followed by fires.

The procedure φ_3' is devoted to considering ways of fighting fires and emergency cooling tank, taking into account properties of the substance that burns, and develop measures to prevent the formation of explosive concentrations of vapor in a cloud for substances that do not burn.

Formally this procedure is determined by a tuple:

$$\varphi_3' = \{ S, A, Ex, Col, Ac\},$$

where A-set of properties of matter, which affect the processes of combustion, and of making fire and emergency cooling of the rolling stock (A=$\{A', A'', A'''\}$).

Subset $A' \subset A$ characterized by physical and chemical properties of substances. Signs of fire explosion properties of matter form a subset of A'' , $A'' \subset$A. The danger to humans characterize subsets of features $A''', A''' \subset A$.

The second phase of recommendations designed to determine the measures for the evacuation of military servants, railway workers, equipment and rolling stock from dangerous areas of the accident and measures aimed at eliminating leakage alarm substance.

The procedure φ_1^2 for determining the necessary measures to evacuate people, equipment and rolling stock and eliminate leakage of dangerous substances is given a tuple:

$$\varphi_1^2 = < S, Ex, Ac, \Delta, X, T, Ev, Liq, Ob >.$$

The third stage of automated recommendations designed to determine the required number of fire departments for fire suppression and protection of the rolling stock, the definition of probabilistic assessments of operations of fire departments and optimization plan to focus their combat.

Determine number of fire departments to extinguish fires in rolling stock is carried out by known techniques, and using algorithms developed for the DSS leader fighting a fire in rolling stock and rail transport facilities.

The procedure φ_1^3 for determining the required number of capabilities for fire suppression and protection of the rolling stock is given a tuple:

$$\varphi_1^3 = < \varphi_3^1, Det''', T, N_\rho >.$$

Det^{III} subset includes recommendations for determining capabilities necessary for fire suppression. N_ρ - number of units required for fire extinguishing.

Procedure of the probabilistic estimates of operations of fire departments is given by a tuple:

$$\varphi_2^3 = < \varphi_1^3, T, Det^{IV}, J_м >.$$

Det^{IV} is decide on a probabilistic assessment of operations of fire departments in fire suppression and optimization plan to focus fire departments. $J_м$ - the likelihood of possible states of the system, which consists of fire units, facilities and rolling stock.

After defining the principal possibilities of success of fire suppression and protection facilities, optimization plan is to focus on capability, taking into account losses from fire and focus of time and input the required number of fire departments in each of the objects (rolling stock).

The procedure for optimizing the plan focus and putting fire departments is given a tuple:

$$\varphi_3^3 = < \varphi_1^3, c_\rho, Det^{IV} >,$$

where c_ρ - losses of time to concentrate fire departments.

To determine the probabilistic assessment of subdivisions fire department while managing traffic accidents in the DSS applies mathematical tools of queuing theory, including mathematical models, based on closed stochastic networks [7].

The system of "fire units - objects (rolling stock) railway affected by fires in DSS filed a stochastic closed network - a set of interrelated systems of mass service (CMO).

The view of the network depends on the particular priority designation capabilities to sites affected by fire.

A general view of such a closed stochastic network presented in Figure 1.

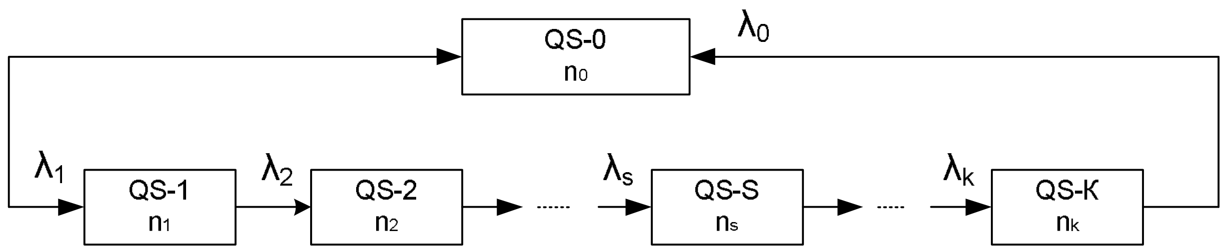

Fig. 1. Closed stochastic network system "fire units - objects (rolling stock), affected by fire. "

Number of fire departments Mrs required for all objects defined tactical calculations. Not allowed income units from outside the network and outputs the limits of their network. It is clear that

$$n = \sum_{j=1}^{K} n_j, \tag{3}$$

where k - number of facilities affected by the fire; nj - number of fire units required for fire suppression and-so on the site.

The items affected by fire are the QS as storage systems with limited service applications. QS-0 is a model fire station, where fire units arrive. In theory queuing system with n parallel handling equipment queue. QS that describe the objects that suffer from fires, are a limited period of storage applications service.

Assuming that the number of fire units that travel to sites is limited, a closed network is always stationary.

Calculating the closed stochastic network made the assumption that the duration of service applications in the QS, which part of the network are random variables, which are distributed by the exponential law.

An important justification for studies of this network scheme is also that it can be considered as a set of independent QS with simpler input.

Closed stochastic network is determined by the following parameters:

- number of (K +1) queuing systems that are part of the network;
- quantity of devices for each QS (n^0,.......,n^K);

- matrix of transition probabilities P=$\|P_{ij}\|$, where P_{ij} - probability that the application and after passing th QS will receive the input of that j-QS;
- n number of applications that circulate in a closed network;
- flux λ_j applications for the entrance of that j-QS.

The sequence of calculating the numerical parameters of stochastic networks [8].

1. Defining the matrix of transition probabilities.

The matrix of transition probabilities - square, the size of (K +1) x (K +1). Index 0 in the transition probabilities related to the source applications QS-0 (fire station):

It P_{oi} - a possibility that application of QS-0to the input and th QS, and P_{jo} -- the likelihood that an application after the passage of that j-QS tends to QS-0.

$$P=\begin{pmatrix} P_{00} & P_{01} & ..... & P_{0\kappa} \\ P_{10} & P_{11} & ..... & P_{1\kappa} \\ .... & .... & ..... & .... \\ P_{\kappa 0} & P_{\kappa 1} & ..... & P_{\kappa\kappa} \end{pmatrix}$$

Thus P_{oi} – is the probability of that the request from QS-0 comes to the entrance of i-th QS, and P_{jo} – is the probability of that the request after coming of j-th QS goes to QS-0.

It is obvious that elements of the matrix P sum of each tape is 1.

2. Determination of the intensity of flow at the entrance of each QS. Assuming that the intensity of the stationary mode flow of applications for the entrance and exit of any cleanse the same:

$$\lambda'_j = \sum_{i=0}^{\kappa} P_{ij}\lambda'_i, \quad j=\overline{o,\kappa};\ \lambda'_i = n\lambda_0,$$

we can write the system of homogeneous linear $\lambda_0, \ldots\ldots, \lambda_\kappa$,which will look like:

$$\left.\begin{aligned} &(P_{00}-1)\lambda'_0 + P_{10}\lambda'_1 + \ldots. + P_{\kappa 0}\lambda'_\kappa = 0, \\ &P_{01}\lambda'_0 + (P_{11}-1)\lambda'_1 + \ldots. + P_{\kappa 1}\lambda'_\kappa = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &P_{0\kappa}\lambda'_0 + P_{1\kappa}\lambda'_1 + \ldots. + (P_{\kappa\kappa}-1)\lambda'_\kappa = 0. \end{aligned}\right\}$$

3. Calculating the probability of network status.

Network status determined by the number of applications in each QS network. Denote the number of entries in the j-and QS (j = $\overline{0,\kappa}$) by n_j. The set of values $\{n_0,\ldots,n_\kappa\}$ determines the state of the network. Given (3), the parameter that is calculated to be determined by the expression:

$$J_n\{n_0,\ldots..n_\kappa\} = \frac{P^{(0)}_{n0}\ldots..P^{(n)}_{nk}}{\sum\limits_{A(n,k)}\prod\limits_{j=0}^{\kappa} P^{(j)}_{nj}}.. \tag{4}$$

In the numerator (4) - the product of probabilities of states of QS, which part of the network, and $P^{(j)}_{nj}$ - the likelihood of each j of that QS.

In the denominator (4) The summation is over all states for which $n = \sum_{j=1}^{\kappa} n_j$ (the set of such states indicated A (n, k)). The denominator in (4) - is a normalizing factor that is introduced to the sum of probabilities of all possible states of a network unit equal.

According to the method of DSS considered various options for priority focus fire units at sites affected by fire, and determined the likelihood of possible states of the network.

From a comparison of the probabilities chosen focus option with a maximum value of such magnitude. Technique is used, you can determine the success of concentrating fire departments for fire suppression on objects (rolling stock) Railway provided when known only time characteristics of "fire departments - objects (rolling stock) railway affected by fire”.

With DSS automatized process of assessing the situation, the parameters of transport developments in fire zones availability of rolling stock and rail facilities and to identify the required number of capabilities for fire suppression.

CONCLUSIONS

Using such a DSS can significantly reduce the time to evaluate the situation and making decisions on the organization of firefighting units to eliminate fire techniques to produce efficient thinking at training officers fire departments railroads, and develop knowledge base DSS.

Future directions of the development of DSS are its use in research networks, comprising some QS with different characteristics, and models that take into account the loss of combat vehicles and servants, and the sequence of different disciplines to focus on capabilities rolling stock, affected by fire.

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ASYMPTOTIC OPTIMUM QUANTIZATION OF THE CASUAL SIGNAL WITH BLANKS BETWEEN QUANTA

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ABSTRACT

In article generalization of a problem of optimum quantization of a casual signal with blanks between quanta is presented. Unlike known works the law of distribution of a casual signal with quanta is received. Instead of the integer decision of a problem the approached asymptotic decision is offered at a great number of quanta and the estimation of its accuracy is given. Besides, the decision of the given problem is received at fuzzy values of parameters of a blank and a population mean of an initial random variable with the normal law of distribution. Bibl. 8 nam., 5 fig.

Keywords: a random variable, asymptotic the optimum decision, quantum, probability density, uniform and normal distributions, fuzzy parameter, accessory function.

Introduction. The problem of quantization of a casual signal, apparently, has been put for the first time and asymptotic is solved in article [1]. Article has been connected with a problem of value of the information in the theory of the information [2], it differed high mathematical level. Application of results of this article to the decision of applied problems of a technical profile inconveniently enough. However, the idea of the asymptotic approach to the decision can form a basis for the decision of various practical problems.

In article [3] as criterion function of quantization of a casual signal the population mean, the most simple and rough characteristic of a random variable is accepted. The problem consisted in search of size of such quantum at which the population mean quantized random variable would reach a minimum. It has been shown that the problem of search of a minimum is a problem of integer optimization. For the numerical decision of a problem original enough algorithm is offered. Use of this algorithm, in our opinion, is expedient for the decision of unique problems of the raised accuracy, especially at small number of quanta. The basic lack – a combination of rough criterion function and enough bulky algorithm of the decision of a problem. The algorithm of the decision represents the basic value of article. It can be used as the sample of programming of the decision of challenges by students of technical colleges. For the decision of engineering problems it restrictedly is suitable because of high labor input of search of result, especially at rough criterion function.

The idea of article [3] was used successfully at the decision of various problems of protection of the information [4,5], and also in problems of reliability of switching structures of systems.

Interest to a quantization problem, in our opinion, is connected now with a problematics of the theory of fuzzy sets. Really, many parametry technical, program and social systems can't be having appearedleny unequivocally, accurately, "is not washed away". The author of given article in [6] the decision of a problem of optimization by a method uncertain mnozhitelja was offered to Lagranzha at fuzzy restriction.

The decision of a problem of quantization at fuzzy parameters practically it is necessary. However, to promote in this area it is not simple enough. In is given the article some generalization criterial the approach to quantization of a casual signal, and also model of the account of an illegibility parametra a blank between quanta is offered.

The author realizes that article is connected only with a question kvantovanija a casual signal, but not quantizations in the information theory.

Density of probability of size quantized a signal. Despite the stated critical remark concerning roughness of criterion function, we will use value of its formal representation and we will make attempt of development of idea of authors [3]. So, the size population mean quantezed a casual signal is presented by authors as:

$$M(x) = (x+c)\int_0^{\infty}[trunc(\frac{z}{x})+1]dF(z), \tag{1}$$

Where size of quantum of a signal, blank size between quanta, the greatest whole part of number, and function of distribution of a random variable no quantized a signal. Further we will be a floor-corduroy road that distribution function has continuous density and a final population mean. We will construct formally following expression for probability density, using application delta-function [7]:

$$\varphi(w) = \int_0^{\infty}\Delta(w-(x+c)[trunc(\frac{z}{x})+r])f(z)dz\,, \tag{2}$$

where $\Delta-$ delta-function, $f(z) = F^{'}(z)$ and $r-$ let some constant number, satisfying to a condition $0 \le r \le 1$. Further, we will be released from integer relations $\frac{z}{x}$, that is we believe that division is presented with some on-sinfulness from integer value. At $r = 0$ size of the relation of the tsebark, some lack, at $r = 1-$ whole with some surplus. It is asked, at enough great number of quanta as itself will conduct conducted-rank r? Obviously, we have the right to believe with the big degree of confidence that the random variable $\hat{r}$ between the next quanta will be distributed in regular intervals with probability density $\frac{1}{x}$ on an interval x. Certainly, at small number of quanta and necessity of search of the integer decision of a problem of quantization it is an assumption unacceptably. Thus, it is a question of asymptotic representation of density of probability (2). For search of density (2) it is necessary to execute double integration at a random $\hat{r}$, but we won't do it, and we will be limited to size population mean expectations for it $v_{1,r} = \frac{1}{2}$. Then (2) it will be presented in a kind:

$$\varphi(w) = \int_0^{\infty}\Delta(w-(x+c)[\frac{z}{x}+\frac{1}{2}])f(z)dz\,. \tag{3}$$

As to transform the size in integral is monotonous, after performance of simple transformations (2) is received:

$$\varphi(w) = \frac{x}{x+c}f(\frac{x}{x+c}w - rx), \quad x+c \le w \langle \infty\ . \tag{4}$$

It is easy to be convinced, that $\int_{r(x+c)}^{\infty}\varphi(z)dz = 1,$ having made variable replacement $\frac{x}{x+c}w - rx = u$.

Let's find a size population mean $\hat{w},$ using (4):

$$v_{1,\varphi}(x) = \frac{x+c}{x}v_{1,Z} + r(x+c), \tag{5}$$

where $v_{1,Z} = \int_0^{\infty}f(z)dz\,.$

We investigate special cases for $r = 0, \frac{1}{2}, 1,$. At $r = 0$ it is had $\nu_{1,\varphi}(x) = \frac{x+c}{x}\nu_{1,Z}$, $\nu'_{1,\varphi}(x) = -\frac{c\nu_{1,Z}}{x^2}$. Function of $\nu_{1,\varphi}(x)$ a minimum has no. At $r = \frac{1}{2}$ $\nu_{1,\varphi}(x) = \frac{x+c}{x}\nu_{1,Z} + \frac{1}{2}(x+c)$, $\nu'_{1,\varphi}(x) = -\frac{c\nu_{1,Z}}{x^2} + \frac{1}{2}$.

There is a minimum at $x_0 = \sqrt{2c\nu_{1,Z}}$, equal $\nu_{1,\varphi}(x_0) = (1 + \frac{c}{\sqrt{2c\nu_{1,Z}}})\nu_{1,Z} + \frac{1}{2}(\sqrt{2c\nu_{1,Z}} + c)$. At $r = 1$ $\nu_{1,\varphi}(x) = \frac{x+c}{x}\nu_{1,Z} + x + c$, $\nu'_{1,\varphi}(x) = -\frac{c}{x^2}\nu_{1,Z} + 1$. There is a minimum a $x_0 = \sqrt{c\nu_{1,Z}}$, equal $\nu_{1,\varphi}(x_0) = (1 + \frac{c}{\sqrt{c\nu_{1,Z}}})\nu_{1,Z} + \sqrt{c\nu_{1,Z}} + c$.

Making calculations under the resulted formulas for normal distribution with parameters $\nu_{1,Z} = 100$ *ед.*, $\sigma_Z = 20$ *ед.*, and $c = 5$ *ед.* , we receive:

- $r = \frac{1}{2}$, $x_0 \approx 31{,}6$ *ед.*, $\nu_{1,\varphi}(x_0) \approx 134{,}1$ *ед.*, $\approx 3{,}7$ *квантов*;
- $r = 1$, $x_0 \approx 22{,}4$ *ед.*, $\nu_{1,\varphi}(x_0) \approx 149{,}7$ *ед.*, $\approx 5{,}5$ *квантов*;
- $r = 2$, $x_0 \approx 15{,}8$ *ед.*, $\nu_{1,\varphi}(x_0) \approx 173{,}2$ *ед.*, $\approx 8{,}3$ *квантов* .

The calculation executed under the formula (1), shows that for resulted values parameters of the normal law is received $x_0 \approx 31{,}6$ *ед.*, $M(x_0) \approx 134{,}3$ *ед.*, $\approx 3{,}7$ *квантов* . It testifies about satisfactorym coincidence of calculation to the calculation executed under the formula (4) at $r = \frac{1}{2}$.

Let's find expression for the second initial moment quantized a size case:

$$\nu_{2,\varphi}(x) = \int_{r(x+c)}^{\infty} z^2\varphi(z)dz = \frac{(x+c)^2}{x^2}(\nu_{2,Z} + x\nu_{1,Z} + r^2x^2), \tag{6}$$

where $\nu_{2,Z} = \int_0^{\infty} z^2 f(z)dz$. We will result expression for an average quadratic deviation at $r = \frac{1}{2}$ – $\sigma_\varphi = \sqrt{\frac{(x+c)^2}{x}(\frac{\sigma_Z{}^2}{x} + \frac{\nu_{1,Z}}{2})}$, we will calculate its value and value variation factor in a minimum point $x_0 = 31{,}6$ *ед.*, it is received $\sigma_\varphi = 51{,}5$ *ед.*, $\eta_\varphi = 0{,}38$.

For determined distributions, that is at $f(z) = \Delta(z - \nu_{1,Z})$, $\nu_{1,Z} = 100$ *ед.*, we will receive the same values for a population mean as it depends only from $\nu_{1,Z}$ and doesn't depend from σ_Z.

Likelihood estimation of size quantized a random variable to mozh the approximately to define, using the formula of density of probability:

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma_\varphi} e^{-\frac{(x-\nu_{1,\varphi}(x_0))^2}{2\sigma_\varphi^2}}. \tag{7}$$

Thus its third initial moment will be equal:

$$\nu_{3,\varphi} = \int_{r(x+c)}^{\infty} z^3 \varphi(z) dz = \frac{(x+c)^3}{x^3} (\nu_{3,Z} + 3rx\nu_{2,Z} + 3r^2x^2\nu_{1,Z} + r^3x^3) \ . \qquad (8)$$

The relative error of value of the third initial moment, defined under the formula (8), in comparison with the third moment found on method [3] is equal $(2{,}674 - 2{,}629)\cdot 10^6 \cdot 100/2{,}674\cdot 10^6 \approx 1{,}7\%$.

Any initial moment i an order for probability density can be defined under the formula :

$$\nu_{i,\varphi} = \frac{(x+c)^i}{x^i} \sum_{j=0}^{i} C_i^j \nu_{i-j,Z} (rx)^j \ . \qquad (10)$$

Believing size of optimum quantum a random variable, it is possible to find its approached value mean squared deviations σ_κ and using the normal law of distribution to receive a necessary likelihood estimation of size of quantum. Exact definition σ_κ is inconvenient enough, as quanta as random variables, are dependent. Believing their independent, the approached estimation from above for σ_κ can be found from following reasons. We will find transformation of Laplas of density of probability (4)

$$L(s) = e^{-r(x+c)} \Psi(\frac{x+c}{x} s), \qquad (11)$$

where $\Psi -$ transformation of Laplas $f(z)$. Further, believing known x_0 and average of quanta n_0, at the given optimum decision we will write down:

$$[g(s)]^{n_0} = L(s), \qquad (12)$$

where $g(s)-$the image of Laplasa of density of probability of size of quantum. Let's find it $g(s) = \sqrt[n_0]{L(s)}$. The first and second initial moments will be defined as $\nu_{1,k} = -g'(0), \nu_{,k} = g''(0),$ and $\sigma_k = \sqrt{\nu_{2,k} - \nu_{1,k}^2}$. Without resulting bulky calculations for our example with $r = \frac{1}{2}$, $x_0 = 31{,}62$ *ед.*, $n_0 = 4$ we will receive $\sigma_k \approx 13{,}58; \eta = \frac{\sigma_k}{\nu_{1,k}} = 0{,}43$.

For real, dependent quanta, these values will be slightly less.

Algorithm of fuzzy quantization of a casual signal. In many cases quantizations of a casual signal it is necessary to take into consideration a fuzzy of the separate parameters influencing received decisions of a problem. Such parameters can be a little. In our example in such parameters can to be parameters of the law of distribution quantized a random variable, blank size between quanta.

The decision of a problem of fuzzy quantization at several fuzzy parameters inconveniently enough because of necessity of construction of multidimensional function of an accessory. Therefore we will be limited to influence consideration only one fuzzy parameter. As such parameter we will accept blank size between quanta c. We will be set by an interval of fuzzy and function of an accessory of parameter in the conditions of a considered example with normal function of distribution of a random variable. Let they are represented by the function $\mu(c) = 0{,}111\cdot(8-c)\cdot(c-2), \quad c \in [2,8]$ which schedule is represented in drawing 1.

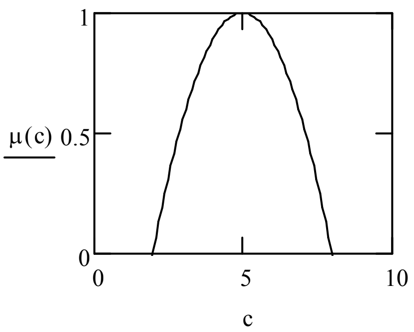


Fig. 1

To construct schedules of functions of an accessory to a population mean (5) and $\nu_{1,\varphi}(x_0) = \dfrac{x_0 + c}{x_0}\nu_{1,Z} + r(x_0 + c)$, $x_0 = \sqrt{2c\nu_{1,Z}}$ and at $r = \dfrac{1}{2}$. Then it is necessary to be set by degree « illegibility» these functions and to define their admissible borders illegibility».

For construction of calculations in the environment of Mathcad it is necessary to use indexes representations of functions [8]:

$$i = 2,3:8;\ \delta = 1;\ c_i = \delta \cdot i;\ \mu_i = 0.111 \cdot (8 - c_i) \cdot (c_i - 2);\ \nu_{1,\varphi,i} = (x + c_i) \cdot (\frac{\nu_{1,Z}}{x} + \frac{1}{2});\ x_0 = \sqrt{2 \cdot c_i \cdot \nu_{1,Z}}.$$

Calculating, we will receive two vectors – a vector - argument and function vector. Substituting in a vector-argument numerical values $\nu_{1,Z} = 100\ e\partial.,\ r = \dfrac{1}{2}$, writing down values both functions in shape it is transposed vectors-lines, we will construct the functions of an accessory of size population mean quantezed the random variable, represented on figure 2.

$\nu 1\phi := (0\ \ 122.7\ \ 126.5\ \ 130.3\ \ 134.2\ \ 138\ \ 141.8\ \ 145.7)^T$

$\mu := (0\ \ 0\ \ 0.56\ \ 0.89\ \ 1.00\ \ 0.89\ \ 0.56\ \ 0)^T$

We will be set, for example, by trust level to the size $\nu_{1,\varphi}$, equal 0,8 and we will find values of the bottom and top borders for this level:

$$A(t) = l\,\mathrm{int}\,erp(\nu_{1,\varphi}, \mu, t);\ l\,\mathrm{int}\,erp(\nu_{1,\varphi}, \mu, 129.26) = 0.8;\ l\,\mathrm{int}\,erp(\nu_{1,\varphi}, \mu, 139.04) = 0.8.$$

Thus, at level trust a size $\alpha = 0{,}8$ population mean quantezed a signal will be in limits $129{,}26\ un. \le \nu_{1,\varphi} \le 139{,}04\ un.$.

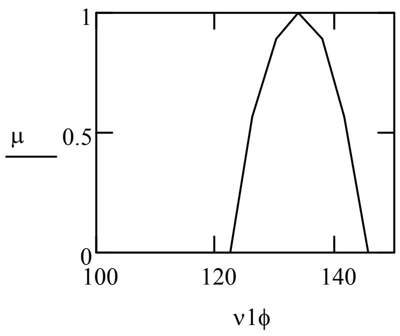


Fig. 2.

$$
\nu 1\phi \rightarrow \begin{bmatrix} 0 \\ 0 \\ (x+2)\cdot\left(\frac{\nu_{1,z}}{x}+r\right) \\ (x+3)\cdot\left(\frac{\nu_{1,z}}{x}+r\right) \\ (x+4)\cdot\left(\frac{\nu_{1,z}}{x}+r\right) \\ (x+5)\cdot\left(\frac{\nu_{1,z}}{x}+r\right) \\ (x+6)\cdot\left(\frac{\nu_{1,z}}{x}+r\right) \\ (x+7)\cdot\left(\frac{\nu_{1,z}}{x}+r\right) \\ (x+8)\cdot\left(\frac{\nu_{1,z}}{x}+r\right) \end{bmatrix} \qquad \mu \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ .555 \\ .888 \\ .999 \\ .888 \\ .555 \\ 0 \end{pmatrix}
$$

Arriving similarly, we find the function an accessory for optimum size of quantum at fuzzy size of a blank between quanta which is represented in figure 3. Function and argument vectors are equal:

$$x_0 = (0\ 20\ 24.49\ 28.28\ 31.62\ 34.64\ 37.42\ 40)^T,$$
$$\mu = (0 \quad 0 \quad 0.56 \quad 0.89 \quad 1.00 \quad 0.89 \quad 0.56 \quad 0)^T.$$

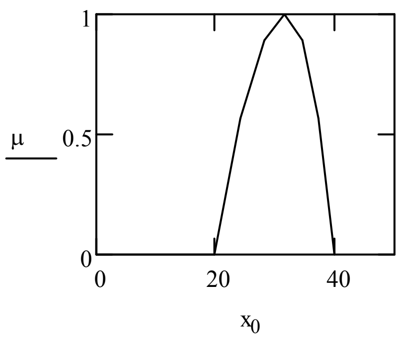


Fig. 3.

Interval accessories of optimum size of quantum at $\alpha = 0{,}8$ trust level should satisfy to inequality $27{,}25\ un. \le x_0 \le 35{,}40\ un.$.

For comparison we will calculate function an accessory for a population mean (1) resulted in [3]. The schedule of this function is shown in figure 4, and at the left it is an interval of an accessory of argument $M(x_0)$ for $x_0 = 31{,}62$ *ed.*, at which $M(x_0) = \min$.

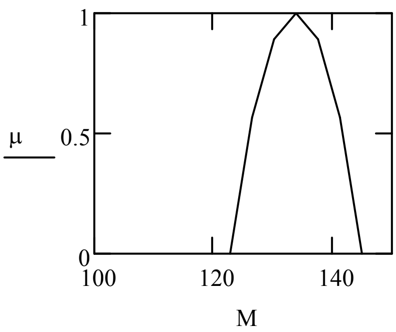
At trust level $\alpha = 0.8$ $129.41\ un. \le M \le 138.73\ un.$

The received results coincide practically with split-hair accuracy that testifies to a correctness of the offered model quantization at the account of an fuzzy of parameter.

$$M := (123.09\ \ 126.75\ \ 130.41\ \ 134.07\ \ 137.73\ \ 141.39\ \ 145.05)^T$$

$$\mu := (0\ \ 0.56\ \ 0.89\ \ 1.00\ \ 0.89\ \ 0.56\ \ 0)^T$$

Fig. 4.



Let's give an example calculations of two-dimensional function of an accessory to fuzzy definition of size of optimum quantum:

$$r_{i,j} = \int_{2}^{8} \Delta(u - c_i) \int_{80}^{120} \sqrt{2uz}\,\Delta(z - m_j)\,dzsu, \quad \omega_{i,j} = \min(\mu_i, \nu_j)\,, \tag{13}$$

where $r_{i,j}$ − an element of a two-dimensional matrix of values of arguments, and $\omega_{i,j}$ − an element of a matrix of function of the accessory, corresponding i, j − to value of argument. Both matrixes are shown in drawing 5.

Fig. 5.

$$\omega := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.56 & .056 & 0.56 & 0 \\ 0 & 0.75 & 0.89 & 0.75 & 0 \\ 0 & 0.75 & 1.00 & 0.75 & 0 \\ 0 & 0.75 & 0.89 & 0.75 & 0 \\ 0 & 0.56 & 0.56 & 0.56 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad r := \begin{pmatrix} 17.9 & 19.0 & 20.0 & 21.0 & 21.9 \\ 21.9 & 23.3 & 24.5 & 25.7 & 26.8 \\ 25.3 & 26.8 & 28.3 & 29.7 & 31.0 \\ 28.3 & 30.0 & 31.6 & 33.2 & 34.6 \\ 31.0 & 32.9 & 34.6 & 36.3 & 37.9 \\ 33.5 & 35.5 & 37.4 & 39.2 & 41.0 \\ 35.8 & 38.0 & 40.0 & 42.0 & 43.8 \end{pmatrix} \tag{14}$$

From matrixes (14) follows that value the accessory function, equal 1.00, corresponds to optimum value of size of quantum $x_0 = 31.6$ *ед.* On values of elements of a matrix r it is possible to define degree fuzzy quantum sizes at certain level of trust to accessory function in a matrix ω . For example, for $\alpha_c = 0.75$ on parameter of a blank c the quantum size should be in limits $30.0\ un. \le x_0 \le 33.2\ un.$, and for $\alpha_\nu = 0.89$ on value of a population mean ν_{1Z} – in limits $28.3\ un. \le \nu_{1Z} \le 34.6\ un.$. For achievement it is more necessary for accuracy to raise accuracy of calculations under formulas (13). Similarly it is possible to define requirements to fuzzy size of a population mean with quanta $\nu_{1\varphi}$.

The conclusion. On the basis of use of expression for a population mean of the random variable presented in the form of sequence of equal quanta on size with blanks between them, expression for density of probability of a random variable with quanta is received.

Asymptotic representation of the given density of probability under condition of replacement of integer number of quanta with the sum of the relation of realization of an initial random variable to size of quantum and a population mean of in regular intervals distributed random variable on an interval of size of quantum is offered. It allows to define optimum size of quantum, to find a likelihood estimation of a random variable with quanta and values of its initial moments at optimum size of quantum.

On an example for an fuzzy blank on size between quanta definition of functions of an accessory of fuzzy values of sizes of optimum quantum and a population mean of a random variable with quanta is shown.

Given article has no direct relation to the information. Further it is expedient to consider the problem on quantization of a casual signal with syntactic, semantic and pragmatical forms of the static information.

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A GENERAL ANALYTICAL SOLUTION FOR THE OCCURRENCE PROBABILITY OF A SEQUENCE OF ORDERED EVENTS FOLLOWING A POISON STOCHASTIC PROCESS

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ABSTRACT

The author presents a general analytical solution determining “the Occurrence probability of a sequence of events each following Poison Stochastic Process”. Generally, this probability is described under the form of an integral equation of order “n”. Where “n” is number of the elementary events in the examined sequence.

As far as the author can tell, the solution is original. It will be of a great interest to a wide range of system reliability problems such as: sequential calculations, dominos effects, dynamics fault trees, Markov systems, priority AND gates, events trees, stochastic optimisation, acceleration techniques for Monte-Carlo simulation, …

Key words: Ordered events, sequential events, Poisson stochastic Process, Markov, probability

1 INTRODUCTIONIS

The author is interested in determining “the occurrence probability of a well-defined sequence of ordered events obeying a Poisson stochastic process”, $p_n(t)$.

One meets often ordered events in system reliability analyses. Analysts may use “Dynamic Fault Tree” with “Priority Gates”, “Markov Graphs” or “Monte-Carlo Simulation” tools in order to deal with the dynamic aspect of this problem.

A fault tree can be described by means of some cut sets. One may calculate the occurrence probability of each cut set. However, the calculated probability does not tell about the occurrence order of the involved events in the cut set. A cut-set, containing n-independent events, may be expressed in $n!$ different ordered sequences. In many reliability and risk assessments, only some given ordered sequences may be of specific concerns. Consequently, it is of great interest to determine the occurrence probabilities of these sequences and their occurrence rates.

In the paper, one describe the problem in the form of a given integral in §3 and a differential equation in §4. In [1], Fussell use the same integral equation as we use in §3, but the events are given in the opposite order. He uses Laplace transformation to find out an exact solution. Although the solution is exact, Fussell switched on to use the asymptotic form of the solution. In [2], Yunge uses the same differential equation given previously in §4 in order to model the sequential occurrence of events in a given priority AND gate (PAG). He uses Laplace transformation and find out the exact solution of the occurrence probability $p_n(t)$. However, in both cases, the authors used complicated forms such that it did not allow putting in evidence the recurrence aspect of the solution. They were more concerned by inserting their models (PAG, ...) in a Dynamic Failure Tree than by other aspects of the solution.

However, if they had not excluded the use of an equivalent Markov Graph, they would have noticed this interesting recurrence aspect of the model, and maybe, they would have used a simpler expression of the solution.

Many other authors followed almost the same way on modelling and produced very interesting applications, [3]-[7], without perceiving that the exact solution could be put in a simpler form.

This would have extended the solution to other categories of problems rather than just the modelling of the PAG's and their inclusion in Dynamic Failure Trees.

Some other researchers could solve the same problem in using numerical techniques such as Petri Nets of Dynamic Bayesian Net (DBN). The use of a numerical technique prevents all possibility to underline the analytical solution and its interest. This is the case of Montani, [8] in an application relative to an active heat rejection system (AHRS) that he used to validate the methodology. The case would have been solved immediately if one had applied the analytical solution given herein. The problem contained only 8 sequences with a maximum length of 4 successive failures.

One may equally mention the case of the applications in [9]-[11], where the general analytical solution would have helped in treating the problems in exact way.

Regarding rare sequential events, many researchers sought answers in developing methods based on Monte-Carlo simulation, [12]-[13]. Some other papers are given in [14]-[21] which have developed interesting methods as well as they developed solutions very close to the one proposed her. Two papers should particularly be underlined are [19] and [20].

The work presented here is limited to Poisson stochastic processes. However, it is of high interest because it will obviously improve the numerical procedures used to treat practical industrial cases.

The analytical solution of this problem presents a particular interest by itself, because of its originality and simplicity. Besides, it suggests some interesting directions of investigation so that it may help in developing analytical solutions for some other specific time distributions different from Poison ones.

2 PROBLEM DEFINITION

Let T describe a top event which results from the occurrence of some n basic events e_i ($i = 1,2,...,n$) in a well determined sequential order. Basic events e_i are following Poisson stochastic processes and each is fully characterised by a constant occurrence rate λ_i and by its occurring order 'i'. The e_1 is the 1st occurred and e_n is the last event.

One would like to determine the occurrence probability of the top event T and its occurrence rate.

3 PROBLEM MODELLING

A will defined top event T will occur if and only if some discrete and independent events e_i happen according to a well specified order $[e_1, e_2, e_3, ..., e_n]$. The corresponding occurring instants are defined by $[t_1, t_2, t_3, ..., t_n]$, where $[t_1 < t_2 < t_3 < ... < t_n]$. Each of these instances $[t_1, t_2, t_3, ..., t_n]$ has its distribution probability function (pdf). The first event is e_1 and the last one is e_n.

The probability $p_n(t)$ that Event T happens within the interval [0,t] is given by:

(1) $$p_n(t) = \int_0^t \rho_1(\xi_1) d\xi_1 * \int_{\xi_1}^t \rho_2(\xi_2) d\xi_2 * \int_{\xi_2}^t \rho_3(\xi_3) d\xi_3 * \ldots * \int_{\xi_{n-2}}^t \rho_{n-1}(\xi_{n-1}) d\xi_{n-1} * \int_{\xi_{n-1}}^t \rho_n(\xi_n) d\xi_n$$

Where: $0 \le \xi_1 \le \xi_2 \le \xi_3 \le ... \le \xi_n \le t$ and,

ρ_i is the Poisson density function characterising the event e_i [$\rho_i = \lambda_i * e^{-\lambda_i t}$].

Many authors could previously developed analytical solutions fo Equation (1) when it was a matter of limited number of ordered events obeying a Poison's Stochastic Process, e.g. [1][2][19][20].

Here, the paper develops a simpler form of the exact solution of Equation (1).

4 ANALYTICAL SOLUTION

It is obvious that Equation (1) can equally be expressed on the following differential form:

$$(2) \quad \frac{d}{dt} p_{n+1}(t) \quad = \quad \rho_{n+1}(t)\ p_n(t)$$

Let $p_n(t)$ be the occurrence probability of a sequence T, a set of chronologically ordered events $[e_1, e_2, e_3, ..., e_n]$. The probability $p_n(t)$ is the solution of the Equation (1) and (2).

Let $p_n(t)$ be expressed by following expression:

$$(3) \quad p_n(t) \quad = \sum_{j=1}^{n} C_j^n * (1 - e^{-(\sum_{l=n-j+1}^{n} \lambda_l)t}),\ C_1^1 = 1.0.$$

Where, each event e_i is defined by a constant occurrence rate λ_i, $\{i \in [1,2,...,n]\}$.

The solution of the problem resumes in determining the coefficients C_i^n .

In appendix (1), we demonstrate the solution proposed in Equation (3) and show that the coefficients C_j^i are fully determined thanks to some recurrence pattern, as following:

$$(4) \quad C_1^{i+1} \quad = \sum_{j=1}^{i} C_j^i, \qquad C_{j+1}^{i+1} = -\frac{\lambda_{i+1}}{\sum_{l=i-j+1}^{i+1} \lambda_l} C_j^i, \qquad j = 1,2,...,i, \quad \text{and} \quad i \in [1,2,...,n]$$

Some examples for calculating the parameters C_j^i are given in appendix (2).

4.1 Occurrence Density and Occurrence rate

By definition, the corresponding occurrence density function $\Theta_i(t)$ can directly be deduced via the first derivative of the occurrence probability function as following:

$$(5) \quad \Theta_n(t) \quad = \quad \frac{dp_n(t)}{dt}$$

The occurrence density function will then be defined by:

$$(6) \quad \Theta_n(t) = \sum_{j=1}^{n} C_j^n * (\sum_{l=n-j+1}^{n} \lambda_l) e^{-(\sum_{l=n-j+1}^{n} \lambda_l)t}$$

We may also define an equivalent occurrence rate Λ_i of the whole sequence T, such as:

$$(7) \quad \Lambda_i(t) = \frac{1}{p_i(t)} * \frac{dp_i(t)}{dt} = \frac{\sum_{j=1}^{i} (\sum_{l=i-j+1}^{i} \lambda_l) * C_j^i e^{-(\sum_{l=i-j+1}^{i} \lambda_l)t}}{\sum_{j=1}^{i} C_j^i * (1 - e^{-(\sum_{l=i-j+1}^{i} \lambda_l)t})}$$

As we may expect, although the ordered events are individually governed by a Poisson Stochastic Process, the sequence T is not. The occurrence rate of the sequence T is time dependent, Eq.(7).

4.2 Mean Occurrence Time

One may also determine the mean occurrence time τ_n corresponding to a given sequence (S_n) of n-events $\{e_1, e_2, e_3, ..., e_n\}$, such as:

$$(8) \quad \tau_n = \int_{t=0}^{\infty} t * dp_n(t)$$

The solution of Eq.(8) is elementary and described by:

$$(9) \quad \tau_n = \sum_{j=1}^{n} \frac{C_j^n}{(\sum_{l=n-j+1}^{n} \lambda_l)}$$

4.3 Asymptotic Behaviour

Having demonstrated that the occurrence probability $p_n(t)$ of a given sequence of n-well defined ordered events can be described by Equation (3), it is straightforward to demonstrate that the occurrence probability $p_n(t)$ has an asymptotic value equal to:

$$(10) \quad p_n(t \to \infty) \quad \to \quad \sum_{j=1}^{n} C_j^n$$

The occurrence probability density function Θ_n of the sequence S_n has an asymptotic value equal to:

$$(11) \quad \Theta_n(t \to \infty) \quad \to \quad 0.$$

Similarly, the equivalent occurrence rate Λ_n of the sequence S_n has an asymptotic value equal to:

(12) $\Lambda_n(t \to \infty) \quad \to \quad 0.$

5 NUMERICAL APPLICATION

Two illustrative numerical applications are given in the following in order to help in sizing the interest of having a generic analytical solution determining the occurrence probability of a given sequence (S_n) of n-events $\{e_1, e_2, e_3, ..., e_n\}$ in the given order.

5.1 Occurrence Order

In this application, we are focusing on the dependence of the occurrence probability on the occurrence order of the basic events.

A very simple example is illustrated in figure (1) where the time evolution of the occurrence probability of a sequence of four basic events $[e_1, e_2, e_3, e_4]$ whose occurrence rates are constant and having the following values: 10^{-4} /h, $5*10^{-3}$ /h, $2.5*10^{-2}$ /h, $1.25*10^{-1}$ /h. In Figure (2), we are comparing two configurations represented by a red curve and a blue one.

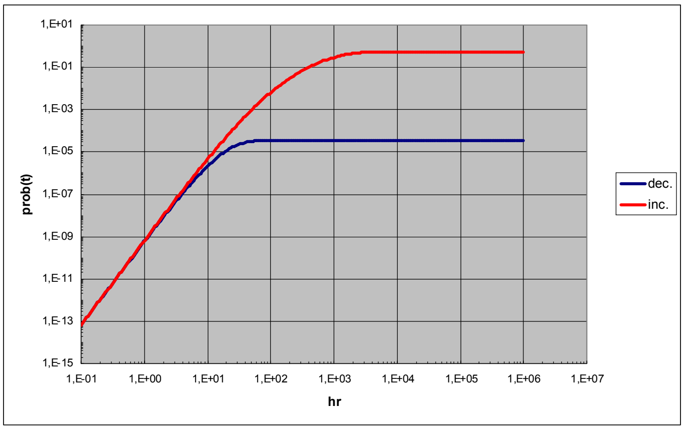

Figure (1) : the occurrence probability of the same set of events in two different orders (dec: decreasing order, inc.: increasing order)

The red curve represents the case where the sequence of events follows the increasing order of the occurrence rates (less frequent occurs first). While, the blue curve describes the case where the sequence of events following the decreasing order of the occurrence rates (more frequent occurs first).

It is obvious that the occurring order of events impacts on the time behaviour of the occurring probability of any sequence of events.

The asymptotic behaviour of the occurrence probability can also be underlined.

5.2 Treatment of a Markov Graph

In this example, a given system is described by Markov graph. The system has 8 possible states. The transitions between different states are fully determined by their transition rates.

In this illustrative example, Figure (2), a unique transitions rate value of 10^{-1} h^{-1} is considered for all transition rates as following:

$$\lambda_{12} = \lambda_{13} = \lambda_{14} == \lambda_{25} = \lambda_{52} = \lambda_{36} = \lambda_{47} = \lambda_{56} = \lambda_{68} = 10^{-1} \ h^{-1}$$

We are interested in the sequences S_n leading to the absorbing states e_7 or e_8, which are the following:

$$S_3 = e_1 \to e_4 \to e_7,$$
$$S_4 = e_1 \to e_3 \to e_6 \to e_8,$$
$$S_5 = e_1 \to e_2 \to e_5 \to e_6 \to e_8,$$
$$S_{5+2n} = e_1 \to (e_2 \leftrightarrow e_5)^n \to e_6 \to e_8, \qquad n = 0,1,2,...$$

Where;

$$(e_2 \leftrightarrow e_5)^0 = e_2 \to e_5$$
$$(e_2 \leftrightarrow e_5)^1 = e_2 \to (e_5 \to e_2) \to e_5$$
$$(e_2 \leftrightarrow e_5)^2 = e_2 \to (e_5 \to e_2) \to (e_5 \to e_2) \to e_5$$

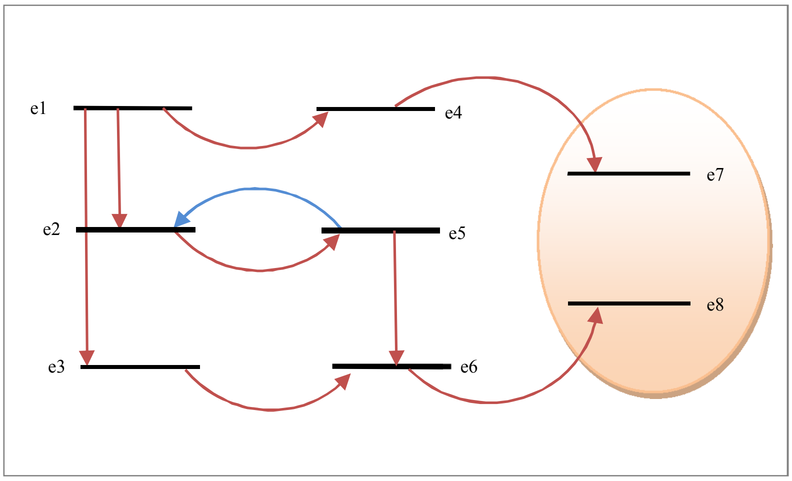

Figure (2) : Schematic presentation of a Markov Graph

One would, then, like to determine for each sequences (S_3, S_4, S_5, S_7) its occurrence probability time-profile, $p_n(t)$. The occurrences probabilities of the sequences S_3, S_4, S_5, S_7 are illustrated in Figure (3). Higher order sequences would have much lower contribution than that of S_7 as illustrated in Figure (3).

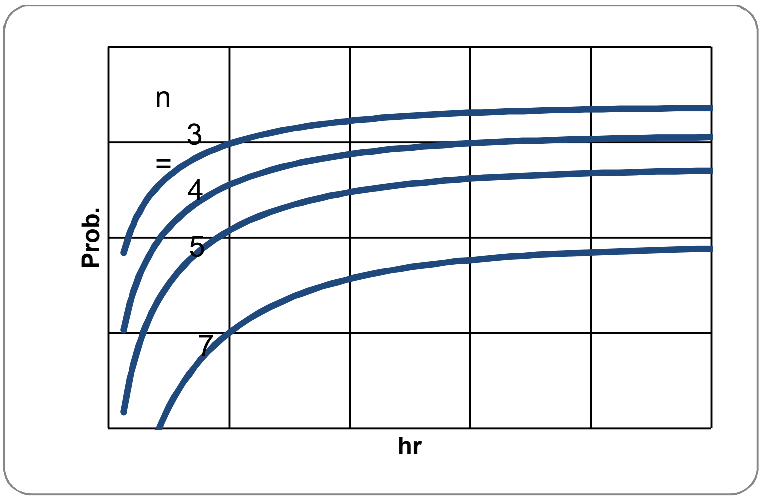

Figure (3) Occurrence probability as a time-function of each sequence rank (S_3, S_4, S_5, S_7)

The asymptotic values of the occurrence probabilities of the sequences ($S_3, S_4, S_5, S_7, ...$) are illustrated for the application in Figure (4).

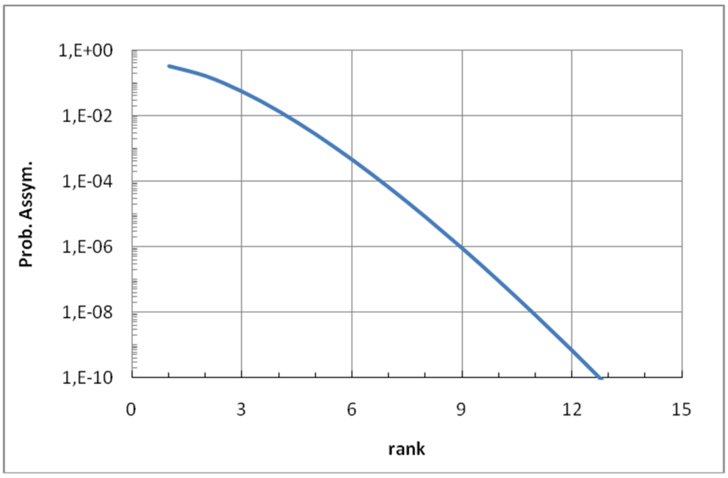

Figure (4) : the asymptotic probability as a function of the sequence rank

Finally, one would also determine the mean occurrence time of the sequences ($S_3, S_4, S_5, S_7, ...$) as a function of the sequence order.

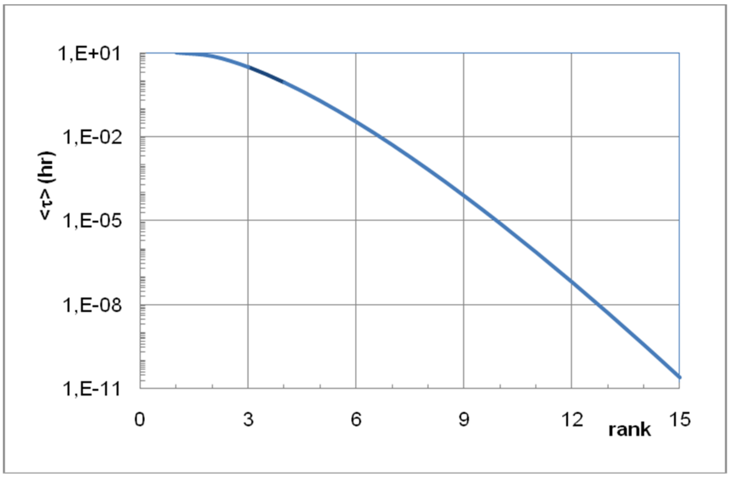

Figure (5) : Meantime of occurrence as a function of the sequence length

6 CONCLUSIONS

The paper proposes a general solution in order to determine the occurrence probability of a given sequence of n-events following Stochastic Poison's Processes. The solution is analytical and original.

Some interesting asymptotic characteristics of this analytical solution have been assessed.

Two simple numerical applications are illustrated in order to underline the interest of possessing an analytical generic solution to this problem.

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Appendix (1)

Let $p_n(t)$, the occurrence probability of the sequence T, be the solution of the Equation (1) and (2) and be expressed as:

$$p_i(t) = \sum_{j=1}^{i} C_j^i * (1 - e^{-(\sum_{l=i-j+1}^{i} \lambda_l)t}),\ C_1^1 = 1.0 \text{ and } i \in [1,2,...,n]$$

$$p_{i+1}(t) = \sum_{j=1}^{i+1} C_j^{i+1} * (1 - e^{-(\sum_{l=i-j+2}^{i+1} \lambda_l)t}),\ C_1^1 = 1.0 \text{ and } i \in [1,2,...,n]$$

And,

$$\frac{d}{dt} p_{n+1}(t) = \rho_{n+1}(t)\ p_n(t)$$

Where Sequence T contains i ordered events, each is defined by an occurrence rate λ_j.

$$\frac{d}{dt} p_{i+1}(t) = \frac{d}{dt} \sum_{j=1}^{i+1} C_j^{i+1} * (1 - e^{-(\sum_{l=i-j+2}^{i+1} \lambda_l)t}) = \sum_{j=1}^{i+1} C_j^{i+1} (\sum_{l=i-j+2}^{i+1} \lambda_l) * e^{-(\sum_{l=i-j+2}^{i+1} \lambda_l)t}$$

$$= C_1^{i+1} * \lambda_{i+1} e^{-\lambda_{i+1}t} + \sum_{j=2}^{i+1} C_j^{i+1} (\sum_{l=i-j+2}^{i+1} \lambda_l) * e^{-(\sum_{l=i-j+2}^{i+1} \lambda_l)t}$$

$$= C_1^{i+1} * \lambda_{i+1} e^{-\lambda_{i+1}t} + \sum_{k=1}^{i} C_{k+1}^{i+1} (\sum_{l=i-k+1}^{i+1} \lambda_l) * e^{-(\sum_{l=i-k+1}^{i+1} \lambda_l)t}$$

While;

$$\rho_{n+1}(t)\ p_n(t) = \lambda_{i+1} * e^{-\lambda_{i+1}t} \sum_{j=1}^{i} C_j^i * \left(1 - e^{-(\sum_{l=i-j+1}^{i} \lambda_l)t}\right)$$

$$= \sum_{j=1}^{i} C_j^i * \lambda_{i+1} * \left(e^{-\lambda_{i+1}t} - e^{-(\sum_{l=i-j+1}^{i+1} \lambda_l)t}\right)$$

$$= \left(e^{-\lambda_{i+1}t} * \sum_{j=1}^{i} C_j^i * \lambda_{i+1}\right) - \left(\sum_{j=1}^{i} C_j^i * \lambda_{i+1} * e^{-(\sum_{l=i-j+1}^{i+1} \lambda_l)t}\right)$$

So,

$$C_1^{i+1} * \lambda_{i+1} e^{-\lambda_{i+1}t} + \sum_{k=1}^{i} C_{k+1}^{i+1} (\sum_{l=i-k+1}^{i+1} \lambda_l) * e^{-(\sum_{l=i-k+1}^{i+1} \lambda_l)t} =$$

$$\left(e^{-\lambda_{i+1}t} * \sum_{j=1}^{i} C_j^i * \lambda_{i+1}\right) - \left(\sum_{j=1}^{i} C_j^i * \lambda_{i+1} * e^{-(\sum_{l=i-j+1}^{i+1} \lambda_l)t}\right)$$

1st Condition

$$\lambda_{i+1} * \left(C_1^{i+1} - \sum_{j=1}^{i} C_j^i \right) * e^{-\lambda_{i+1} t} \quad =0$$

Then;

$$C_1^{i+1} = \sum_{j=1}^{i} C_j^i \text{ , } i \in [1,2,...,n]$$

2nd Condition

$$\sum_{j=1}^{i} \left(C_{j+1}^{i+1} \left(\sum_{l=i-j+1}^{i+1} \lambda_l \right) + C_j^i * \lambda_{i+1} \right) * e^{-\left(\sum_{l=i-j+1}^{i+1} \lambda_l \right) t} \quad =0$$

Then,

$$C_{j+1}^{i+1} \quad = - \frac{\lambda_{i+1}}{\sum_{l=i-j+1}^{i+1} \lambda_l} * C_j^i \text{ , } \forall \; j \in [1,i] \text{ and } i \in [1,2,...,n]$$

Appendix (2):

Consider the case of a sequence containing 4 basic events in some well-determined order, so we have :

$$C_1^{i+1} \quad = \sum_{j=1}^{i} C_j^i \text{ ,} \qquad \text{and} \qquad C_1^1 = 1$$

$$C_{j+1}^{i+1} \quad = - \lambda_{i+1} \cdot \frac{C_j^i}{\left(\sum_{l=i-j+1}^{i+1} \lambda_l \right)} \qquad j = 1,2,...,i$$

We may, then, find out the coefficients C_j^i as following:

$N = 1$

$C_1^1 = 1$

$N = 2$

C_1^2 **= 1,**

$$C_2^2 = - \frac{\lambda_2}{\lambda_2 + \lambda_1}$$

$N = 3$

$C_1^3 = (1 - \frac{\lambda_2}{\lambda_2 + \lambda_1}),$

$C_2^3 = -\frac{\lambda_3}{\lambda_3 + \lambda_2},$

$C_3^3 = +\frac{\lambda_3}{\lambda_3 + \lambda_2 + \lambda_1} * \frac{\lambda_2}{\lambda_2 + \lambda_1}$

$N = 4$

$C_1^4 = (1 - \frac{\lambda_2}{\lambda_2 + \lambda_1}) - (\frac{\lambda_3}{\lambda_3 + \lambda_2}) + (\frac{\lambda_3}{\lambda_3 + \lambda_2 + \lambda_1} * \frac{\lambda_2}{\lambda_2 + \lambda_1})$

$C_2^4 = -\frac{\lambda_4}{\lambda_4 + \lambda_3} * (1 - \frac{\lambda_2}{\lambda_2 + \lambda_1})$

$C_3^4 = \frac{\lambda_4}{\lambda_4 + \lambda_3 + \lambda_2} * \frac{\lambda_3}{\lambda_3 + \lambda_2}$

$C_4^4 = -\frac{\lambda_4}{\lambda_4 + \lambda_3 + \lambda_2 + \lambda_1} * \frac{\lambda_3}{\lambda_3 + \lambda_2 + \lambda_1} * \frac{\lambda_2}{\lambda_2 + \lambda_1}$

So, the occurrence probability of this given sequence will, then, be determined by:

$$
\begin{aligned}
p_4(t) &= \sum_{j=1}^{4} C_j^4 * (1 - e^{-(\sum_{l=4-j+1}^{4} \lambda_l)t}) \\
&= C_1^4 * (1 - e^{-\lambda_4 t}) \\
&+ C_2^4 * (1 - e^{-(\lambda_4 + \lambda_3)t}) \\
&+ C_3^4 * (1 - e^{-(\lambda_4 + \lambda_3 + \lambda_2)t}) \\
&+ C_4^4 * (1 - e^{-(\lambda_4 + \lambda_3 + \lambda_2 + \lambda_1)t})
\end{aligned}
$$

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