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COMPLETE CALCULATION OF DISCONNECTION PROBABILITY IN PLANAR GRAPHS

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ABSTRACT

In this paper complete asymptotic formulas for an disconnection probability in random planar graphs with high reliable arcs are obtained. A definition of coefficients in these formulas have geometric complexity by a number of arcs. But a consideration of planar graphs and dual graphs allow to solve this problem with no more than cubic complexity by a number of graph faces.

1. INTRODUCTION

A problem of a calculation of a connectivity probability in random graphs with unreliable arcs is considered in manifold articles and monographs devoted to the reliability theory [1] - [4] etc. It occurs in an analysis of electro technical devices, computer networks and has manifold applications to a research of honeycomb structures [5], [6], and nanosystems [7] – [9].

In [10] – [12] upper and low estimates of the connectivity probability are constructed for general type networks on a base of maximal systems of disjoint frames. For small numbers of arcs in [13] accelerated algorithms of a calculation of reliability polynomial coefficients are constructed. These algorithms showed good results in a comparison with direct calculations. In [14] this problem is solved using the Monte-Carlo method with some combinatory formulas. To calculate the connectivity probability in rectangle lattices the transfer matrix method is used [15]. But an increasing of arcs number leads to large complexity and so it is worthy to develop asymptotic methods.

In this paper an analog of the Burtin-Pittel asymptotic formula [16] for disconnection probability of random graph with high reliable arcs is constructed. Its parameters are the minimal number D of arcs in cross sections and the number C of cross sections with volume D . A definition of D for a random port demands to find a maximal flow and has cubic complexity [17]. But a definition of C has geometric complexity.

So we consider widely used planar graphs for which we prove that a definition of coefficients D, C has no more than cubic complexity by a number of faces. And there is a lot of graphs [18, Ch. IV] for which this complexity is linear and smaller. These results are based on a consideration of dual graphs [19, [20], in which cross sections generate cycles [21], [22]. Numerical experiment confirms an accuracy and a performance of suggested method.

2. ASYMPTOTIC FORMULAS

Consider non oriented connected graph G with finite sets of nodes U and of arcs W . Suppose that each pair of nodes in G may be connected with no more than single arc and there are not loops. Denote $\mathcal{L}(u,v)$ the set of all cross sections in G which divide nodes $u,v\in U,\ u\neq v,$ and define the set $\mathcal{L}=\bigcup_{u\neq v}\mathcal{L}(u,v)$ of all cross sections in G . Graph cross section is such set of arcs which deletion makes the graph non connected. Put $d(L)$ a number of arcs in the cross section L and

$$D(u,v)=\min\left(d(L):L\in \mathcal{L}(u,v)\right),\ D=\min_{u\neq v} D(u,v),\ \mathcal{L}_*=\left\{L\in \mathcal{L}:d(L)=D\right\},$$

C - is a number of cross sections in the set $\mathcal{L}_*$. Suppose that graph arcs work independently with probabilities $p(w), w\in W$.

Theorem 1. If $\overline{p}(w)=1-p(w)=h,\ w\in W,$ then graph disconnection probability

$$\overline{P}\sim Ch^D,\ h\to 0. \tag{1}$$

Theorem 2. If $\overline{p}(w)\sim c_w h,\ h\to 0,\ w\in W,$ then

$$\overline{P}\sim \sum_{L\in \mathcal{L}_*} h^D \prod_{w\in L} c_w,\ w\in W,\ h\to 0.$$

Theorems 1, 2 are generalizations of the Burtin-Pittel asymptotic formula [16].

3. CALCULATION OF CONSTANTS *C, D*

Theorem 3. The set of arcs which do not belong to any cycle coincides with the set of cross sections $\mathcal{L}_*$ and $D=1$.

Assume that the graph G is planar and its each arc belongs to some cycle. Arcs of planar graph divide a plane into faces [19,Ch. 1]. }. Confront the graph G its dual graph G^*. Each face z in G accords the node z^* in G^*, each arc w in G belonging faces z_1, z_2 accords an arc w^* connecting nodes z_1^*, z_2^* in G.

A set of arcs $\{w_1,...,w_d\}$ in G accords some subgraph R^* in G^*. For its definition each arc w_i, $1\le i\le d$, accords a pair of faces which contain this arc. Then this pair of faces accords a pair of nodes in R^* connected by the arc w^*. Say that the graph R^* is generated by the set of arcs $\{w_1,...,w_d\}$.

Theorem 4. The set of cross sections $\mathcal{L}_*$ consists of all sets of arcs $\{w_1,...,w_d\}$ which generate cycles with minimal length D^* in the dual graph G^* and $D=D^*\le 5$.

This statement is a corollary of the Whithney theorem and the Euler formula [19, Theorem 1.5, Corollary of Theorem 1.6], [20]. In fig. 1 there are examples of planar graphs arranged on a sphere with $D=4,\ D=5$.

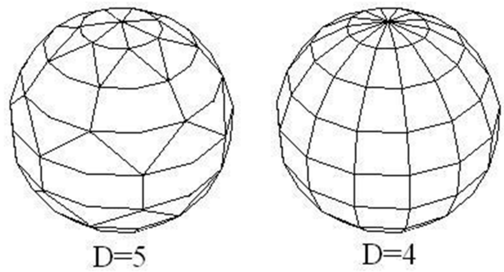

Fig. 1.

Suppose that elements a_{ij} of the matrix *A* define a number of arcs which belong to $z_i \cap z_j$, $i\neq j$, $a_{ii}=0$, in the planar graph G with *n* faces and *m* arcs and without loops and multiple arcs.

Corollary 1. If $\max_{1\le i,j\le n} a_{ij} > 1$, then

$$D = 2\,,\ C = \frac{1}{4}\sum_{1\le i,j\le n} a_{ij}\left(a_{ij}-1\right) \tag{2}$$

and a complexity of constants D, C calculation by the formula (2) is squared by *n*. If for $i < j$ $a_{ij} > 1$ only for $j = n$ then this complexity is linear.

Define c_i the number of cycles with length i, $i = 3,4,5,$ in G^*. Assume that all cycles $u_1 \to u_2 \to \ldots \to u_k \to u_1$, consist of same set of nodes $\{u_1,\ldots,u_k\}$ and differ by an initial node u_1 and by a direction of a bypass coincide. Elements of a power A^l, $l > 1$, of a matrix *A* denote by $a_{ij}^{(l)}$.

Corollary 2. If $\max_{1\le i,j\le n} a_{ij} = 1$, then

$$D = \min\left(i : c_i > 0, i = 3,4,5\right),\ C = c_D\ , \tag{3}$$

$$c_3 = \frac{1}{6} trA^3\ ,\ c_4 = \frac{1}{8}\left(trA^4 - 2m - 2\sum_{1\le i\ne j\le n} a_{ij}^{(2)} \right),$$

$$c_5 = \frac{1}{8}\left(trA^5 - 5trA^3 - 5\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} - 2\right) a_{ij}^{(3)} \right).$$

Complexity of the constants D, C calculation using the formula (3) is cubic by n. The formulas of c_3, c_4, c_5 calculation are obtained in [21], see also [22, Formulas (16), (17)].

Consider a connected graph G' which consists of plane faces in three dimensional space. Suppose that each pair of faces has not joint points or has joint node or has joint arc and each arc belongs at least to two faces. Take a set of arcs $\{w_i,\ 1\le i\le d\}$ from G' and confront each arc w_i a pair of faces z_i, z^i, which contain this arc. Then the set of arcs $\{w_1,\ldots,w_d\}$ accords some (non unique) graph Γ_d with the nodes $z_i, z^i, 1\le i\le d,$ and arcs $\{w_1,\ldots,w_d\}$ which connect these nodes.

Theorem 5. If the graph Γ_d is acyclic then the set of arcs $\{w_1,\ldots,w_d\}$ which generates it is not cross section in G'.

Corollary 3. Suppose that the set $\mathcal{L}'$ of arcs sets which generate cycles with minimal length D^* and which are cross sections in G' is not empty. Then $D = D^*$, $\mathcal{L}_* = \mathcal{L}'$.

4. EXAMPLES

Results of the number D definition and an enumeration of cross sections with minimal volume are based on listed theorems and simple geometric constructions.

Example 1. On fig. 2 there are examples of planar graphs with representatives of cross sections from the set $\mathcal{L}_*$:

1) an integer rectangle with the length *M* an integer rectangle with the length *N* ($\mathcal{L}_*$ consists of arcs pairs connected with angle nodes),

2 a honeycomb structure ($\mathcal{L}_*$ consists of all possible pairs of arcs which belong to internal and external faces simultaneously),

3) a tube which is constructed by a gluing of opposite sides (with a length M) of integer rectangle ($\mathcal{L}_*$ consists of arcs triplets which have common butt node), if N>3.

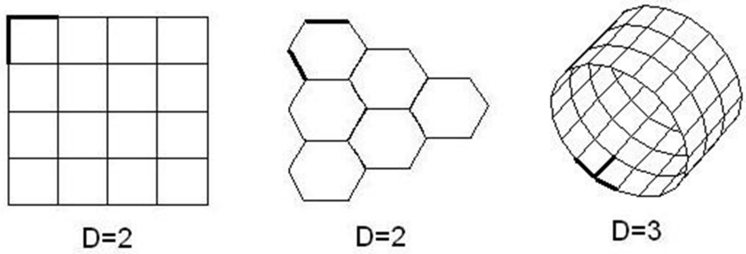

Fig. 2. Planar graphs with cross sections dedicated by bold type.

Example 2. On fig. 3 there are graphs with examples of their cross sections from the set $\mathcal{L}_*$:

1) a graph constructed from integer rectangle by a gluing of pairs of its opposite sides ($\mathcal{L}_*$ consists of arcs quads which have common node),
2) a graph constructed from unit cubes with integer coordinates of their nodes ($\mathcal{L}_*$ consists of arcs triplets which contain a cube node, in this node the cube does not intersect or has only common node with another cube).

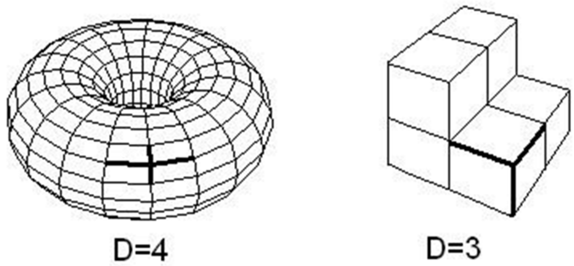

Fig. 3. Graph G' with dedicated cross sections.

5. NUMERICAL EXPERIMENT

Calculate the disconnection probability of honeycomb structure (fig. 1, in center) using Theorem 1 and Corollary 1 and by the Monte-Carlo method with 10^6 realizations. Failure probability of each arc is 0.005. Results of calculations are represented in the table. Time of calculations by asymptotic method is few seconds and by the Monte-Carlo method is some hours.

Size structure	Asymptotic method	Monte-Carlo method	Relative error
2×2	0.00045	0.000439	2.4 %
3×3	0.00055	0.000526	4.3 %
3×4	0.00060	0.000579	3.5 %
3×5	0.00065	0.000621	4.5 %
4×4	0.00065	0.000619	4.8 %
5×5	0.00075	0.000732	2.4 %

The author thanks A.S. Losev for numerical experiment realization.

6. PROOFS OF MAIN STSTEMENTS

Proof of Theorem 1. Suppose that V_L is a random event that all arcs in cross section L fail. Then

$$\overline{P} = P\left(\left(\bigcup_{L\in\mathcal{L}_*} V_L\right)\bigcup\left(\bigcup_{L\in L\setminus\mathcal{L}_*} V_L\right)\right) \sim P\left(\bigcup_{L\in\mathcal{L}_*} V_L\right), h\to 0.$$

As $P(V_L)=o(h^D)$, $L\in\mathcal{L}\setminus\mathcal{L}_*, h\to 0$, so

$$P\left(\bigcup_{L\in L_*} V_L\right)\sim Ch^D, h\to 0\,.$$

Proof of Theorem 5. Suppose that arcs set $\{w_1,..., w_d\}$ from the graph G generates acyclic graph R^*. Prove that each arc $w_i, 1\le i\le d,$ may by bypassed in G by a way which does not contain arcs of this set.

The subgraph R^* consists of trees $S_i^*,...,S_m^*$ which do not connect with each other. Arrange each tree S_i^*, $1\le i\le m,$ on a plane so that in each node z^* arcs connected with this node follow each other as their pre images on the face z if we bypass this face in some direction. Confront each tree S_i^* closed way which bypasses once all its arcs from both sides, $1\le i\le m$ (fig. 4).

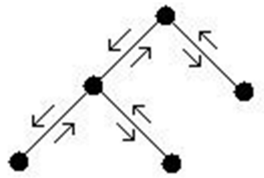

Fig. 4. Bypass of tree arcs.

Accord the way Γ_i^* bypassing tree S_i^* arcs a closed way Γ_i which passes in the graph G through all nodes of arcs $\{w_1,..., w_d\}$, which generate the tree S_i^* (fig. 5). The way Γ_i has not arcs from the set $\{w_1,..., w_d\}$. Consequently each arc from $\{w_1,..., w_d\}$ may be bypassed in G by a way which does not contain arcs from this set. So the set $\{w_1,..., w_d\}$ from the graph G, $d\le D^*$, which does not generate a cycle in G^*, does not belong to the set of cross sections $\mathcal{L}_*$.

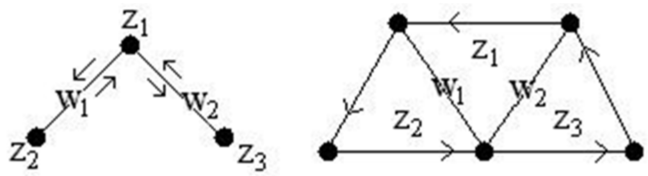

Fig. 5. Bypassing of arcs in a tree and in G .

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