# RECURRENT SEQUENCE OF PARALLEL-SERIAL CONNECTIONS

**G. Tsitsiashvili*, M. Osipova****

•

* IAM FEB RAS, Vladivostok, Russia, ** FEFU, Vladivostok, Russia
* guram@iam.dvo.ru, ** mao1975@list.ru

## ABSTRACT

In this paper a sequence of parallel-serial connections is considered. In this sequence next connection is obtained by parallel or serial linking of new arc to obtained connection. Distributions of random numbers of connectivity components are analyzed. These distributions are considered intensively now. Central limit theorem is proved for these distributions and parameters (mean and variance) of normal limit distribution are calculated.

## 1. INTRODUCTION

In the reliability theory parallel-serial connections play important role [1] – [6] etc. These connections are widely used in electrotechnics, in computer networks etc. A specific of these connections is a possibility to calculate their reliability by algorithms with linear complexity by a number of arcs.

Last years large interest is called to characteristics of networks sparseness. It means that powers of nodes (a number of incident arcs) is bounded by some positive number (see [7] and large bibliography in this article). Stochastic modeling and statistical processing of internet type networks data showed that nodes powers have distribution with heavy tails [8]. Last circumstance makes actual to consider parallel-serial connections which are free of this lack.

Last time a distribution of numbers of connectivity components in different random networks are analyzed intensively now [9] – [11]. In this paper numbers of connectivity components in recurrent sequence of connections obtained by parallel or serial linking of new arc is considered. For this sequence central limit theorem is proved and parameters of limit normal distribution are calculated.

A problem to calculate a mean and mainly a variance of limit normal distribution in this model is technically sufficiently complicated. In this paper it is based on central limit theorem for discrete Markov chains [12] and on a construction of special and sufficiently fast algorithm of such calculations.

## 2. MODEL DESCRIPTION

Consider the sequence $\mathcal{A}_n,\ n \geq 1$, of ports defined recursively by a sequential or parallel connection of new arc $b_n$ to the port $\mathcal{A}_n$. Denote a type of connection by $||$ or $\rightarrow$ accordingly. Suppose that random variable $\omega_n$ characterizes a type of the arc $b_n$ connection to the port $\mathcal{A}_n$ and put

$$\pi_{\rightarrow} = P(\omega_n = \rightarrow), \qquad \pi_{||} = P(\omega_n = ||) = 1 - \pi_{\rightarrow}, \qquad 0 < \pi_{\rightarrow} < 1.$$

Here random variable $\beta_n$ characterizes a state of the arc $b_n$:

$$P(\beta_n = 1) = P(b_n \text{ in working state}) = p,\ \ P(\beta_n = 0) = 1 - p = q,\ \ 0 < p < 1.$$

The sequences of random variables $\{\omega_n,\ n \geq 1\}$, $\{\beta_n,\ n \geq 1\}$ are independent and each of them consists of independent and identically distributed random variables.

The port $\mathcal{A}_n$ with randomly working arcs is characterized by random vector $(\alpha_n, \eta_n)$ there $\alpha_n$ is an indicator of a connectivity between initial and final nodes of parallel-sequential connection $\mathcal{A}_n$ and $\eta_n$ is a number of connectivity components in $\mathcal{A}_n$. Introduce auxiliary random variables

$$\vec{\alpha}_{n+1} = \alpha_n \wedge \beta_n,\ \vec{\eta}_{n+1} = \eta_n + 1 - \beta_n, \tag{1}$$

$$\overline{\overline{\alpha}}_{n+1} = \alpha_n \vee \beta_n,\ \overline{\overline{\eta}}_{n+1} = \eta_n - \beta_n + \alpha_n\beta_n, \tag{2}$$

then

$$(\alpha_{n+1}, \eta_{n+1}) = I(\omega_n = \rightarrow)(\overleftarrow{\alpha}_{n+1}, \overleftarrow{\eta}_{n+1}) + I(\omega_n = ||)(\overline{\overline{\alpha}}_{n+1}, \overline{\overline{\eta}}_{n+1}), \tag{3}$$

where $I(C)$ is an indicator of an event $C$.

## 3. LIMIT THEOREM FOR MARKOV CHAIN CHARACTERIZING CONNECTIVITY OF PARALLEL-SERIAL CONNECTIONS

Denote $\Delta_{n+1} = \eta_{n+1} - \eta_n$, then the sequence $X_k = (\alpha_k, \Delta_k),\ k \geq 1$, is Markov chain with the states set $\mathcal{X} = \{(i,j),\ i = 0, 1,\ j = -1, 0, 1\}$ as follows

$$(\alpha_{n+1}, \Delta_{n+1}) = I(\omega_n = \rightarrow)(\alpha_n\beta_n, 1 - \beta_n) + I(\omega_n = ||)(\alpha_n \vee \beta_n, -\beta_n + \alpha_n\beta_n).$$

From the equalities (1) - (3) and the conditions $0 < p < 1, 0 < \pi_{\rightarrow} < 1$ we see that Markov chain $X_k, k \geq 1$, states are interconnected. Consequently from the central limit theorem for discrete Markov chains with finite states set [12, chapters V,VI} there are normally distributed random vector $N(0, \mathcal{B})$ with the dimension six and with zero mean and with covariance matrix $\mathcal{B}$ and real numbers $A(x),\ x \in \mathcal{X}$, which do not depend on initial state $X_1$ so that for any real $t(x),\ x \in \mathcal{X}$,

$$P\left(\left(\frac{N_n(x) - nA(x)}{\sqrt{n}}, x \in \mathcal{X}\right) > (t(x),\ x \in \mathcal{X})\right) \rightarrow P(N(0, \mathcal{B}) > (t(x),\ x \in \mathcal{X})),\ n \rightarrow \infty. \tag{4}$$

Here $N_n(x) = \sum_{k=1}^{n} I(X_k = x)$ and the inequalities are defined componentwise.

Introduce auxiliary numbers $a(x),\ x \in \mathcal{X}$:

$$a(i, 0) = 0, \qquad a(i, 1) = 1, \qquad a(i, -1) = -1, \qquad i = 0,1.$$

From the formula (4) it is simple to obtain that there is normally distributed random variable $N(0, B)$ with zero mean and with the covariance $B > 0$ so that for any real $t$

$$P\left(\frac{1}{\sqrt{n}}\sum_{x \in \mathcal{X}} a(x)(N_n(x) - nA(x)) > t\right) \rightarrow P(N(0, B) > t),\ n \rightarrow \infty. \tag{5}$$

Using obvious equality $\sum_{x \in \mathcal{X}} a(x)N_n(x) = \sum_{k=1}^{n} \Delta_k = \eta_n,\ n \geq 1$, rewrite the formula (5) as follows

$$P\left(\frac{\eta_n - nA}{\sqrt{n}} > t\right) \to P(N(0,B) > t),\ n \to \infty,\ A = \sum_{x\in\mathcal{X}} a(x)A(x). \tag{6}$$

**Remark 1.** A calculation of the vector $(A(x),\ x \in \mathcal{X}))$ and especially of covariance matrix $\mathcal{B}$ in the formula (4) is sufficiently complicated procedure [12, chapters V, VI}. So to define the mean $A$ and the covariance $B$ we use following limit formulas

$$A = \lim_{n\to\infty} \frac{M\eta_n}{n},\ B = \lim_{n\to\infty} \frac{D\eta_n}{n} \tag{7}$$

which are corollaries of the formula (6) with special initial distribution of $X_1$.

## 4. CALCULATION OF LIMIT NORMAL DISTRIBUTION PARAMETERS

Choose random vector $(\alpha_1, \Delta_1) = (\alpha_1, \eta_1)$ which does not depend on random sequences $\{\omega_n,\ n \geq 1\}$, $\{\beta_n,\ n \geq 1\}$ and satisfies the equalities

$$P((\alpha_1, \eta_1) = (1,1)) = P = \frac{\pi_{||}p}{\pi_{||}p + \pi_{\to}q},\ P((\alpha_1, \eta_1) = (0,2)) = Q = 1 - P \tag{8}$$

with $P(\alpha_n = 1) \equiv P,\ P(\alpha_n = 0) \equiv Q$. Random sequence $\alpha_n,\ \ n \geq 1$, is stationary Markov chain.
**Theorem 1.** The equalities

$$A = Q\pi_{\to}q, \tag{9}$$

$$B = \pi_{\to}qQ(1 - \pi_{\to}qQ + 2PQ) > 0 \tag{10}$$

are true.
**Proof.** To define the constants $A,\ B$ from (7) we construct recurrent algorithm. Denote

$$M_n = M\eta_n,\ A_n = M(\eta_n\,|\alpha_n = 1),\ B_n = M(\eta_n\,|\alpha_n = 0),\ M_n = A_nP + B_nQ, \tag{11}$$

$$M'_n = M\eta_n^2,\ A'_n = M(\eta_n^2|\alpha_n = 1),\ B'_n = M(\eta_n^2|\alpha_n = 0),\ M'_n = A'_nP + B'_nQ \tag{12}$$

where

$$A_1 = 1,\ B_1 = 2,\ A'_1 = 1,\ B'_1 = 4.$$

Using the formulas (1) - (3), (11) obtain for $n \geq 1$:

$$A_{n+1} = \frac{A_nP\pi_{\to}p + A_nP\pi_{||}p + (B_n - 1)Q\pi_{||}p + A_nP\pi_{||}q}{P},$$

$$B_{n+1} = \frac{B_nQ\pi_{\to}p + (A_n + 1)P\pi_{\to}q + (B_n + 1)Q\pi_{\to}q + B_nQ\pi_{||}q}{Q},$$

$$M_{n+1} = A_nP + B_nQ - Q\pi_{||}p + P\pi_{\to}q + Q\pi_{\to}q = M_n + Q\pi_{\to}q = M_1 + nQ\pi_{\to}q, \qquad M_1 = 1 + Q.$$

Then from (7) we obtain the equality (9).

And

$$A_{n+1} - B_{n+1} = (A_n - B_n)\lambda - \left(2\pi_{\to}q + \pi_{||}p\right),\ n \geq 1,\ \lambda = \pi_{||}q + \pi_{\to}p < 1. \tag{13}$$

so

$$A_{n+1} - B_{n+1} = -\left[\lambda^n + \left(2\pi_{\to}q + \pi_{||}p\right)\frac{1 - \lambda^n}{1 - \lambda}\right] = \lambda^nQ - 1 - Q, \qquad A_{n+1}P + B_{n+1}Q = M_{n+1}$$

consequently

$$A_{n+1}=M_{n+1}+Q[\lambda^n Q-1-Q],\ B_{n+1}=M_{n+1}-P[\lambda^n Q-1-Q],\ n\geq 1. \qquad (14)$$

Begin now a calculation of $M'_{n+1}$. Using the formulas (1) - (3), (12) obtain for $n\geq 1$:

$$A'_{n+1}=\frac{A'_nP\pi_{\to}p+A'_nP\pi_{||}p+(B'_n-2B_n+1)Q\pi_{||}p+A'_nP\pi_{||}q}{P},$$

,

$$B'_{n+1}=\frac{B'_nQ\pi_{\to}p+(A'_n+2A_n+1)P\pi_{\to}q+(B'_n+2B_n+1)Q\pi_{\to}q+B'_nQ\pi_{||}q}{Q},$$

$$M'_{n+1}=M'_n+2A_nQ\pi_{||}p+2B_nQ(\pi_{\to}q-\pi_{||}p)+\pi_{\to}q(1+P).$$

So from (14) we obtain

$$M'_{n+1}=M'_1+2Q\pi_{||}p\sum_{k=0}^{n-1}A_{k+1}+2Q(\pi_{\to}q-\pi_{||}p)\sum_{k=0}^{n-1}B_{k+1}+n\pi_{\to}q(1+P)=$$

$$=M'_1+2Q\pi_{\to}q\sum_{k=0}^{n-1}M_{k+1}-2nQP(1+Q)\pi_{||}p+n\pi_{\to}q(1+P)+2\pi_{\to}qP^2Q\frac{1-\lambda^n}{1-\lambda}=$$

$$=M'_1+2Q\pi_{\to}q(n(1+Q)+\pi_{\to}qQn(n-1)/2)-2nQP(1+Q)\pi_{||}p+$$

$$+n\pi_{\to}q(1+P)+2P^2Q^2(1-\lambda^n),\qquad M'_1=1+3Q.$$

Consequently

$$D\eta_{n+1}=M'_{n+1}-M^2_{n+1}=n\pi_{\to}q[1+P-Q^2\pi_{\to}q-2P^2(1+Q)]+2P^2Q^2(1-\lambda^n)+QP.$$

Then from (7), (13) we have

$$B=\pi_{\to}q\big(1+P-Q^2\pi_{\to}q-2P^2(1+Q)\big)=\pi_{\to}qQ(1-\pi_{\to}qQ+2PQ)>0.$$

Theorem is proved.

**Remark 2.** From Remark 1 is possible to replace the condition (8) by more natural suggestion

$$P\big((\alpha_1,\eta_1)=(1,1)\big)=p,\qquad P\big((\alpha_1,\eta_1)=(0,2)\big)=q$$

so that the equalities (6), (9), (10) are true also.

## REFERENCES

1. Barlow R.E., Proschan F. 1965. *Mathematical Theory of Reliability*. London and New York: Wiley.
2. Ushakov I.A. et al. 1985. *Reliability of technical systems*. Handbook. Moscow: Radio and Communication (in Russian).
3. Riabinin I.A. 2007. *Reliability and safety of structural complicated systems.* Sankt-Petersberg: Edition of Sankt-Petersberg university (in Russian).

4. Solojentsev E.D. 2006. Specific of logic-probability risk theory with groups of incompatible events. *Automatics and remote control,* no. 7: 187-203 (in Russian).
5. Satyanarayana A., Wood R.K. 1985. A linear time algorithm for computing k-terminal reliability in series-parallel networks. *SIAM, J. Computing,* v. 14: 818-832.
6. Ball M.O., Colbourn C.J., Provan J.S. 1995. Network Reliability. Network Models. *Handbook of Operations Research and Management Science,* v. 7: 673-762.
7. Raigorodskiy A. 2010. Models of random graphs and their application. *Proceedings of MPhTI*, v. 2, no. 4: 130-140 (in Russian).
8. Kumar R., Raghavan P., Rajagopalan S., Sivakumar D., Tomkins A., Upfal E. 2000. Stochastic model for the web graph. *Proc. 41-st Symposium on Foundation of Computer Science.*
9. Sukach E., Ratobilskaia D., Demuskov A. 2011. Computer system of probability-algebraic modeling of systems with many states. *Mathematical mashines and systems,* no. 3: 32-39 (in Russian).
10. Sukach E., 2011. Approaches to decrease dimension of multi component and structurally complicated systems with many states in estimate of their reliability. *Problems of physics, mathematics and technique.* no. 3: 99-103 (in Russian).
11. Timashev A. 2011. Asymptotic expansions for distributions of numbers of connectivity components in random images and decompositions. *Discrete mathematics,* v. 23, no. 2: 66-75 (in Russian).
12. Romanovsky V. 1949. *Discrete Markov chains.* Moscow-Leningrad: State Publishing House of Technical-Theoretical Literature (in Russian).