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UNIVERSAL GENERATING FUNCTION & OPTIMAL REDUNDANCY PROBLEMS

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1. Introduction

The Method of U-functions, or the Method of the Universal Generating Function (UGF), was introduced in [(1986) Ushakov, (1987) Ushakov] and later developed in [(1988) Ushakov; (1995) Gnedenkon& Ushakov]. Actually this is a generalization and "algebraic" formalization of the well-known Kettelle's Algorithm [(1962) Kettelle]. In turn, Kettelle's Algorithm, is a form of presentation of convolution of discrete random variables. The method of U-functions is very convenient for computerized calculations.

Last years, this method was significantly developed by G. Levitin and A. Lisnianski

2. Briefly about Generating Function

Everybody knows that Generating Function (GF) is very convenient mathematical tool for finding a convolution of discrete random variables.

Consider two non-negative discrete random variables X_1 and X_2 that are characterized by discrete distributions

$$P^{(1)}(x)=\begin{cases} P\{X^{(1)}=x_1^{(1)}\}=p_1^{(1)}, \\ P\{X^{(1)}=x_2^{(1)}\}=p_2^{(1)} \\ \ldots \\ P\{X^{(1)}=x_{n(1)}^{(1)}\}=p_{n(1)}^{(1)}, \end{cases} \quad \text{and} \quad P^{(2)}=\begin{cases} P\{X^{(2)}=x_1^{(2)}\}=p_1^{(2)}, \\ P\{X^{(2)}=x_2^{(2)}\}=p_2^{(2)}, \\ \ldots \\ P\{X^{(2)}=x_{n(2)}^{(2)}\}=p_{n(2)}^{(2)}, \end{cases} \tag{1}$$

correspondingly where $n(1)$ and $n(2)$ are numbers of discrete realizations of values of each type.

If we are interested in the distribution of r.v. $X = X_1 + X_2$, we perform product of generating functions, perform collecting terms, and get:

$$\varphi(z)=\varphi^{(1)}(z)\cdot\varphi^{(2)}(z)=\left(\left(\sum_{1\le i\le n(1)} p_i^{(1)} z^{a_i^{(1)}}\right)\times\left(\sum_{1\le j\le n(2)} p_j^{(2)} z^{a_j^{(2)}}\right)\right)=\sum_{1\le k\le n} p^k z^{A_k} \tag{2}$$

where A_k is a convolution of two r.v.'s. – $a_i^{(1)}$ and $a_i^{(1)}$.

Thus, this transform suggests multiplication of polynomial coefficients and summation of polynomial powers. The method of U-functions suggests a transparent and convenient method of computerized solutions of various enumeration problems where variables are subjects to operations beyond multiplication and summation, for instance, finding distribution of minimum, maximum, geometrical summation, etc., depending on physical nature of the analyzed problem.

Table 1. Examples of object interaction depending on physical nature of unit parameters

Name	$\bigotimes_{SERIES} (\mu_1, \mu_2)$	$\bigotimes_{PARALLEL} (\mu_1, \mu_2)$
PFFO (for “hot” redundancy)	$\mu_1 \times \mu_2$	$(1-\mu_1)\times(1-\mu_2)$
Cost	$\mu_1 + \mu_2$	$\mu_1 + \mu_2$
Weight	$\mu_1 + \mu_2$	$\mu_1 + \mu_2$
el. Resistance	$\mu_1 + \mu_2$	$((\mu_1)^{-1} + (\mu_2)^{-1})^{-1}$
el. Capacity	$((\mu_1)^{-1} + (\mu_2)^{-1})^{-1}$	$\mu_1 + \mu_2$
el. Conductivity	$((\mu_1)^{-1} + (\mu_2)^{-1})^{-1}$	$\mu_1 + \mu_2$
pipeline capacity	$\min(\mu_1, \mu_2)$	$\mu_1 + \mu_2$
random time to failure	$\min(\mu_1, \mu_2)$	$\max(\mu_1, \mu_2)$
…	…	…
number of different redundant units	(μ_1, μ_2)	(μ_1, μ_2)

Here by symbol "$\bigotimes$" we denote interaction of parameters of various physical nature. In particular, the method of U-functions can be effectively applied to solving the optimal redundancy problem.

3. Method of U-functions

Let us consider GF from another viewpoint. Each *k*-th discrete distribution one can represented as a set of triplets:

$$S^{(k)} = \left\{(p_1^{(k)}, c_1^{(k)}, 1), (p_2^{(k)}, c_1^{(k)}, 2), ..., (p_1^{(k)}, c_{n(k)}^{(k)}, n(k))\right\} \tag{3}$$

where $p_j^{(k)}$ and $c_j^{(k)}$ are the probability of failure-free operation (PFFO) of unit *k* with *j*-th variant of redundancy and the cost of this variant, correspondingly. The third component is the number of redundant units of type *k* (or, in more general case, the ordering number of variant of unit *k*).

Indeed, product of two GF's is equivalent to “Descartes interaction” of two sets $S^{(1)}$ and $S^{(2)}$, i.e. each triplet of set $S^{(1)}$ interacts with all triplets of set sets of $S^{(2)}$. Interaction of two triplets can be conditionally written as follows:

$$(p_i^{(1)}, c_i^{(1)}, i^{(1)}) \bigotimes (p_j^{(2)}, c_j^{(2)} j^{(2)}) \tag{4}$$

In turn, interaction of triplets consists of interactions of its components that produce a new triplet

$$\left(p_i^{(1)} \otimes^{\Pi} p_j^{(2)}; c_i^{(1)} \otimes^{\Sigma} c_j^{(2)}; i^{(1)} \otimes^{\cup} j^{(2)}\right) = \left(p^{(2*)}, c^{(2*)}, j^{(2)})\right) \tag{5}$$

Here interaction $\otimes^{\Pi}$ means product, operator $\otimes^{\Pi}$ does summation, and operator $\otimes^{\Pi}$ does union (a vector with corresponding components), i.e.

$$p^{(2*)} = p_i^{(1)} \otimes^{\Pi} p_j^{(2)} = p_i^{(1)} \times p_j^{(2)};$$
$$c^{(2*)} = c_i^{(1)} \otimes^{\Sigma} c_j^{(2)} = c_i^{(1)} + c_j^{(2)};$$
$$j^{(2*)} = i^{(1)} \otimes^{\cup} j^{(2)} = (i^{(1)},\ j^{(2)}) \tag{6}$$

One can easily see that Descartes interaction of duplets that belongs sets $S^{(1)}$ and $S^{(2)}$ is completely equivalent to product of two generating functions $\varphi^{(1)}(z)$ and $\varphi^{(2)}(z)$.

Analogously with the product of GF's one has to collect terms for getting the final set

$$S = S_1 \otimes S_2.$$

Naturally, operator $\otimes$ possesses commutativity property, i.e.

$$\otimes\ (a, b) = \otimes\ (b, a) \tag{7}$$

and associativity property, i.e.

$$\otimes\ (a, b, c) = \otimes\ (a \otimes (b, c)) = \otimes\ ((a \otimes b), c). \tag{8}$$

4. Using U-function for solving of optimal redundancy problems

Let us consider a series system consisting of n units, each of which has PFFO equals p_k and costs c_k units. For increasing reliability of each unit, one can use redundancy of individual units. Each unit k is represented by set of triplets

$$S_k = \left[\{R_0^{(k)}; C_0^{(k)}; 0\}, \{R_1^{(k)}; C_1^{(k)}; 1\}, \{R_2^{(k)}; C_2^{(k)}; 2\}, ..., \{R_s^{(k)}; C_s^{(k)}; s\}...\right] \tag{9}$$

where s is the number of redundant units (any natural number); $C_s^{(k)}$ is the total cost of s redundant units (usually, a linear function of the number s); and $R_s^{(k)}$ is the PFFO of unit k with s redundant units. It is well-known that for loaded redundancy of group including one main and s identical redundant units:

$$R_s^{(k)} = 1 - (1 - p_k)^{s+1};$$

and for an unloaded redundant (spare) units:

$$R_s^{(k)} = \sum_{0 \le j \le s} \frac{(\lambda_k t)^j}{j!} \exp(-\lambda_k t);$$

Now consider a general procedure of optimal redundancy with the use of U-functions. First of all, take units 1 and 2 and arrange the Descartes interaction procedure between sets S_1 and S_2. In our case

$$R_i^{(1)} \otimes R_j^{(2)} = R_i^{(1)} \times R_j^{(2)} = R_K^{(2*)};$$
$$C_i^{(1)} \otimes C_j^{(2)} = C_i^{(1)} + C_j^{(2)} = C_K^{(2*)};$$
$$i \otimes j = (i,\ j) = \vec{K},$$

i.e. interaction between two numbers produces vector, containing numbers of redundant units of the 1st and 2nd types..

Here symbol "*" relating to the number means that this "aggregated" unit includes all previous units.

At the next step of sets S_1 and S_2 interaction one takes "aggregated unit 2* and unit 3:

$$R_K^{(2*)} \otimes R_k^{(3)} = R_K^{(2*)} \times R_k^{(3)} = (R_i^{(1)} \times R_j^{(2)}) \times R_k^{(3)} = R_L^{(3*)};$$
$$C_K^{(2*)} \otimes C_k^{(3)} = C_K^{(2*)} + C_k^{(3)} = C_i^{(1)} + C_j^{(2)} + C_k^{(3)} = C_L^{(3*)};$$
$$\vec{K} \otimes k = (\vec{K},\ k) = (i,\ j,\ k) = \vec{L}.$$

Vector $\vec{L}$ shows that in a series system of 3 units the 1st unit has i redundant ones, the 2nd unit has j redundant ones and the 3rd units has k redundant ones.

This procedure continues until necessary final triplets will have been generated. Instead of further abstract presentation of the procedure, let us turn to a simple illustrative numerical example.

The result of interaction is presented in the table below.

Example 1. Consider a series system of four units with parameters given in the table below. Assume that "hot" redundancy is used for the system reliability improvement.

Table 2. System unit parameters

	Unit-1	Unit-2	Unit-3	Unit-4
PFFO	0.6	0.6	0.7	0.7
Cost	3	5	3	5

In accordance with the description given above, the block diagram of the using U-functions in this particular case can be presented as follows (see Figure 1).

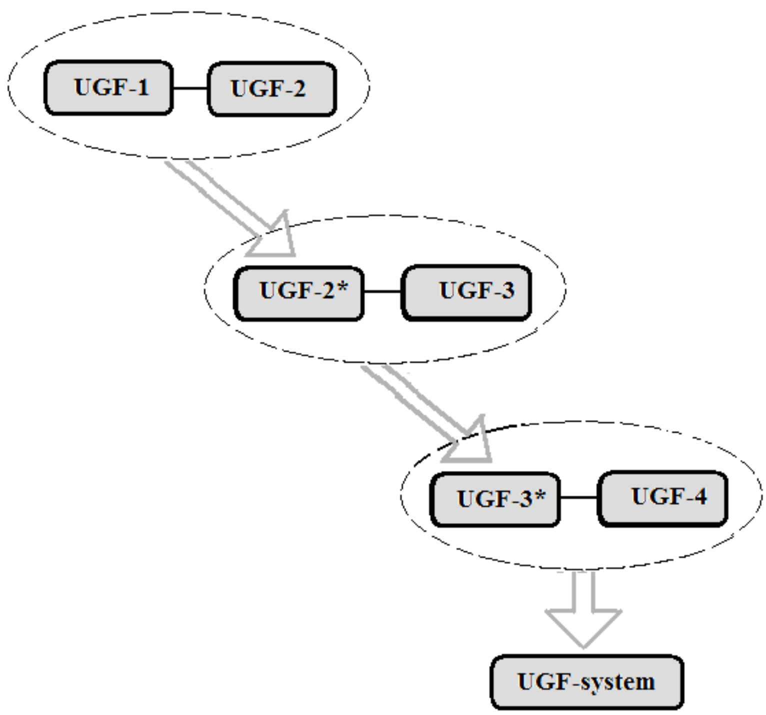

Figure 1. Block-diagram of the solution procedure for Example 1

Let us solve two problems of optimal redundancy:

(a) *Direct problem*: Find the optimal allocation of redundant units to reach required PFFO level of the system equals to 0.97;

(b) *Inverse problem*: Find the optimal allocation of redundant units to reach maximum possible PFO level under condition that the total cost of the system does not exceed 70 units of cost.

In this case, the UGF for each unit is defined by a set of triplets (Cost, PFFO, Number of redundant units). Solution for the first step of the solution (interaction of set 1 and set 2) can be presented in the form of the table below.

First of all, restrict ourselves with possible solutions for the Direct Problem: since the system PFFO has to be not less than 0.97, it means that PFFO of each of four redundant groups has to be not less than 0.97.

Since cost restriction equals 70 cost units, the total cost of redundant units in each redundant group has to be not larger than, say 20-30.

Keeping this in mind let us construct the table with triplets obtained in the result of interaction sets for Unit-1 and Unit -2.

Table 3. Result of interaction of UGF's for Unit-1 and Unit-2

		***S*₁**					
		9 0.936 3	**12 0.9744 4**	**15 0.9898 5**	**18 0.9959 6**	**21 0.9984 7**	**24 0.9993 8**
	15 0.936 3	24 0.8761 (3; 3)	27 0.912 (4; 3)	30 0.9264 (5; 3)	33 0.9322 (6; 3)	36 0.9345 (7: 3)	39 0.9354 (7:3)
	20 0.9744 4	29 0.912 (3; 4)	32 0.9495 (4; 4)	35 0.9644 (5; 4)	38 0.9704 (6; 4)	41 0.9728 (7; 4)	44 0.9738 (8; 4)
***S*₂**	**25 0.9898 5**	34 0.9264 (3; 5)	37 0.9644 (4; 5)	40 0.9796 (5; 5)	43 0.9857 (6; 5)	46 0.9881 (7; 5)	49 0.9891 (8; 5)
	30 0.9959 6	39 0.9322 (3; 6)	42 0.9704 (4; 6)	45 0.9857 (5; 6)	48 0.9918 (6; 6)	51 0.9943 (6; 7)	54 0.9953 (6;8)
	35 0.9984 7	44 0.9345 (3; 7)	47 0.9728 (4; 7)	50 0.9881 (5; 7)	53 0.9943 (6; 7)	56 0.9967 (7; 7)	59 0.9977 (7; 8)
	40 0.9993 8	49 0.9354 (3; 8)	52 0.9738 (4; 8)	55 0.9891 (5; 8)	58 0.9953 (6; 8)	61 0.9977 (7; 8)	64 0.9987 (8; 8)

In this table triplets that are dominated by others are marked with grey shadowing. One can observe that dominating sequence occupies an area around "diagonal of the table. This property can be successfully used for minimizing the calculations: as soon as a dominated triplet appears below this "diagonal area", the further calculation in cells located below this cell can be stopped. Analogously, if a dominated triplet appears upper this "diagonal area" , the further calculation in cells located to the right from this cell can be also stopped. We will use this property in further calculating.

Thus, the dominating sequence characterizing an "equivalent" Unit-2*is presented in non-shadowed area of table 1. On the basis of data for Unit-2*, we can construct an analogous table for "equivalent Unit-3* (see Table 4).

Table 4. Result of interaction of UGF's for Unit-2* and Unit-3

		***S*₃**					
		9 0.973 3	**12 0.9919 4**	**15 0.9976 5**	**18 0.9993 6**	**21 0.9998 7**	**24 0.9999 8**
	24 0.8761 (3; 3)	33 0.8524 (3; 3; 3)	36 0.869 (3; 3; 4)	xxx	xxx	xxx	xxx
	27 0.912 (4; 3)	36 0.8874 (4; 3; 3)	39 0.9046 (4; 3; 4)	42 0.9098 (4; 3; 5)	xxx	xxx	xxx
	30 0.9264 (5; 3)	39 0.9014 (5; 3; 3)	42 0.9189 (5; 3; 4)	45 0.9242 (5; 3; 5)	xxx	xxx	xxx
S*₂	**32 0.9495 (4; 4)**	41 0.9239 (4; 4; 3)	44 0.9418 (4; 4; 4)	47 0.9472 (4; 4; 5)	xxx	xxx	xxx
	35 0.9644 (5; 4)	44 0.9384 (5; 4; 3)	47 0.9566 (5; 4; 4)	50 0.9621 (5; 4; 5)	xxx	xxx	xxx
	38 0.9704 (6; 4)	0.9442 (6; 4; 3)	50 0.9625 (6; 4; 4)	53 0.9681 (6; 4; 5)	xxx	xxx	xxx
	40 0.9796 (5; 5)	0.9532 (5; 5; 3)	52 0.9717 (5; 5; 4)	55 0.9772 (5; 5; 5)	xxx	xxx	xxx
	43 0.9857 (6; 5)	0.9591 (6; 5; 3)	55 0.9777 (6; 5; 4)	58 0.9833 (6; 5; 5)	61 0.9850 (6; 5; 6)	xxx	
	46 0.9881 (7; 5)	xxx	58 0.9801 (7; 5; 4)	61 0.9857 (7; 5; 5)	64 0.9874 (7; 5; 6)	67 0.9879 (7; 5; 7)	xxx
	49 0.9891 (8; 5)	xxx	xxx	64 0.9867 (8; 5; 5)	67 0.9884 (8; 5; 6)	70 0.9889 (8; 5; 7)	73 0.9885 (8; 5; 8)

Table 5. Final result of calculating

		***S*₄**					
		15 0.973 3	**20 0.9919 4**	**25 0.9976 5**	**30 0.9993 6**	**35 0.9998 7**	**40 0.9999 8**
	41 0.9239 (4; 4; 3)	56 0.899 (4; 4; 3; 3)	61 0.9164 (4; 4; 3; 4)	xxx	xxx	xxx	xxx
	44 0.9418 (4; 4; 4)	59 0.9164 (4; 4; 4; 3)	64 0.9342 (4; 4; 4; 4)	69 0.9395 (4; 4; 4; 5)	xxx	xxx	xxx
	47 0.9566 (5; 4; 4)	62 0.9308 (5; 4; 4; 3)	67 0.9489 (5; 4; 4; 4)	72 0.9543 (5; 4; 4; 5)	xxx	xxx	xxx
S*₃	**50 0.9625**	65 0.9365	70 0.9547	75 0.9602	xxx	xxx	xxx

	(6; 4; 4)	(6; 4; 4; 3)	(6; 4; 4; 4)	(6; 4; 4; 5)			
	52 **0.9717** **(5; 5; 4)**	67 0.9455 (5; 5; 4; 3)	72 0.9638 (5; 5; 4; 4)	77 0.9694 (5; 5; 4; 5)	xxx	xxx	xxx
	55 **0.9777** **(6; 5; 4)**	70 0.9513 (6; 5; 4; 3)	75 0.9698 (6; 5; 4; 4)	80 0.9754 (6; 5; 4; 5)	85 0.9770 (6; 5; 4; 6)	xxx	xxx
	58 **0.9833** **(6; 5; 5)**	0.9568 (6; 5; 5; 3)	78 0.9753 (6; 5; 5; 4)	83 0.9809 (6; 5; 5; 5)	88 0.9826 (6; 5; 5; 6)	xxx	xxx
	61 **0.9857** **(7; 5; 5)**	0.9591 (7; 5; 5; 3)	81 0.9777 (7; 5; 5; 4)	86 0.9833 (7; 5; 5; 5)	91 0.985 (7; 5; 5; 6)	xxx	xxx
	64 **0.9874** **(7; 5; 6)**	xxx	84 0.9794 (7; 5; 6; 4)	89 0.985 (7; 5; 6; 5)	94 0.9867 (7; 5; 6; 6)	99 0.9872 (7; 5; 6; 7)	xxx
	67 **0.9884** **(8; 5; 6)**	xxx	87 0.9804 (8; 5; 6; 4)	92 0.9860 (8; 5; 6; 5)	97 0.9877 (8; 5; 6; 6)	102 0.9882 (8; 5; 6; 7)	107 0.9883 (8; 5; 6; 8)

All calculations have been done with a simple Excel program.

Solutions of the problems above can be easily found from the last table. First time PFFO exceed level of 0.97 when X=(6; 5; 5; 4) and the corresponding system cost is 78 cost units. The inverse problem solution for restriction on the cost equals 70 cost units reaches when X=(4,4,5,3) and corresponding PFFO is equal to 0.9547.

Notice that due to associativity property of U-functions it is possible to get the same solution using another order of units' interaction.

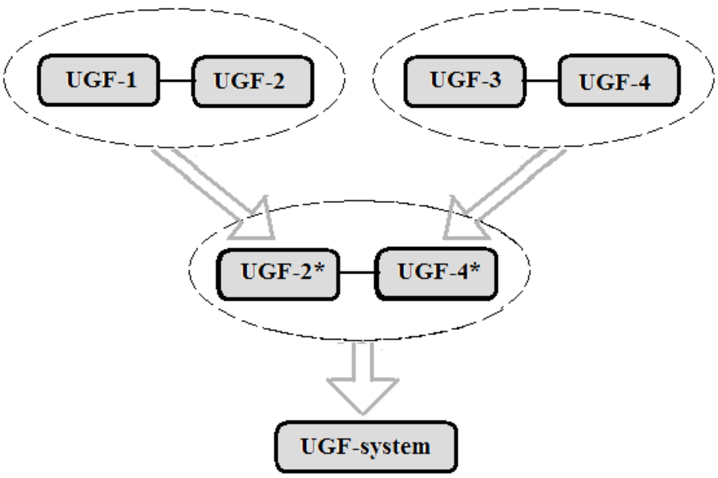

Figure 2. Second type of units' interaction procedure

Remark. By the way, this example shows with transparency that one can consider not only redundancy as a method of system reliability increase. For instance, one can consider a set of variants of the units with various reliability and cost. Actually, Unit-2* and Unit-4* can be considered as "black boxes" that are characterized by corresponding dominating sequences of triplets $\{Q_s^{(k)}; C_s^{(k)}; s\}$ where *s* is just a number of variants of considered Unit-*k*.

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(This bibliography on the subject is presented in chronological order to show the steps of the U-functions method development).

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