BASIS OF CONTINUUM APPROXIMATION IN MODELS OF GROWING RANDOM NETWORKS

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ABSTRACT

Accuracy asymptotics for differences between prelimit and limit distributions of nodes powers in models of random growing networks are constructed. A rate of a convergence in these relations is power. Obtained formulas allow to ground the continuum approximations for considered models.

1. INTRODUCTION

Main aim of this paper is to estimate a rate of a convergence to limit distributions in models of growing random networks [1]. One of the most convenient methods to define limit distribution of node exponent is the continuum approximation [2], [3]. It is based on asymptotic behavior of considered distribution when a number of steps tends to the infinity. In [4, p. 124] it is marked that ``a problem of strict formal description of statistical ensemble of random networks with fixed distribution of nodes exponents is not yet solved``. An absence of correct mathematical base of the continuum approximation makes susceptible results obtained using this approximation in spite of its calculation convenience.

In this paper a ground of the continuum approximation in a calculation of limit distributions for main models of growing random networks is made. Here the model of growing exponential network, the model of Barabasi and the model of Dorogovtsev are analyzed. Exact asymptotic of a difference between prelimit and limit distributions when a number of steps tends to the infinity is obtained and this asymptotic is power. Such results are based on a construction of recurrent relations for prelimit distributions of nodes exponents distributions and on asymptotic series in these relations.

2. EXPONENTIAL NETWORKS

Consider a model of growing exponential network in which new node is connected with each existing node with equal probabilities. Denote $p(k,s,t)$ the probability that on the step $t \geq 1$ the node $s, 1 \leq s \leq t,$ is connected with k arcs of non oriented graph of exponential network. Then k is called the power of the node s. In [2] the following relations are obtained (here δ_{ij} is the Kroneker index):

$$p(k,t,t) = \delta_{k1}, \quad p(k,s,t+1) = \frac{p(k-1,s,t)+(t-1)p(k,s,t)}{t}, \quad k \geq 1.$$

Designate

$$P(k,t) = \frac{1}{t}\sum_{s=1}^{t} p(k,s,t) \tag{1}$$

then the recurrent relation

$$(t+1)P(k,t+1)=P(k-1,t)+(t-1)P(k,t)+\delta_{k1},\ k>1,\ t\ge 1,$$

is true and $P(1,1)=1$, $P(k,t)=0$, $k>t\ge 1$. So for $k=1$

$$P(1,t+1)=\frac{t+t(t-1)P(1,t)}{(t+1)t}=\frac{t+(t-1)+(t-1)(t-2)P(1,t-1)}{(t+1)t}=...=$$

$$=\frac{1+...+t}{(t+1)t}=\frac{1}{2},\ t\ge 1, \tag{2}$$

and for $k\ge 2$

$$P(k,t+1)=\frac{tP(k-1,t)+t(t-1)P(k,t)}{(t+1)t}=$$

$$=\frac{tP(k-1,t)+(t-1)P(k-1,t-1)+(t-1)(t-2)P(k,t-1)}{(t+1)t}=...=$$

$$=\frac{1}{(t+1)t}\sum_{j=1}^{t} jP(k-1,j). \tag{3}$$

From the last formula

$$P(2,t+1)=\frac{1+...+t}{2t(t+1)}=\frac{1}{4},\ t\ge 1, \tag{4}$$

$$P(3,t+1)=\frac{2+...+t}{4t(t+1)}=\frac{1}{8}-\frac{1}{4t(t+1)},\ t\ge 1. \tag{5}$$

Denote $f_k(t)=P(k,t+1)-2^{-k}$, $k\ge 1$, then from the formulas (2) – (5) we have

$$f_1(t)\equiv f_2(t)\equiv 0,\ f_3(t)\sim -\frac{1}{4t^2},\ t\to\infty. \tag{6}$$

Theorem 1. For $t\to\infty$ the relations

$$f_k(t)\sim -\frac{C_k \ln^{k-3} t}{t^2},\ C_k=\frac{1}{4(k-3)!},\ k\ge 3, \tag{7}$$

take place.

Proof. For $k=3$ the formula (7) is a corollary of the formula (6) then by an induction we obtain from (3) that

$$f_{k+1}(t)=P(k+1,t+1)-\frac{1}{2^{k+1}}=\frac{1}{t(t+1)}\sum_{j=1}^{t} jP(k,j)-\frac{1}{2^{k+1}}=$$

$$=\frac{1}{t(t+1)}\sum_{j=1}^{t} j\left(\frac{1}{2^k}+f_k(j-1)\right)-\frac{1}{2^{k+1}}=\frac{1}{t(t+1)}\sum_{j=1}^{t} jf_k(j-1)\sim -\frac{C_k \ln^{k-2} t}{(k-2)t^2},\ t\to\infty.$$

The formula (7) is proved.

Remark 1. In all sections of this paper the continuum approximation is based on the limit

$$t\big(P(k+1,t)-P(k,t)\big)=t\big(f_k(t)-f_k(t-1)\big)\to 0,\ t\to\infty. \tag{8}$$

In this section the formula (8) is a corollary of the formula (7).

3. MODEL OF BARABASI-ALBERT

Consider Barabasi-Albert model of growing network [1] in which new node is connected with each existing node with probability proportional to a power of existing node. Denote $p(k,s,t)$ the probability that on the step $t\ge 1$ the node $s, 1\le s\le t,$ is connected with k arcs of non oriented graph of Barabasi-Albert network. In [2] the following relations are obtained

$$p(k,t,t)=\delta_{k1},\ p(k,s,t+1)=\frac{k-1}{2t}p(k-1,s,t)+\left(1-\frac{k}{2t}\right)p(k,s,t),\ k\ge 1.$$

From the formula (1) we have

$$(t+1)P(k,t+1)=\frac{k-1}{2}P(k-1,t)+\left(t-\frac{k}{2}\right)P(k,t)+\delta_{k1},\ k>1,\ t\ge 1,$$

$$P(1,1)=1,\ P(k,t)=0,\ k>t\ge 1,\ P(0,t)\equiv 0, t\ge 1.$$

Analogously with (2), (3) it is not difficult to obtain

$$P(1,t+1)=\frac{1}{t+1}\left[1+\sum_{j=1}^{t}\prod_{s=j}^{t}\left(1-\frac{1}{2s}\right)\right],\ t\ge 1, \tag{9}$$

$$P(k,t+1)=\frac{k-1}{2(t+1)}\sum_{j=1}^{t}P(k-1,j)\prod_{s=j+1}^{t}\left(1-\frac{k}{2s}\right),\ k\ge 2,\ t\ge 1, \prod_{t+1}^{t}=1. \tag{10}$$

Lemma 1. For $A>0$ the equalities

$$\sum_{j=1}^{t}\prod_{s=j}^{t}\left(1-\frac{A}{s}\right)=\frac{t-A}{1+A}+\frac{\psi(t)}{(1+A)^2\Gamma(-1-A)},\ \psi(t)=\frac{\Gamma(1-A+t)}{\Gamma(t+1)}, \tag{11}$$

$$\sum_{j=1}^{t}\prod_{s=j+1}^{t}\left(1-\frac{A}{s}\right)=\frac{t+1}{1+A}-\frac{\psi(t)}{(1+A)\Gamma(1-A)} \tag{12}$$

are true. Here $\Gamma(z)$ is the gamma function.

Proof. Denote $S(t)=\sum_{j=1}^{t}\prod_{s=j}^{t}\left(1-\frac{A}{s}\right)$ then

$$S(1)=1-A=\frac{1-A}{1+A}+\frac{\psi(1)}{(1+A)^2\Gamma(-1-A)},$$

consequently the formula (11) for $t=1$ is proved. Suppose that this formula is true for t and prove it for $t+1$

$$S(t+1)=\sum_{j=1}^{t+1}\prod_{s=j}^{t+1}\left(1-\frac{A}{s}\right)=\frac{t+1-A}{t+1}(S(t)+1)=$$

$$=\frac{t+1-A}{t+1}\left(\frac{t-A}{1+A}+\frac{\psi(t)}{(1+A)^2\Gamma(-1-A)}\right)=\frac{t+1-A}{t+1}+\frac{\psi(t+1)}{(1+A)^2\Gamma(-1-A)}.$$

The relation (11) is proved for all natural t. The formula (12) may be proved similar.

Remark 2. In left sides of the formulas (11), (12) we have functions defined for $A>0$, $t>0$ and on right sides - the gamma functions which may be non defined for some $A>0$, $t>0$. But ratios of these gamma functions in such points may be redefined using limit transition to these points.

Designate

$$f_k(t) = P(k, t+1) - \Pi(k),\ \Pi(k) = \frac{4}{k(k+1)(k+2)},\ k \geq 1. \tag{13}$$

Theorem 2. For $t \to \infty$ and $k \geq 1$ the following relations are true

$$f_k(t) \sim \frac{t^{-3/2}}{3\sqrt{\pi}}. \tag{14}$$

Proof. It is known [5, subsection 1.18] that for $A > 0$

$$\psi(t) = \frac{\Gamma(1 - A + t)}{\Gamma(t+1)} \sim t^{-A}, t \to \infty. \tag{15}$$

Assume that $k = 1$ then the relation (14) is a corollary of the formulas (9), (10), (15) for $A = 1/2$

$$f_1(t) = P(1, t+1) - \Pi(1) = \frac{1}{t+1}\left(1 + \frac{t - 1/2}{3/2} + \frac{\psi(t)}{(3/2)^2 \Gamma(-3/2)}\right) - \frac{2}{3} \sim$$

$$\sim \frac{4t^{-1/2}}{9\Gamma(-3/2)t} = \frac{t^{-3/2}}{3\sqrt{\pi}},\ t \to \infty.$$

Suppose that the formula (14) is true for $k-1$ and prove it for k, $k > 1$. Represent $f_k(t)$ in the form $f_k(t) = a_k(t) + b_k(t)$ where

$$a_k(t) = \frac{k-1}{2(t+1)} \sum_{j=1}^{t} \Pi(k-1) \prod_{s=j+1}^{t} \left(1 - \frac{k}{2s}\right) - \Pi(k),$$

$$b_k(t) = \frac{k-1}{2(t+1)} \sum_{j=1}^{t} f_{k-1}(j-1) \prod_{s=j+1}^{t} \left(1 - \frac{k}{2s}\right).$$

Consequently from the formulas (12), (13) for $A = k/2$ we have

$$a_k(t) = \frac{4}{k(k+1)}\left[\frac{1}{2(t+1)} \sum_{j=1}^{t} \prod_{s=j+1}^{t} \left(1 - \frac{k}{2s}\right) - \frac{1}{k+2}\right] =$$

$$= \frac{4}{k(k+1)}\left[\frac{1}{2(t+1)}\left(\frac{t+1}{1+k/2} + \frac{\psi(t)}{(1+k/2)\Gamma(1-k/2)}\right) - \frac{1}{k+2}\right] =$$

$$= \frac{4\psi(t)}{2(t+1)k(k+1)(1+k/2)\Gamma(1-k/2)} \sim \frac{\Pi(k)t^{-1-k/2}}{\Gamma(1-k/2)},\ t \to \infty.$$

From the induction assumption $f_{k-1}(t) \sim \frac{t^{-3/2}}{3\sqrt{\pi}}$, $t \to \infty$, and from the gamma function properties and from the formula (15) for $A = k/2$ we obtain

$$b_k(t) = \frac{k-1}{2(t+1)} \sum_{j=1}^{t} f_{k-1}(j-1) \frac{\psi(t)}{\psi(j)} \sim \frac{k-1}{2(t+1)} \int_1^t \frac{j^{k/2}}{3\sqrt{\pi} j^{3/2} t^{k/2}} dj \sim \frac{t^{-3/2}}{3\sqrt{\pi}},\ t \to \infty.$$

Then asymptotic relation (14) is true for arbitrary natural k.

4. MODEL OF DOROGOVTSEV

Consider Dorogovtsev model of growing network in which new node is connected with each existing node with the probability proportional to a sum of its power (a number of arcs incoming to existing node) and some constant $a>0$. Here $a>0$ is model parameter. Denote $p(k,s,t)$ the probability that on the step $t\geq 1$ the node $s,\ 1\leq s\leq t,$ has the power k. In [2] we obtain the relations

$$p(k,s,t+1)=\frac{k-1}{t(1+a)}p(k-1,s,t)+\left(1-\frac{k+a}{t(1+a)}\right)p(k,s,t),\ p(k,t,t)=\delta_{k0},\ k\geq 0.$$

Designate

$$P(k,t)=\frac{1}{t}\sum_{s=1}^{t}p(k,s,t)$$

then

$$P(k,t+1)=\frac{1}{t(a+1)}\left[P(k,t)\left((a+1)(t-1)-k-a\right)+P(k-1,t)(k-1+a)+(1+a)\delta_{k0}\right],$$

$P(0,1)=1$, $P(k,t)=0$, $k\geq t$, $P(-1,t)\equiv 0$, $t\geq 0$.

Analogously with the formulas (9), (10) it is not difficult to obtain

$$P(0,t+1)=\frac{1}{t}\left[1-\frac{a}{a+1}\prod_{j=1}^{t-1}\left(1-\frac{a}{j(a+1)}\right)+\sum_{s=1}^{t-1}\prod_{j=s}^{t-1}\left(1-\frac{a}{j(a+1)}\right)\right], \qquad (16)$$

$$P(k,t+1)=\frac{k-1+a}{t(a+1)}\sum_{s=1}^{t}P(k-1,s)\prod_{j=s}^{t-1}\left(1-\frac{k+a}{(a+1)j}\right),\ k>0. \qquad (17)$$

Denote $A_k=(k+a)/(a+1)$, $k\geq 0$,

$$\Pi(k)=(1+a)\frac{\Gamma(1+2a)\Gamma(k+a)}{\Gamma(a)\Gamma(k+2+2a)},\ f_k(t)=P(k,t+1)-\Pi(k),\ k\geq 0.$$

Theorem 3. The formulas

$f_k(t)\sim C_k t^{-1-A_0}$, $t\to\infty$, $k\geq 0$,

$$C_0=\frac{1}{(1+A_0)^2\Gamma(-1-A_0)}-\frac{A_0}{\Gamma(1-A_0)},\ C_k=\frac{C_{k-1}(k-1+a)}{(a+1)(A_k-A_0)},\ k>0, \qquad (18)$$

are true.

Proof. From the formulas (11), (15), (16) for $A=A_0$ we have

$$f_0(t)=P(0,t+1)-\Pi(0)=\frac{1}{t}\left[1-\frac{A_0\psi(t-1)}{\Gamma(1-A_0)}+\frac{t-(1+A_0)}{1+A_0}+\frac{\psi(t-1)}{\Gamma(-1-A_0)(1+A_0)^2}\right]-$$

$$-\frac{1+a}{1+2a}=\frac{\psi(t-1)C_0}{t}\sim C_0 t^{-1-A_0},\ t\to\infty.$$

For $k>0$ we seek $f_k(t)=P(k,t+1)-\Pi(k)$ in the form $f_k(t)=a_k(t)+b_k(t)$,

$$a_k(t)=\frac{k-1+a}{t(a+1)}\sum_{s=1}^{t}\Pi(k-1)\prod_{j=s}^{t-1}\left(1-\frac{A_k}{j}\right)-\Pi(k),$$

$$b_k(t)=\frac{k-1+a}{t(a+1)}\sum_{s=1}^{t}f_{k-1}(s-1)\prod_{j=s}^{t-1}\left(1-\frac{A_k}{j}\right).$$

Then from the formula (15) for $t \to \infty$ and $A = A_k$ we have

$$a_k(t) = \frac{(k-1+a)\Pi(k-1)}{t(a+1)}\left[\sum_{s=1}^{t-1}\prod_{j=s}^{t-1}\left(1-\frac{A_k}{j}\right)+1\right]-\Pi(k)=$$

$$=\frac{(k-1+a)\Pi(k-1)\psi(t-1)}{t(a+1)\Gamma(-1-A_k)(1+A_k)^2}\sim\frac{(k-1+a)\Pi(k-1)t^{-A_k-1}}{(1+A_k)^2(a+1)\Gamma(-1-A_k)}.$$

From the induction assumption $f_{k-1}(t)\sim C_{k-1}t^{-1-A_0}$ and from the formula $A_k > A_0$ and from the formula (15) for $A = A_k$ we obtain

$$b_k(t)=\frac{k-1+a}{t(a+1)}\sum_{s=1}^{t} f_{k-1}(s-1)\prod_{j=s}^{t-1}\left(1-\frac{A_k}{j}\right)=\frac{k-1+a}{t(a+1)}\sum_{s=1}^{t} f_{k-1}(s-1)\frac{\psi(t-1)}{\psi(s-1)}\sim$$

$$\sim\frac{C_{k-1}(k-1+a)}{t^{1+A_0}(a+1)(A_k-A_0)}=C_k t^{-1-A_0},\ t\to\infty.$$

Consequently asymptotic relation (18) is proved for arbitrary natural k.

Remark 3. A consideration of Dorogovtsev model [2] is connected with its wide application to modern models of growing random networks. For small a this model gives sufficiently simple and convenient description of the Internet network with power distribution network nodes exponents

$$\Pi(k)\sim Dk^{-2-a},\ D=\frac{(1+a)\Gamma(1+2a)}{\Gamma(a)},\ k\to\infty$$

with the parameter close to two [6].

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