MULTISTATE COHERENT SYSTEMS WITH MULTIPLE STATE TRANSITION AT A TIME

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Abstract

A key requirement in defining a multistate coherent system (MCS) is the relevance condition of its components. A new class of MCSs is introduced with a new component relevance condition. Also we introduce a more general relevance condition. They are compared with some existing component relevance conditions. Based on the two new relevance conditions, two component importance measures for MCSs are defined. They are most appropriate for comparing components when certain type of system improvement is sought. We introduce new joint importance measures for two or more components with respect to the proposed relevance conditions. The new MCS classes include several existing MCSs as special case. An illustrative example of the proposed MCSs is also provided.

Keywords: Reliability, MCS, relevance condition, component importance, joint importance.

1. Introduction

Let us consider a coherent system with n components C={1,2,…,n}. Furthermore, suppose that each component can be in one of M+1 states, {0,1,2,…,M}, where '0' is the failed state and 'M' is the maximal or "perfect" state. To describe such a multistate system (MSS), a general theory has been developed in the literature.[5,9,11] A binary state system (BSS) of n components can be described by a structure function $\phi:\{0,1\}^n \to \{0,1\}$, which presents the state of system as a function of states of its n components.[4] A binary system is statistically coherent if it satisfies the following conditions;[4]

(i) $\phi(\bar{x})$ is non-decreasing in each argument, where $\bar{x}=(x_1,x_2,...,x_n)$ with $x_i \in \{0,1\}$, and

(ii) for each i, there exist a vector $(\cdot_i,\bar{x})$, such that $\phi(1_i,\bar{x}) > \phi(0_i,\bar{x})$, where $(\cdot_i,\bar{x}) = (x_1,x_2,...,x_{i-1},\cdot_i,x_{i+1},...,x_n)$.

Note that the condition (i) and (ii) gives, $\phi(\bar{j}) = j,\ j=0,1$ where $\bar{j}=(j,j,...,j)$.

In practice, a system and its components often have more than two states of performance.[5] The structure function of the MSS is $\phi: S^n \to S,\ S=\{0,1,...,M\}$, which relates the level of performance of system to level of performance of each of its components. There are various approaches which extends the structure function from the binary case to the multistate case.[5,9,10,11] The effort resulted in, extension of the requirement of non-decreasing binary structure function to MSS structure function. Also note the condition $\phi(\bar{j}) = j,\ j=0,1$ of the binary coherent system (BCS) is extended to the MSS requiring $\phi(\bar{j}) = j,\ j \in \{0,1,...,M\}$.[11] The condition (ii) of relevancy in BCS is extended in various different ways. Some extensions can be seen in Refs. 1, 2 and 12.

In this paper, we extend the relevance condition to the MSS case in a general way, which includes several existing relevance conditions as special cases. Section 2 introduces the new class of multistate coherent system(MCS)s and its generalization by introducing a reasonable component relevance condition. The two new classes are compared with the some existing classes. Section 3 introduces two new component importance and joint importance measures for the proposed MCSs. Section 4 provides an example of an offshore electrical power generation system. Discussion and conclusion are given in section 5.

2. Component relevancy and the new classes of MCSs

In this section we discuss the new relevance condition and its generalization on which two new classes of MCSs are defined. Consider the following component relevance conditions.

NAT[13] : For every component *i* and level *j*>0, there exist $(\cdot_i, \bar{x})$ such that $\phi(j_i, \bar{x}) \geq j$ and $\phi((j-1)_i, \bar{x}) < j.$

GRI.1[11] : For every component *i* and level *j*>0, there exist $(\cdot_i, \bar{x})$ such that $\phi(j_i, \bar{x}) > \phi((j-1)_i, \bar{x}).$

GRI.2[11] : For every component *i*, there exist $(\cdot_i, \bar{x})$ such that $\phi(0_i, \bar{x}) < \phi(M_i, \bar{x}).$

and **EP**[9] **:** For every component *i* and level *j*≥1, there exist $(\cdot_i, \bar{x})$ such that $\phi(j_i, \bar{x}) > \phi(0_i, \bar{x}).$

NAT and **GRI.1** indicate degree of relevance of each component to every level of performance; while **GRI.2** merely states that ϕ is not a constant in any of its arguments.

Now consider a situation in which some component is not relevant to every level of performances, i.e., the system degrades from state *j* to *j-1* or *j-2* etc when the component degrades only from state *j* to *j-2* or *j-3* etc. In order to degrade the system, component must degrade more than one level of performance. For example,[14] let $S = \{0,1,2,3,4\}$, and the component can take 0, 2, and 4 when the system can take 0, 1, 2, 3 and 4. Consider the structure function ϕ_2 having 5 components in Ref.14. From the minimal path vectors of ϕ_2, we have, $\phi_2(4_1,4_2,2_3,4_4,2_5) = 4 > \phi_2(4_1,4_2,2_3,2_4,2_5) = 3$, when the 4th component degrades from state 4 to state 2, the system degrades from state 4 to state 3. Now consider the structure function ϕ_1 with three components in Ref.14. We have, $\phi_1(4_1,0_2,4_3) = 4 > \phi_1(4_1,0_2,2_3) = 2,$ when the third component degrades from state 4 to state 2, the system degrades from state 4 to state 2. Here fourth component must degrade from state 4 to state 2 for the system to degrade from state 4 to state 3 with respect to ϕ_2. The third component must degrade from state 4 to state 2 for the system to degrade from state 4 with respect to ϕ_1.

We define a new component relevance condition as, degrading a component from state *j* to state *j-2* can cause system failure or degradation while degradation of the component from state *j* to *j*-1 cannot cause system failure or degradation.

Now the new class of MCSs can be defined as follows.

Definition.1.: A multistate system of *n* components with structure function ϕ belonging to class **CM.1** if ϕ is non-decreasing, $\phi(\bar{j}) = j,$ and for each component, there exist $(\cdot_i, \bar{x})$ such that $\phi(j_i, \bar{x}) > ((j-2)_i, \bar{x}).$

Now consider the generalization of the new relevance condition, one or more than one level of degradation of the component can cause the system degradation, i.e., when the component 'i' degrades from state *j* to state *j'*∈*{j-1, j-2, j-3,…., 1, 0},* the system degrades from state *j* to any lower state. Thus we define the generalized class of MCSs with this relevance condition.

Definition 2.: A multistate system of *n* components with structure function ϕ belonging to class **CM.2** if ϕ is non-decreasing, $\phi(\bar{j}) = j$, and for each component, there exist $(\cdot_i, \bar{x})$ such that $\phi(j_i, \bar{x}) > (j'_i, \bar{x})$ where $j' \in \{j-1, j-2, j-3, ..., 1, 0\}$.

In the following section we introduce the component importance and joint importance measures to the new classes of MCSs.

3. Component importance and Joint importance measures

We consider the problem of measuring the reliability importance and structural importance of individual components, and the joint reliability importance and joint structural importance of two or more components in the new classes of the MCSs. The main advantage of defining a new relevance condition is to obtain the importance measures. [6] At the reliability design phase, the joint importance can improve system designer's understanding of the relationship between the components and the system, and among the components, [3] which is quite desirable**.** Birnubaum measure provides the importance of a component in the BSS. [6] It is further extended to the MSS. [7,15] Now we consider $X = (X_1, X_2, ..., X_n)$ as a random vector with component states X_i as random variables and $p_{ij} = \Pr\{X_i = j\}$ where $j \in S = \{0,1,...,M\}$. For the BSS with structure function ϕ, the Birnubaum reliability importance [6] of component i is

$$I_i(B) = P(\phi(1_i, \bar{x}) - \phi(0_i, \bar{x}) = 1) = h(1_i, \bar{p}) - h(0_i, \bar{p}).$$

where $h(\bar{p})$, $\bar{p} = (p_1, p_2, ..., p_n)$ and $\forall i \; p_i = (x_i = 1)$, is the reliability function of the BCS,

$$h(\bar{p}) = p_i h(1_i, \bar{p}) + (1 - p_i) h(0_i, \bar{p}) = p_i I_i(B) + h(0_i, \bar{p}).$$

Therefore,

$$\frac{\partial h(\bar{p})}{\partial p_i} = I_i(B).$$

We propose the following component importance measures for the two classes of new MCSs.

1. $I_i(CM.1) = P(\phi(j_i, \bar{x}) > \phi((j-2)_i, \bar{x})).$
2. $I_i(CM.2) = P(\phi(j_i, \bar{x}) > \phi(j'_i, \bar{x})), \quad j' \in \{j-1, j-2, ..., 1, 0\}.$

Let the distribution of X_i be described by $p_i = (p_{i0}, p_{i1}, ..., p_{iM})$. The reliability function of the MCS with minimum satisfactory system level j, is $P(\phi(\bar{x}) \geq j) = \sum_{j \in S} P(\phi(j_i, \bar{x}) \geq j),$ since $p_{i0} + p_{i1} + p_{i2} + ... + p_{iM} = 1$. Now we prove the following theorems.

Theorem 1. For the **CM.1** class, $I_i(CM.1)$ is the rate of improvement of $P(\phi(\bar{x}) \geq j)$ with respect to p_{ij}.

Proof. Clearly,

$$P(\phi(\bar{x}) \geq j) = \sum_{S \setminus \{j-2\}} p_{ij} [P(\phi(j_i, \bar{x} \geq j) - P(\phi((j-2)_i, \bar{x}) \geq j)] + P(\phi((j-2)_i, \bar{x}) \geq j),$$

since $1 - p_{ij-2} = p_{i0} + p_{i1} + ... + p_{ij-3} + p_{ij-1} + ... + p_{iM}$. Differentiating $P(\phi(\bar{x}) \geq j)$ partially with respect to p_{ij}, we get

$$I_i(CM.1) = \frac{\partial P(\phi(\bar{x}) \geq j)}{\partial p_{ij}} = P(\phi(j_i, \bar{x}) \geq j) - P(\phi((j-2)_i, \bar{x}) \geq j) = P(\phi((j-2)_i, \bar{x}) < \phi(j_i, \bar{x})).$$

Theorem 2. For the **CM.2** class, $I_i(CM.2)$ is the rate of improvement of $P(\phi(\bar{x}) \geq j)$ with respect to p_{ij}.

Proof. Clearly,

$$P(\phi(\bar{x}) \geq j) = \sum_{S \setminus \{j'\}} p_{ij}[P(\phi(j_i, \bar{x}) \geq j) - P(\phi(j'_i, \bar{x}) \geq j)] + P(\phi(j'_i, \bar{x}) \geq j),$$

where $1 - p_{ij'} = p_{i0} + p_{i1} + ... + p_{ij'-1} + p_{ij'+1} + ... + p_{iM}$ and differentiating the $P(\phi(\bar{x}) \geq j)$ partially with respect to p_{ij} we get,

$$I_i(CM.2) = \frac{\partial P(\phi(\bar{x}) \geq j)}{\partial p_{ij}} = P(\phi(j'_i, \bar{x}) < \phi(j_i, \bar{x})), \quad j' \in \{j-1, j-2, ..., 1, 0\}.$$

Now define the structural definition of the component importance (when reliabilities of components are not given) with respect to the new relevance conditions.

Consider $\phi(\bar{x}) = 1 \;\; if \;\; \phi(\bar{x}) \geq j$ and 0 otherwise. We define the structural importance of a component as follows.

Definition.3.: Let $\phi : S^n \to S$ be the MCS structure function in **CM.1** class. Then ϕ is said to have the following measures of structural importance for the level *j* of component *i*:

$$I_{ij}(CM.1) = \frac{1}{(M+1)^{n-1}} \sum_{\{\bar{x}: x_i = j\}} Max\{0, \phi(j_i, \bar{x}) - \phi((j-2)_i, \bar{x})\}.$$

Definition.4.: Let $\phi : S^n \to S$ be the MCS structure function in **CM.2** class. Then ϕ is said to have the following measures of structural importance for the level *j* of component *i*:

$$I_{ij}(CM.2) = \frac{1}{(M+1)^{n-1}} \sum_{\{\bar{x}: x_i = j\}} Max\{0, \phi(j_i, \bar{x}) - \phi(j'_i, \bar{x})\}, \quad j' \in \{j-1, j-2, ..., 1, 0\}.$$

In order to define the joint importance measures for two or more components in the new classes of MCSs, we recall the joint structural importance measure(*JSIM*)s [8] and joint reliability importance measure(*JRIM*)s [8] for the MSS with relevance condition in **GRI.1**. The *JSIM* (*i,j*) for two components *i* and *j* with the new relevance conditions can be obtained by replacing $\tilde{m}$ with *m-2* or $m' \in \{m-1, m-2, ..., 2, 1, 0\}$ in,

$$JSIM(i,l) = \sum_{m=1}^{M} \sum_{k=1}^{M} \{ SIM(i,l;m,k) - SIM(i,l;m,\tilde{k}) \}$$

where $SIM(i,l;m,k) = \dfrac{\sum_{\vec{X}_{il}} \sum_{q=1}^{j} \chi(\phi(m_i, k_l, \vec{x}_{il}) = j, \phi(\tilde{m}_i, k_l, \vec{x}_{il}) = j - q)}{(M+1)^{n-2}}$.

Here χ(true)=1 and χ(false)=0, and $\phi(m_i, k_l, \bar{x}_{il}) = j, \phi(\tilde{m}_i, k_l, \bar{x}_{il}) = j - q$ where $\bar{x}_{ij} = (x_1, ..., m_i, ..., k_l, ..., x_n)$ determines the critical path vector to the level *j* with state *m* of component *i*. The *JSIM* (*i,j,k*) for three components can be obtained as, for $\tilde{k} = n-2$ or $\tilde{k} \in \{n-1, n-2, ..., 2, 1, 0\}$,

$$JSIM(i,l,r) = \sum_{k=1}^{M} \sum_{n=1}^{M} \sum_{m=1}^{M} \{ JSIM(i,l,r;m,k,n) - JSIM(i,l,r;m,k,\tilde{n}) \},$$

where $JSIM(i,l,r;m,k,n) = JSIM(i,l,r;m,k,n) - JSIM(i,l,r;m,\tilde{k},n)$.

Thus we can find *JSIM* of any number of components w. r. t. both relevance conditions in the MCS classes **CM.1** and **CM.2**. Thus *JSIM* [8] holds with new relevance conditions by replacing $\tilde{m}$ with suitable m-*2* or $m' \in \{m-1, m-2, ..., 2, 1, 0\}$.

Now we consider the *JRIM* for *k* components in the MSS. [8] The *JRIM* of state b_1 of component a_1, state b_2 of component a_2,..., state b_k of the component a_k $(k \leq n)$ of the MSS is

$$JRIM(a_1,...,a_k;b_1,...,b_k) = \frac{\partial^k E_s}{\partial R_{a_1} b_1 \partial R_{a_2} b_2 ... \partial R_{a_k} b_k}, \ k = 2,...,n,$$

where $E_s = \sum_{j=0}^{M} P(\phi(x) \geq j)$ is the expected system performance and $R_{a_i} b_i$ is the reliability function with respect to level b_i of component a_i.

Here $\tilde{m}$*=m-1* in the expansion of E_s. [8] The results also holds true with the other values of $\tilde{m}$*=m', m'*$\in$ *{m-1,m-2,m-3,...,0}*. Hence the *JRIM* with the new MCSs can be obtained with appropriate *m'* values for $\tilde{m}$, i.e., *m'=m-2* or $m' \in \{m-1, m-2,...,2,1,0\}$.

Now consider some implications based on the new relevancy definitions. In fact one can easily prove the following implications.

Theorem 3. CM.1=>EP, CM.2=>NAT=>GRI.1=>GRI.2=>EP, CM.2=>GRI.1, CM.2=>GRI.2 and **CM.2=>EP**.

It is clear that all the existing relevance conditions are special cases of **CM.2** (or **CM.1)** relevance condition. Hence the existing MCSs are special cases of the proposed MCSs.

4. Example

Ref.14 considered an offshore electrical power generation system, which supply two nearby oilrings with electrical power. Both oilrings have their own main generation, represented by equivalent generators A_1 and A_3 each having capacity of 50MW. In addition the oilrings has a standby generator A_2 that is switched into the network in case of outage of A_1 or A_3, or may be used in extreme load situations in either of the oilrings. The A_2 also has capacity 50MW. The control unit, U, continuously supervises the supply from each of the generators with automatic control of the switches. If for instance the supply from A_3 to oilring 2 is not sufficient, whereas the supply from A_1 to oilring 1 is sufficient, *U* can activate A_2 to supply oilring 2 with electrical power through the subsea cables L. The components have states {0, 2, 4} and the system have states {0, 1, 2, 3, 4}, where 0, 1, 2, 3 and 4 represents the states of the system at capacities 0MW, 12.5MW, 25MW, 37.5MW, and 50MW respectively.

Table I .Minimal path vectors of ϕ_1

Levels	U	A_1	A_2
2	2	2	0
2	4	0	2
4	2	4	0
4	4	0	4
4	4	2	2

Table II .Minimal path vectors of ϕ_2

Levels	U	A_1	L	A_2	A_3
1	4	4	2	2	0
1,2	2	0	0	0	2
2	4	4	2	4	0
2	4	4	4	2	0
3	4	4	2	2	2
3,4	2	0	0	0	4

4	4	4	2	4	2
4	4	4	4	2	2
4	4	4	4	4	0

The minimal path vectors to the levels are given in table 1 and table II, of the structure functions,

$\phi_1(U, A_1, A_2) = I(U > 0)\min(A_1 + A_2 I(U = 4), 4),$ the amount of power that can be supplied to platform 1, and

$\phi_2(U, A_1, L, A_2, A_3) = I(U > 0)\min(A_3 + A_2 I(U = 4) I(A_1 = 4) L / 4, 4),$ the amount of power that can be supplied to platform 2, when $I(.)$ is the indicator function. One may easily verify from the tables that the new relevance condition of **CM.1** and **CM.2** are found to be holding good w. r. t. the structure functions considered in the example. As done for *JSIM* and *JRIM* for two components[11] and *JSIM* for three components,[8] we can compute the concerned joint importance measures for any number of components in the power generation system.

5. Discussion and Conclusion

The theory of MSS reliability models has been developed to cope with many real-life situations. The present paper introduced two classes of MCSs with a new relevance condition and its generalization. It is shown that many MCSs introduced earlier in the literature are included in the new classes. Structural definitions of importance and joint structural importance measures are given, and new reliability importance and joint reliability importance measures are introduced. In system engineering, a practical and difficult problem is the identification of those groups of components that mostly influence the system behavior with respect to safety and reliability. In this respect, the main advantage of our importance and joint importance measure with respect to the new MCS models is the information provided by them for the reliability theoreticians and design analysts. It gives useful information for safe and efficient operation of the system, where existing importance measures gives information about individual component importance and joint importance of components with some limited number of relevance conditions.

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