LOGICAL ANALYSIS OF FAILURES GRAPH

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ABSTRACT

In this paper a problem to define direct and inverse sets of nodes connected with failed node is considered. This problem is solved by a calculation of connectivity matrix. To simplify initial network a problem of a minimization of its numbers of nodes and arcs is solved also. Calculation complexity of this solution is approximately cubic by a number of nodes.

1. INTRODUCTION

Failures graphs or failures trees are widely used in an analysis of different engineering networks: communication, internet, software, in parallel and distributed systems and in technology processes control, etc. (see for example [1]-[3]). This concept is developed presumably in engineering literature and so needs some mathematical interpretation and generalization by modern methods of discrete mathematics: mathema-tical logic, graph theory, general algebra, theory of algorithms and discrete optimization. A purpose of such generalization is to decrease dimension of failures graphs and trees without a change of their main characteristics: direct and inverse sets of nodes connected with failed node. A combination of such approaches allows to obtain new results in this classical area of the reliability theory.

2. DIRECT AND INVERSE SETS OF NODES CONNECTED WITH FAILED NODE

Our aim is to analyze how a failure in some node of oriented graph leads to failures in another nodes. This problem is connected with a necessity to investigate a spread of failures in technological networks. The problem is formulated analogously to a problem of failures tree analysis. But in a case of failures graph which appeared in manifold applications to systems of energy supplement this problem is significantly more complicated. Later we assume that if there is the arc (i,j) in the failures graph then a failure in the node i spreads to the node j.

Consider oriented graph Gwithout fold arcs and without loops. Suppose that its nodes set I consists of $n<\infty$ elements and its arcs set V consists of m elements. This graph may be described by incidence matrix $\|g_{ij}\|_{i,j\in I}$where $g_{ij}=1$ if the arc$(i,j)\in V$ and $g_{ij}=0$ if the arc $(i,j)\notin V$. Further we assume that $g_{ii}=1,\ i\in I$.

Introduce on the nodes set I of the graph G binary relation [4] " $\geq$ ".

Definition1 .Say that $i_1\geq i_2$ if in the graph G there is a way from the node i_1 to the node i_2.

It is obvious that this binary relation is reflexive: $i\geq i,\ i\in I$ and transitive: if $i_1\geq i_2,\ i_2\geq i_3$ then $i_1\geq i_3,\ i_1,i_2,i_3\in I$.

Denote by $I_1(i)=\{j: j\geq i\},\ I_2(i)=\{j: i\geq j\}$ direct and inverse sets of nodes connected with failed node i. To define the family of sets $\{I_1(i),\ I_2(i),\ i\in I\}$ we suggest the following algorithm.Suppose that the graph$\|g_{ij}^k\|_{i,j\in I}$ is defined by the following conditions: $g_{ij}^k=1$ if in the graph G there is a way from the node i to the node j with a length not larger than k. In opposite case $g_{ij}^k=0$. It is obvious that $g_{ij}^1=g_{ij}$.

Theorem 1. The following equalities are true

$$g_{ij}^{k+s} = \max_{r\in I} \min\left(g_{ir}^k, g_{rj}^s\right),\ i \in I, j \in I, k \geq 1, s \geq 1. \tag{1}$$

Proof. Suppose that $g_{ij}^{k+s} = 1$ then there is a way from the node i to the node j with a length not larger than $k + s$. This way may be divided into two ways: from the node i to the node r with a length not larger than $k$and from the node r to the node j with a length not larger that . Consequently $g_{ir}^k = 1$, $g_{rj}^s = 1$ and the formula (1) is true.

Vice-versa assume that $g_{ij}^{k+s} = 0$ and suppose that $\max_{r\in I} \min\left(g_{ir}^k, g_{rj}^s\right) = 1$. Then there is $r \in I$ so that $g_{ir}^k = 1$, $g_{rj}^s = 1$ and consequently there is a way from the node i to the node j with a length not larger than $k + s$ and so $g_{ij}^{k+s} = 1$. This contradiction proves the equality $\max_{r\in I} \min\left(g_{ir}^k, g_{rj}^s\right) = 0$. The equality (1) is true.

Theorem 2. The relation $i \geq j,\ i, j\ \in I$ is true if and only if

$$g_{ij}^{2^l} = 1,\ l = \min(t: 2^t \geq n). \tag{2}$$

Proof. The formula (2) leads to the inequalities $g_{ij}^1 \leq g_{ij}^1 \leq \cdots$. Any way from the node i to the node j with a length not larger than n contains cycles because the number of nodes in the graph Gequals n. So by a deletion of all cycles it is possible to transform a way between the nodes i, j with a length larger than n to a way with a length not larger than n.Consequently the equalities $g_{ij}^n = g_{ij}^{n+1} = \cdots$are proved and so $i \geq j$ is true if $g_{ij}^n = g_{ij}^{2^l} = 1$.

Corollary 1. From Theorems 1,2 we obtain that to calculate the matrix $\left\| g_{ij}^{2^l} \right\|_{i,j\in I}$ it is possible to use the following algorithm

$$g_{ij}^{2^{k+1}} = \max_{r\in I} \min\left(g_{ir}^{2^k}, g_{rj}^{2^k}\right), i \in I, j \in I, 1 \leq k < l \tag{3}$$

with calculation complexity $2ln^3$ where $l \sim log_2 n, n \to \infty$.

Corollary 2. The following equalities are true:

$$I_1(i) = \left\{j: g_{ji}^{2^k} = 1\right\}, I_2(i) = \left\{j: g_{ij}^{2^k} = 1\right\}, i \in I. \tag{4}$$

Remark 1. A specific of the algorithm described by the formulas (3), (4) is that analogously to the Floid and Steinberg algorithm [5] we calculate complete family $\{I_1([i]),\ I_2([i]),\ i \in I\}$ not its representatives. Given algorithm contains logic product of matrices described by the formula (3) and permitting a paralleling by well known methods [6]. To economy of computer memory it is worthy to describe elements of all matrices by logic variables not decimal ones.

3. REPRESENTATIN OF THE SETS $I_1(i), I_2(i)$ BY CLASSES OF NODES

Consider now a structure of results obtained from the formula (4).

Definition2. On the set I introduce binary relation " $\equiv$ "by the condition $i_1 \equiv i_2$ if and only if $i_1 \geq i_2, i_2 \geq i_1$.

Lemma 1. The relation" $\equiv$ " is equivalence relation.

Proof. From Definition 2 we have that binary relation " $\equiv$ " is reflexive and transitive and symmetric.

Consequently the set I may be divided into equivalence classes

$\hat{\imath} = \{i': i' \equiv i\} = \left\{i': g_{i'i}^{2^l} = g_{ii'}^{2^l} = 1\right\}$.

Denote by $\mathcal{I} = \{\hat{\imath}: i \in I\}$ the set of such equivalence classes.

Definition 3. On the set $\mathcal{I}$ introduce binary relation$\hat{\imath}_1 \geq \hat{\imath}_2 \Leftrightarrow \exists i_1' \in \hat{\imath}_1, i_2' \in \hat{\imath}_2 : i_1' \geq i_2'$.

Lemma 2. Binary relation " $\geq$ "on the set $\mathcal{I}$ is a relation of partial order and is defined by the condition $\hat{\imath} \geq \hat{\jmath} \Leftrightarrow g_{\hat{\imath}\hat{\jmath}}^{2^l} = 1$.
2. If $\hat{\imath}_1 \geq \hat{\imath}_2$ then for any $i_1' \in \hat{\imath}_1, i_2' \in \hat{\imath}_2$ the formula $i_1' \geq i_2'$ is true.
Proof. 1. From Definitions 2, 3 we have that partial order " $\geq$ " on the set $\mathcal{I}$ of equivalence classes is refle-xive and transitive and antisymmetric and consequently is the relation of partial order. The formula $\hat{\imath} \geq \hat{\jmath} \Leftrightarrow g_{\hat{\imath}\hat{\jmath}}^{2^l} = 1$ is obvious.
2. Assume that $\hat{\imath}_1 \geq \hat{\imath}_2$ consequently for any $i_1' \in \hat{\imath}_1, i_2' \in \hat{\imath}_2$ it is possible to construct in the graph G a way from the node i_1' to the node $i_2'$and so $i_1' \geq i_2'$.
Corollary 3. The following formulas are true

$$I_1(i) = \cup_{\hat{\jmath} \geq \hat{\imath}} \hat{\jmath}\,,\; I_2(i) = \cup_{\hat{\imath} \geq \hat{\jmath}} \hat{\jmath}\;, i \in I. \tag{5}$$

On the set $\mathcal{I}$ of equivalence classes using the relation of partial order " $\geq$ " it is possible to construct the oriented graph $\mathcal{G}$ with the set of arcs $\mathcal{V}$ using the following procedure. Suppose that $\hat{\imath} \geq \hat{\jmath}$ and there are nodes $i' \in \hat{\imath}, j' \in \hat{\jmath}$ so that $(i', j') \in V$ then the arc $(\hat{\imath}, \hat{\jmath}) \in \mathcal{V}$. It is obvious that the graph $\mathcal{G}$ is acyclic and
without fold arcs. It is possible to restore the relation " $\geq$ "using an analog of Definition1. Denote by $\|g_{\hat{\imath}\hat{\jmath}}\|_{\hat{\imath}\hat{\jmath} \in \mathcal{I}}$ incidence matrix of the graph $\mathcal{G}$ and define $\left\|g_{\hat{\imath}\hat{\jmath}}^{2^l}\right\|_{\hat{\imath}\hat{\jmath} \in \mathcal{I}}$ as connectivity matrix of the graph $\mathcal{G}$ nodes. It is obvious that a calculation of the matrices $\|g_{\hat{\imath}\hat{\jmath}}\|_{\hat{\imath}\hat{\jmath} \in \mathcal{I}}$, $\left\|g_{\hat{\imath}\hat{\jmath}}^{2^l}\right\|_{\hat{\imath}\hat{\jmath} \in \mathcal{I}}$ by known matrices $\|g_{ij}\|_{ij \in I}$, $\left\|g_{ij}^{2^l}\right\|_{ij \in I}$ demands not larger than $2n^3$ arithmetical operations.

4. MINIMIZATION OF ARCS NUMBER IN GRAPH $\mathcal{G}$

Consider the graph $\mathcal{G}$ with the set $\mathcal{I}$ of nodes and the relation of partial order " $\geq$ " and the set of arcs $\mathcal{V}$. Our problem is to remove from the set $\mathcal{V}$ maximal possible subset of arcs so that the matrix of nodes connectivity $\left\|g_{\hat{\imath}\hat{\jmath}}^{2^l}\right\|_{\hat{\imath}\hat{\jmath} \in \mathcal{I}}$ does not change and consequently the sets $\{\hat{\jmath}: \hat{\jmath} \geq \hat{\imath}\}$, $\{\hat{\jmath}: \hat{\imath} \geq \hat{\jmath}\}$, $\hat{\imath} \in \mathcal{I}$ from the formula (5) do not change also. This procedure of a minimization of arcs number is necessary to make failures graph maximally transparent and compact.

Assume that the set $\mathcal{V}$ contains the arc $(\hat{\imath}, \hat{\jmath})$.

(A). Suppose that there is the node $\hat{k} \in \mathcal{I}$, $\hat{k} \neq \hat{\imath}$ such that $g_{\hat{\imath}\hat{k}}^{2^l} g_{\hat{k}\hat{\jmath}}^{2^l} = 1$. Then the arc $(\hat{\imath}, \hat{\jmath})$ is to be deleted from the set $\mathcal{V}$. As the graph $\mathcal{G}$ is acyclic so a way $\hat{\imath}, \hat{k}, \hat{\jmath}$ does not contain the arc $(\hat{\imath}, \hat{\jmath})$ and consequently a deletion of this arc from the set $\mathcal{V}$ does not change the connectivity matrix $\left\|g_{\hat{\imath}\hat{\jmath}}^{2^l}\right\|_{\hat{\imath}\hat{\jmath} \in \mathcal{I}}$. If the condition (A) is not true then the arc $(\hat{\imath}, \hat{\jmath})$ remains in the set $\mathcal{V}$ and the matrix $\left\|g_{\hat{\imath}\hat{\jmath}}^{2^l}\right\|_{\hat{\imath}\hat{\jmath} \in \mathcal{I}}$ does not change also.

As a deletion of different arcs from the set $\mathcal{V}$ is realized independently so we obtain minimal possible number of arcs in the set $\mathcal{V}$and this solution is unique. Algorithm of the number of arcs in the graph $\mathcal{G}$ minimization may be easily parallelized because a rejection of arcs is realized independently. It is not difficult to obtain that this procedure demands not larger than$2n^3$arithmetic operations.

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