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ASYMPTOTICS OF CONNECTIVITY PROBABILITY OF GRAPH WITH LOW RELIABLE ARCS

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ABSTRACT

In this paper a problem of asymptotic estimate for connectivity probability of non oriented connected graph with fold and low reliable arcs is solved. An algorithm of a calculation of asymptotic constants with cubic complexity by a number of nodes is constructed. This algorithm is based on Kirchhoff`s theorem for a calculation of a number of spanning trees and relative characteristics.

1. INTRODUCTION

A problem of connectivity probability calculation for random graphs with low reliable arcs is considered in manifold articles. In [1], [2] upper and lower bounds for connectivity probability (reliability polynomial) in general type networks are constructed using maximal systems of non intersected skeletons. Networks for which these bounds equal zero are constructed. In [3] connectivity probability for multi graphs (graphs with fold arcs) is analysed. For relatively small number of arcs in [4] accelerated algorithms of reliability polynomial coefficients calculation are constructed. These algorithms showed sufficiently good results in a comparison with Maple 11. In [5] this problem is solved using Monte-Carlo method with some specific combinatory formulas.

But for large number of arcs these calculations become sufficiently complicated. So a problem of a construction of sufficiently convenient asymptotic formulas for a calculation of connectivity probability in graphs with high reliable or low reliable arcs becomes actual. First problem is solved in [6] for planar graphs using Burtin-Pittel asymptotic formula [7], Whithney theorem about a relation between cross sections in planar graphs and cycles in dual graphs [8] and Harary formulas [9] for numbers of simple (without self intersection) cycles with length not larger than 5 in non oriented connected graph.

In this paper a problem of asymptotic estimate for connectivity probability of non oriented connected graph with fold and low reliable arcs is solved. An algorithm of a calculation of asymptotic constants with cubic complexity by a number of nodes is constructed. This algorithm is based on Kirchhoff`s theorem for a calculation of a number of spanning trees [11] and relative characteristics. It is worthy to note that last years this theorem and its manifold generalizations are widely used in different disciplines: physics of elementary particles, electromagnetism, acoustics, elasticity theory, sociology, biology etc. [10] – [12]. In this paper Kirchhoff`s theorem is used in reliability theory.

2. MAIN RESULTS

Assume that non oriented and connected and simple (without loops and fold arcs) graph G has the nodes set $U=\{1,\ldots,n\}$ and the arcs set V. Each arc $v=(i,j)$, $1\leq i\neq j\leq n$, of the graph G works with the probability $p(v)$ and fails with the probability $1-p(v)$ and arcs work independently. We shall analyse connectivity probability $p(G)$ of the graph G with randomly working and low reliable arcs. Define Kirchhoff`s matrix $K=\|k(i,j)\|_{i,j=1}^{m}$ of the graph G :

$$k(i,j)=\begin{cases} degree\ of\ \ node\ i, & i=j, \\ \quad -1, & (i,j)\in V, \\ \quad 0, & else. \end{cases}$$

Here node`s degree is a number of its incident arcs. Spanning tree of the graph G is a tree with the nodes set U and with the arcs set which contains in V. Denote m the number of spanning trees of the graph G.

Theorem 1. If $p(v)=h,\ h\in V$ then for $h\to 0$ we have the equality

$$P(G)=mh^{n-1}\big(1+O(h)\big) \tag{1}$$

where calculation complexity of the coefficient m definition is $O(n^3)$.

Proof. Formula (1) is proved in [2, Formula (5)]. From Kirchhoff`s-Trent theorem (see for example [10]) algebraic complements of all elements of Kirchhoff`s matrix K are equal to the number m of all spanning trees in the graph G. It is well known (see for example [13]) that a calculation of a determinant with the order $n-1$ and consequently a calculation of the coefficient m by the Gauss method demands $O((n-1)^3)$ arithmetical operations.

Corollary 1. If for some natural $l(v)$ the probability $p(v)=h^{l(v)}$ then each arc in the graph G may be replaced by $l(v)$ serial connected arcs , $v\in V$, and then Theorem 1 may be applied to this graph .

This corollary may be used to analyse random time to a failure of network connectivity if analogous characteristic for an arc is $p(v)=P(\tau(v)>T)$, $T\to\infty$.

Theorem 2. If for some positive $s(v)$ the probabilities $p(v)\sim s(v)h,\ h\to 0$, $v\in V,$ then

$$P(G)=m_1h^{n-1}\big(1+O(h)\big)\,,\ h\to O\,, \tag{2}$$

Here m_1 is algebraic complement of each element (which are equal) of the matrix $K_1=$ $=\|k_1(i,j)\|_{i,j=1}^{m}$ where

$$k_1(i,j)=\begin{cases} \displaystyle\sum_{t\in V,(i,t)\in V} s(i,t)\,, i=j \\ -s(i,t),(i,j)\in V \\ 0, else. \end{cases}$$

Proof. Denote $G_1,\ldots,G_m$ spanning trees of the graph G. Each of them has $n-1$ arcs. Put A_k the event that all arcs in the tree G_k work, $1\leq k\leq m$. Then we have: $P(G)=P(\bigcup_{k=1}^{m}A_k)$ and so $\sum_{k=1}^{m}P(A_k)-\sum_{1\leq k_1<k_2\leq m}P\big(A_{k_1}A_{k_2}\big)\leq P(G)\leq\sum_{k=1}^{m}P(A_k)$.

From the conditions on $p(v)$ we obtain that

$\sum_{k=1}^{m}\prod_{v\in G_k}hs\,(v)-O(h^n)\leq P(G)\leq\sum_{k=1}^{m}\prod_{v\in G_k}hs\,(v)$,

$$\sum_{k=1}^{m}\prod_{v\in G_k}hs\,(v)=h^{n-1}\sum_{k=1}^{m}\prod_{v\in G_k}s\,(v)$$

Designate $m_1 = \sum_{k=1}^{m} \prod_{v \in G_k} s(v)$ and obtain Formula (2). From a generalization of Kirchhoff`s-Trent theorem (see for example [11, Theorem 1]) we obtain that m_1 coincides with algebraic complement of each element of the matrix K_1 and calculation complexity of its definition is $O(n^3)$.

Remark 1. Theorem 2 statement may be spread onto multi graph constructed from the graph G by a replacement of each arc $v \in V$ by $s(v)$ parallel and independently working copies of this arc. This parallel connection has working reliability $\sim s(v)h$, $h \to 0$.

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