MARKOV MODELS FOR TENSILE AND FATIGUE RELIABILITY ANALYSIS OF UNIDIRECTIONAL FIBER COMPOSITE

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Abstract

This paper is a review integrating, amending, and developing the approach applied in authors' previous works devoted to the tensile and fatigue reliability analysis of unidirectional composite material considered as a series system the links of which are, in general case, complex parallel systems with redistribution of load after failure of some items. By processing experimental data it is shown that the models based on the Markov chains (MCh) theory allow (1) to describe connection of cdf of tensile strength of fibers (strands) and a composite specimen, (2) to perform nonlinear regression analysis of fatigue curve and prediction of its changes due to a change of tensile strength characteristics of the composite components, (3) to predict the fatigue life at a program loading, (4) to estimate the cdf of the residual strength and residual life after a preliminary fatigue load.

Keywords: Composite, Fiber, Fatigue life, Strength

1. INTRODUCTION

The distribution of static strength, the fatigue curve, and the accumulation of fatigue damage under a program loading are often described by poorly interconnected hypotheses. The distribution of static strength is usually analyzed by the Weibull or lognormal distributions, while the fatigue curve is described by formal regression dependences. The accumulation of fatigue damage under a program loading, as a rule, is carried out by using the Palmgren-Miner hypothesis or its modifications. Here we consider the application of the Markov chain (MCh) theory for a unified approximate solution of the mentioned problems. Application of the Markov process theory for specific problems is discussed in several publications (see, for example, [1]-[2] as the most interesting) but the idea of connection of cdfs of tensile strength, fatigue life, residual strength and residual fatigue life (after some preliminary fatigue loading) with the cdf of tensile strength of a composite material component is relatively new. First steps in that direction were made in 1980 [3]. This paper is a review integrating, amending, and developing the approach applied in authors' previous works [4-7] "furnished" with examples of solution of the above-mentioned problems for unidirectional fibrous composite within the framework of some specific case of unified mathematical model.

Actually this paper is an addition to [4]. The main new idea of the paper is to show that nearly the same type of MCh model can be used for the case of tensile strength, as well as for the case of fatigue life analysis of a composite material. A new idea of using random Daniels sequence is discussed also.

2. MODEL OF A UNIDIRECTIONAL FIBER-REINFORCED COMPOSITE MATERIAL. MAIN IDEAS

We consider the composite specimen as a series system with n_L links, a random number of which, K_L, $1 \le K_L \le n_L$ have defects. We call them weak micro-volumes (WMV). Contrary to a general set of probability structure (ps) for a single fiber (strand), described in [4], in which the failure of both types of links (with defect and without defect) can be the cause of failure of specimens here we suppose that failure of the specimens can be only as the consequence of failure of some WMV. This assumption is equivalent to the assumption that the strength of links without defects is equal to infinity or very large.

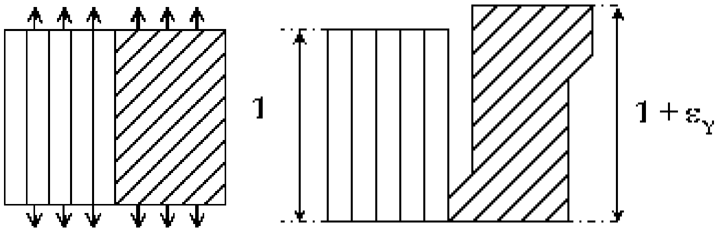

Fig.1. Model of the weak microvolume of a composite under a load and after removal of the load .

We suppose that in general case the WMW consists of n_C perfectly elastic (brittle) longitudinal items (LI) (fibers or strands) and a matrix where plastic deformations are accumulated if cyclic loading takes place (Fig.1). We assume that, except for the LIs, the plastic part includes all other composite components, i.e. the matrix itself and all the layers with stacking different from the longitudinal one! And we assume finally that, if the number of LIs in the WMV decreases by r_R items, the elastic part of the specimen breaks down, which is followed by the failure of the specimen as a whole. The total number of LIs in one WMV, n_C, in general case can be more than r_R but we suppose that failure of (r_R / n_C)-th part of elastic LIs of WMV is considered as failure of elastic part and the whole WMV also by definition. The value of $\bar{f}_C = r_R / n_c$ is a parameter of the model. The slanted hatching in Fig.1 symbolically points to the possibility of accumulating an irreversible plastic strain. If it exceeds some critical level ε_{YC}, the failure of the WMV and the specimen as a whole takes place also. We emphasize that this graphic image, as applied to a composite, should be understood symbolically. It is more suitable for metals, where the accumulation of plastic strains is associated with some "act of flow" (for metals - a shift over slip planes). We assume that one act of flow leads to the appearance of a constant plastic strain ε_{Y1}. The failure of WMV takes place after the accumulation of a "critical" number of such acts, r_Y, i.e., after the accumulation of a critical plastic strain, ε_{YC}, for which the relation $\varepsilon_{YC} = \varepsilon_{Y1}\, r_Y$ is valid, where ε_{YC} and r_Y are model parameters. Since the elastic and plastic parts are integrated in a unit, the accumulation of plastic strains (irreversible deformation of the plastic part) leads to the appearance of residual stress: tension in the elastic and compression in the plastic part of the specimen [4].

For description of the process of failure of WMV using MCh theory we should provide the description of space of states of MCh and its connection with the structure of the composite WMV, the corresponding structure of the matrix of transition probabilities and its connection with the cdf of mechanical characteristic of the component of WMV, the process of loading.

2.1. Probability description of WMV

2.1.1. Description of space of states and transition probability matrix

As it was already mentioned, a set of probability structures (ps) for description of **fiber (or strand) as series system** is considered in [4]). Now we consider probabilities structure of specimen but, first, ps of one WMV.

Let us, in general case, associate the process of gradual failure of a WMV with an absorbing MCh the set of states of which is determined by the number of broken LIs and the number of acts of flow. The matrix of transition probabilities is presented as a totality of (r_Y+1) blocks with (r_R+1) states within each of them. Then, the indices of input and output states, *i* and *j*, can be expressed in terms of the corresponding local indices i_Y, i_R, j_Y and j_R: $i=(r_R+1)(i_Y-1)+i_R$; $j=(r_R+1)(j_Y-1)+j_R$.

Table 1 shows an example of (symbolic) filling of the matrix for the case where $r_Y=r_R$=2 for independent failure of LI and act of flow. In this case the destruction of a specimen occurs if two LIs fail (event A), or two acts of flow are accumulated (event B), or events A and B take place simultaneously. The absorbing states of the MCh correspond to these events. In the example considered, there are (r_Y+1)(r_R+1) = 9 states. The symbols p_{R0}, p_{R1}, … designate the probabilities of failure of the corresponding numbers of elastic (rigid) elements; p_{Y0}, p_{Y1}, … are the probabilities of the corresponding numbers of acts of flow (yielding).

TABLE I

EXAMPLE OF THE TRANSITION PROBABILITY MATRIX

		j_Y	*1*			*2*			*3*		
		j_R	*1*	*2*	*3*	*1*	*2*	*3*	*1*	*2*	*3*
i_Y	i_R	i \ j	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	*1*	**1**	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	$p_{R2}p_{Y0}$	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$	$p_{R2}p_{Y1}$	$p_{R0}p_{Y2}$	$p_{R1}p_{Y2}$	$p_{R2}p_{Y2}$
	2	**2**	0	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	0	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$	0	$p_{R0}p_{Y2}$	$p_{R1}p_{Y2}$
	3	**3**	0	0	1	0	0	0	0	0	0
2	*1*	**4**	0	0	0	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	$p_{R2}p_{Y0}$	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$	$p_{R2}p_{Y1}$
	2	**5**	0	0	0	0	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	0	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$
	3	**6**	0	0	0	0	0	1	0	0	0
3	*1*	**7**	0	0	0	0	0	0	1	0	0
	2	**8**	0	0	0	0	0	0	0	1	0
	3	**9**	0	0	0	0	0	0	0	0	1

Since the local order number of state is defined by the number of failed LIs, it is connected also with the local stress in intact LIs. The set of states of MCh can be connected also not only with the number intact LIs but with the set of corresponding values of local stress. For tensile test it is more convenient to use the connection with the intact (or failed) LIs. For fatigue test it is more convenient to consider the connection with the local stress (see the definition of Daniels' sequence in the following).

Consider now the simplest **special case** (the most interesting for **tensile** test of a unidirectional composite) when there are only n_C LIs (fibers or strands). Equality $r_R=n_C$ is used for the definition of failure of WMV, and the existence of composite matrix is not taken into account. In this case the WMV is a parallel system with (n_C-K_C) LIs, where n_C is a constant (initial number of LIs without any defect), K_C, $0\le K_C\le n_C$, is the number of failures of LIs in the link. Note that in this case the equality $K_C=n_C$ means the failure of link (WMV) and the specimen also.

Note also, that for this type of WMV there is only one block in the transition probability matrix P corresponding to $r_Y = 0$, $r_R = n_C$. The number of states of MCh is equal to ($r_R + 1$).

Note again, that it is not necessary to connect the failure of WMV with equality $K_C = n_C$. We can in general case to define the failure of unidirectional WMV as the event when the intact part of LIs becomes less than some critical value $\bar{f}_C$: $(n_C - K_{Ci})/n_c < \bar{f}_c$. In this case $r_R = [\bar{f}_C n_C] + 1$, where $[x]$ is the integer part of x.

For the considered type of WMV the four main versions (hypotheses) of the structure of matrix P, denoted as P_a, P_{an_C}, P_b and P_c, are considered in [4]. Matrix P_a corresponds to the assumption that, in one step of MCh, only one LI can fail, and it is the nearest one to the already failed LIs (in some way this corresponds to development of a crack); P_{an_C} corresponds to the failure of the weakest item in the considered WMV(cross section); P_b corresponds to a binomial distribution of the number of failures at every step of MCh; P_c corresponds to the case when the stress concentration function is known.

Total initial load of this type of WVM is equal to Sn_C, where S is the initial stress (load of one LI). For the three first types of matrix a uniform distribution of load between intact LIs is supposed. Then, if the number of failed LIs is equal to i, the stress in the still intact LIs will be equal to $Sn_C/(n_C - i)$. For the matrix of type P_c the function of stress distribution across the cross section of WMV should be known (see the details in [4]). This connection of the local stress and the number of states of MCh should be taken into account for calculation of matrix of transition probabilities.

The corresponding set of states can be used also for modeling of **fatigue** test (again, see details in [4]). But in [6,7] the set of MCh states is connected with the random Daniels' sequence (RDS). The components of RDS, $\{s_0, s_1, s_2, ...\}$, correspond to the **random** process, a realization of which has the following form: $s_{i+1} = S/(1 - \hat{F}_{X_s, n_C}(s_i)),\ i = 0,1,2,...,n_C$, where $s_0 = S$, S is the initial stress (initial load of one LI), $\hat{F}_{X_s, n_C}(.)$ is the estimate of cdf of strength of a LI, which is defined by some sample $(x_{s1}, ..., x_{sn_C})$ of observations of n_C random variables (random strength of n_C LIs) with the same cdf $F_{X_s}(x)$. Here we use the following definition: $\hat{F}_{X_s, n_C}(x) = k(x)/n_C$, where $k(x)$ is the number of observations which are lower than x. or equal to x.So here $(x_{s1}, ..., x_{sn_C})$ is a vector of observations of random strengths, $X_{s1}, ..., X_{sn_C}$, of components of some WMV, $s_{i+1} = S/(1 - k(s_i)/n_C)$, $i = 0,1,2,...,n_c - 1$. In following we suppose that $(x_{s1}, ..., x_{sn_C})$ is the vector of ordered statistics: $x_{s1} \le x_{s2} \le ... \le x_{sn_C}$.

The increase of local stress corresponds to decreasing of local cross section (reduction of the number of intact components of WMV). Let us again define that the failure of WMV takes place if local cross section become less than some value $\bar{f}_C$ (initial total cross section area of WMV is equal to one). Here $\bar{f}_C$ is a constant, a parameter of the considered model. Then critical local stress corresponding to this event, S^*_{UT}, is defined as minimum of stress, s, for which the part of LIs with strength less than s is more than $\bar{f}_C$: $S^*_{UT} = \min\{s : \hat{F}_{X_s}(s) \ge \bar{f}_C,\ s \in \{s_0, s_1, s_2, ...\}\}$, where $\{s_0, s_1, s_2, ...\}$ is RDS. The random number $N_{RDS} = \max\{i : s_i < S^*_{UT},\ s_i \in \{s_0, s_1, s_2, ...\}\}$, we call as RDS fatigue life (RDSFLf) at stress S. Here s_i is an item of RDS (for specific S, for specific realization $x_{s,1-n_C} = (x_{s1}, ..., x_{sn_C})$ of random vector of ordered statistics $X_{s,1-n_C} = (X_{s1}, ..., X_{sn_C})$).

Let us define the random function $S_{DFLm}(k_S, x_{s,1-n_C})$ the value of which is equal to the maximum value of S for which in RDS with $s_0 = k_S S$ and specific $x_{s,1-n_C}$ there is some i for which $s_i = s_{i+1} = s_{i+2} = ... = \max(s_0, s_1, s_2, ...) = s^*(S, k_S, x_{s,1-n_C}) < \infty$. Growth of stress in RDS is stopped after it reaches $s^*(S, k_S, x_{s,1-n_C})$ which is the solution of the equation $x = k_S S / (1 - \hat{F}_{X_s}(x))$ or $k_S S = x(1 - \hat{F}_{X_s}(x))$. We call $S_{DFLm}(k_S, x_{s,1-n_C})$ the k_S-RDS-fatigue-limit (k_S-RDSFLm)) because if initial stress, S, is lower its value then corresponding k_S-RDSFLf (for $s_0 = k_S S$ and specific $x_{s,1-n_C}$) is equal to infinity. Existence of k_S-RDSFLm explains the phenomenon of random fatigue limit.

Solution of the equation $k_S S = x(1 - \hat{F}_{X_s}(x))$ exists if $k_S S \le \max x(1 - \hat{F}_{X_s}(x))$. In accordance with Daniels [8,9] the strength of a parallel system of n_C LIs is random variable $S_D = \max x(1 - \hat{F}_{X_s}(x))$ with asymptotically normal distribution $N(\mu_D, \sigma_D^2)$, where parameters μ_D and σ_D are defined by the cdf of strength of LI . By fitting the fatigue life data by k_S-RDS-MCh model we can find an appropriate estimate parameters of the model including k_S. Then we can make estimate the probability that k_S-RDSFLf is equal to infinity: $p_{\inf} = P(T_C = \infty \mid S) = P(S \le S_{DFLm}(k_S, x_{s,1-n_C})) = \Phi\left((k_S S - \mu_D)/\sigma_D\right)$.

Example of Monte Carlo calculation of "normalized" RDS, ($(1/k_S)$ $\{s_0, s_1, s_2, ...\}$, $s^*(S, k_S, x_{s,1-n_C})$ and $p_{\inf}$ for different k_S and 10 random realizations of $x_{s,1-n_C}$ was provided in [7] . Similar result for the same initial data is shown in Fig. 2.

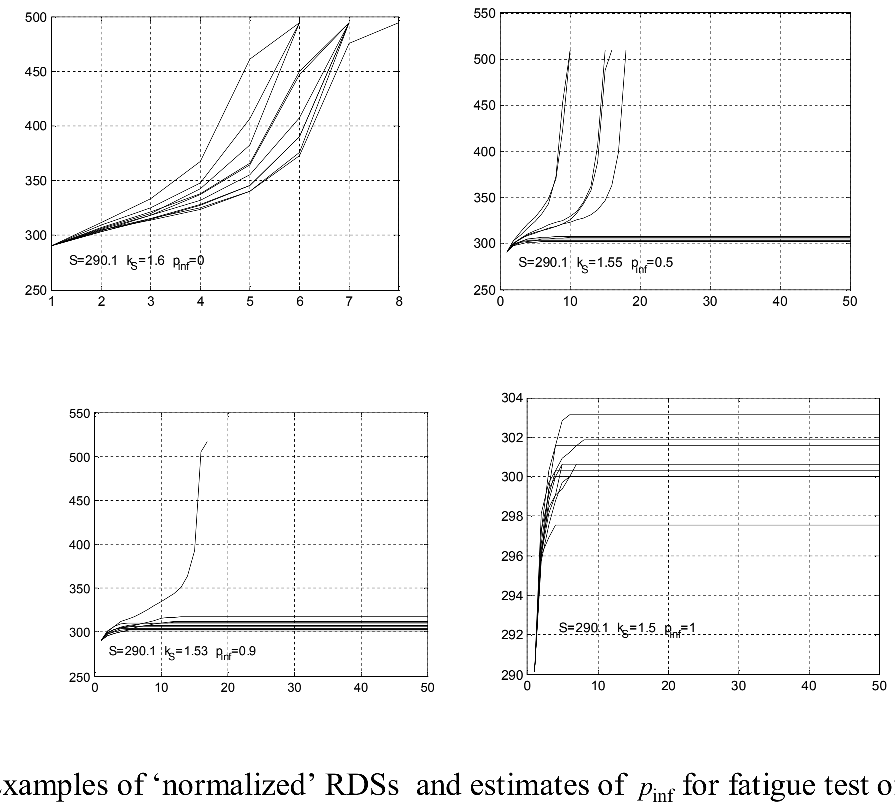

Fig. 2. Examples of 'normalized' RDSs and estimates of $p_{\inf}$ for fatigue test of carbon-fiber composite [5] for S = 290.1 MPa and different k_S (see [7])

If for calculation RDS we use $k_S = 1$ then RDSFLfs are very small, RDSFLm is very high (see [4,6,7]). So although the use of RDS provides a qualitative explanation of fatigue failure of a composite material and can also explain the phenomenon of fatigue limit, the quantitative prediction is very poor.

But the possibility of explanation of the existence of phenomenon of fatigue limit is very attractive. The very high value of RDSFLm can be explained by existence of local stress concentration, i.e. instead of equality $s_0 = S$ (see [6]) the initial stress in RDS should be defined by equality $s_0 = k_S S,$ where k_S is a local stress concentration coefficient. The probability characteristics of fatigue life and appropriate description of fatigue curve in the framework of this model can be fitted to the real characteristics of fatigue life using the theory of MCh with space of states based on RDS.

For this type of MCh the first r states of state space are related with the items of RDS, $\{s_0, s_1, ..., s_{r-1}\}$, s_r is connected with the absorbing state . In Fig.2 we see two types of RDS. For RDS of first type items of RDS grow up to infinity. For this type of RDS-MCh model, absorbing state is connected with the event that the local stress is equal or larger than S_{UT}^*. For the second type of RDS there is a final limit and absorbing state should be connected with $s^*(S, k_S, x_{s,1-n_C})$. In the simplest case it can be assumed that only transitions to the nearest 'senior' state can take place. So the following simple matrix of transition probabilities can be considered:

$$P = \begin{bmatrix} q_1 & p_1 & 0 & & & \cdots & 0 \\ 0 & q_2 & p_2 & 0 & & \cdots & 0 \\ 0 & 0 & q_3 & p_3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & & \cdots & 0 & q_r & p_r \\ 0 & & & \cdots & 0 & 0 & 1 \end{bmatrix},$$

where $p_i = (k(s_i) - k(s_{(i-1)}) / (n_C - k(s_{(i-1)}))$, $q_i = 1 - p_i$. Note that here P is a realization of random matrix $P = P(x, X_{s,1-n_C})$. It is a function of load x and random vector of strength of n_C LIs, $X_{s,1-n_C} = (X_{s,1}, ..., X_{s,n_C})$. So all results of calculation using this matrix will be random if n_C is not large enough. In order to get mean results the Monte Carlo method can be used. But if n_C is large enough then, for example, if there is normal distribution, $N(\theta_0, \theta_1^2)$, of logarithm of strength of LI then the items of matrix P , approximately, can be defined in following way : $p_1 = \Phi((\log(k_S S) - \theta_0) / \theta_1)$; $s_2 = k_S S / (1 - p_1)$; $p_i = (p_{ic} - p_{(i-1)c}) / (1 - p_{(i-1)c})$,

where $p_{ic} = \Phi((\log(s_i) - \theta_0) / \theta_1)$, , $s_{i+1} = k_S S / (1 - p_{ic})$, $i = 1, 2, ..., r$.

and the corresponding nonrandom matrix P can be used. A numerical example for this special case is considered in [6].

2.1.2. Description of the process of loading. CDF of tensile strength and fatigue life of WMV

By renumbering the states, the matrix of transition probabilities of any absorbing MCh can be reduced to the form

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix},$$

where *I* is the unit matrix and 0 is the matrix consisting of zeros.

As it was told already the matrix *P* is a function of **loading** stress, x. **Loading process** in **tensile** test is described by an ascending up to infinity sequence $x1\infty = \{x_1, x_2, ..., x_t, ...\}$. The cdf of the number of steps up to absorption, T_A, is defined by matrix P, a priori probability distribution on MCh states, π, and by an sequence of loads, $x1\infty$:

$$F_{T_A}(t) = \pi(\prod_{j=1}^{t} P(x_j))u, \quad x_j \in \{x_1, x_2, ..., x_t, ...\} \tag{1}$$

where vector-column $u = (0,0,...,1,...,1)'$ has only zeros and units (for absorbing states). By the choice of π we can model different levels of a priori quality of tested specimens (for example, the modified binomial distribution of initial number of intact LIs in one WMV can be modeled).

The load corresponding to the time to absorption, x_{T_A} is a random tensile strength. If $x_t = g(t)$, where $g(.)$ is monotonicaly increasing function for which there is the inverse function $g^{-1}(.)$, then cdf of random variable $X = x_{T_A}$ is

$$F_X(x) = P(g(T_A) \le x) = F_{T_A}(g^{-1}(x)), \quad x \in x1\infty. \tag{2a}$$

For **fatigue** test for estimation of fatigue life at a stress level x, all items in the sequence $x1\infty = \{x_1, x_2, ..., x_t, ...\}$ are equal: $x_1 = x_2 = x_3 = ... = x$ where x is some parameter of cycle (for example, x is a maximum stress of pulsating cycle). Then fatigue life (cycle number up to failure) is equal to $T_C = k_m T_A$ cycles, where k_m, $k_m > 0$, is a scale factor, i.e. it is the number of cycles corresponding to one MCh step. In this case $\prod_{j=1}^{t} P(x_j) = P^t(x)$ and $F_{T_A}(t) = \pi P^t(x)u$, $t = 0,1,2,...$. Cdf of the number of cycles up to failure, T_C, is defined by equation

$$F_{T_C}(n) = F_{T_A}(n / k_m) = \pi P^{n/k_m}(x)u, \quad n = 0, k_m, 2k_m, 3k_m, ... \tag{2b}$$

Note again that we have random results of equations (1, 2a, 2b) if we have random matrix $P = P(x, X_{s,1-n_C})$ (if MCh state space is defined by RDS) but using Monte Carlo method (or if n_C is large enough) we can get approximately determined results.

Examples of equations for calculation of P for fatigue loading by pulsating cycle taking into account presence of the plastic part of WMV are given in [4]. A version of "translation" of any cycle with any other asymmetry into a pulsating cycle is given in [5].

Let us denote by S_n the **conditional** fatigue limit at n cycles of load. Then cdf of S_n is

$$F_{S_n}(x) = P(T_C \le n \mid S = x) = \pi P^{[n/k_m]}(x)u,$$

where P is defined by stress x, $[x]$ is the integer part of x, π and u are the same as previously.

It is worth to note that probability that fatigue life is larger than n at stress S is defined by equation $P(T_C > n \mid S = x) = 1 - \pi P^{[n/k_m]}(x)u$. This probability we can estimate relatively easy. But it is very difficult to estimate function $F_{S_n}(x)$.

As it was shown already, using RDS models we can estimate the probability that fatigue life at a stress level S is equal to infinity.

Let us denote $P^t = \begin{bmatrix} Q_t & R_t \\ 0 & I \end{bmatrix}$. Different columns of the matrix $B_t = (I - Q_t)R_t$ define the probabilities of absorption (failure) in different absorbing states for different initial states (rows).

For example, matrix element in the right upper corner of B_t corresponds to probability of failure of the WMV as consequence of failure of all LIs (or, in general case, for the composite matrix the relation $\varepsilon_{YC} = \varepsilon_{Y1}\, r_Y$ is reached) at MCh step number t if the initial state $i = 1$ (see Table 1).

But if we do not need this detailed information then instead of matrix P we can use the matrix P_U in which all the absorbing states are united in one absorbing state. The number of states of matrix P_U is equal to $h = r_Y r_R + 1$. The matrix P_U is useful for analysis of fatigue life in program loading. Cdf of fatigue life for a program loading defined by the sequence $x1\infty$ corresponding to some specific program of loading can be calculated again using equations (2a) and (2b). Consider specific two-stage fatigue loading: $x1t = \{x1t_1, xt_{1+1}\infty\} = \{\{x_1, x_2, ..., x_{t_1}\}, \{x_{(t_1+1)}, x_{(t_1+2)}, ..., x_{(t_1+\Delta t)}, ...\}\}$; $x_1 = x_2 =, ..., = x_{t_1} = x^I$, $x_{(t_1+1)} = x_{(t_1+2)} =, ..., = x_{(t_1+\Delta t)} = ... = x^{II}$, x^I is the stress in the first fatigue loading stage, x^{II} is the stress in the second fatigue loading stage. After preliminary loading $x1t_1 = \{x_1, x_2, ..., x_{t_1}\}$ an a priori distribution π_0 is transformed into „a posteriori" distribution $\pi_{t_1} = \pi_0 \prod_{j=1}^{t_1} P(j)$. Using π_{t_1} instead of π in equations (2a) and (2b) we can get the cdf of both residual strength and residual fatigue life ΔT in two-stage fatigue loading.

It is necessary to note that usually we are interested in these characteristics only for the specimens which are still intact after the preliminary fatigue loading. The components of „a posteriori" conditional distribution for them are: $\pi_{t_1 c}(k) = \pi_{t_1}(k)/(1 - \pi_{t_1}(h))$ for $k = 1, 2, ..., (h-1)$, $\pi_{t_1 c}(h) = 0$. This distribution, matrix P_U and the second stage loading $xt_{1+1}\infty = \{x_{(t_1+1)}, x_{(t_1+2)}, ..., x_{(t_1+\Delta t)}, ...\}$ should be used in (2a) and (2b) instead of π, P and $x1\infty$ for calculation of cdf of residual step number ΔT_A up to absorption (WMV failure). Now the residual strength is rv $x_{(t_1+\Delta T_A)}$ (see also in [4] the definition of the so called k_m-residual strength which defines the stress of fatigue cycle which produces the fatigue failure in k_m fatigue cycles which are equivalent to one step of MCh. Its cdf is defined by equation

$$F_{S_{x1t_1}}(x_j) = \pi_{t_1 c} P(x_j) u ,$$

where $x_j \in \{x_{t_1+1}, x_{t_1+2}, ...\}$, $x_{t_1+1} \ge x^I$, $\{x_{t_1+1}, x_{t_1+2}, ...\}$ is an ascending up to infinity sequence of stresses in "residual" tensile strength test) ,

2.2. Probability description of a specimen

We suppose that in the simplest case, neglecting the presence of composite matrix in a unidirectional composite (sequence of links), there are two types of links: there are K_L, $1 \le K_L \le n_L$, links with defects and ($n_L - K_L$) without defects. In damaged links, we call them as WMV, there are only ($n_C - K_C$) LIs, $1 \le K_C \le n_C - 1$; K_C LIs are failed. So now WMV is a parallel system with ($n_C - K_C$) items. Recall that the equality $K_C = n_C$ means the failure of WMV and the specimen also. There are a priori probability mass functions (pmf) of random variables K_L and K_C : p_{K_L} and p_{K_C}.

In general case some WMVs can appear before but another during tensile or fatigue loading.

2.2.1. All WMV appear before test

Let $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, be the number of failures of LIs in the *i*-th link at the tensile load x_t, $x_t \in \{x_1, x_2, ..., x_t, ...\}$, $x_1 < x_2 < x_3 ... < x_t < x_{t+1} < ...$ Load increases up to infinity. Then the number of steps of load increasing up to tensile failure of the *i*-th WMV

$$T_{Ai} = \max(t : n_C - K_{Ci}(t) \geq 0), \tag{3}$$

Two approaches for describing the second stage of total specimen failure can be considered: (1) the failure process development takes place in every WMV; (2) the failure process development takes place only in one WMV in which there is the maximum value of a priori failed LIs due to technological defects.

For the first hypothesis, which is studied in [4] (see also reference in [4]), we have the following definitions of the number of steps of tensile loading up to failure of the specimen

$$T_A = \min_{1 \leq i \leq K_L} T_{Ai} = \min_{1 \leq i \leq K_L} \max_t (t : n_C - K_{Ci}(t) \geq 0).$$

For corresponding cdf we have

$$F_{T_A}(x) = \sum_{k=1}^{n_L} p_{K_L}(k)(1-(1-F_{T_{A1}}(x))^k),$$

where $F_{T_{A1}}(x)$ is cdf of T_A of one WMV; p_{K_L} is pmf of rv K_L (modified ($K_L \geq 1$) binomial or Poisson distribution; see [4]). In the following we call this hypothesis as MinMax hypothesis and the corresponding family of cdf (for different versions of MChs) as MinMax cdf family. In the simplest case, if $P(K_L = n_L) = 1$, we have $F_{T_A}(x) = (1-(1-F_{T_{A1}}(x))^{n_L}$.

In this paper we consider the second hypothesis: the failure process development takes place only in one WMV in which there is a minimum of intact LIs, $N_{Cmn} = \min_i (n_C - K_{Ci}(0))$, where $K_{Ci}(0)$ is the initial number of (technological) defects in i-th WMV. Then

$$T_A = \max(t : \min_i (n_C - K_{Ci}(0)) - K_{Ci^*}(t) \geq 0) = \max(t : N_{Cmn} - K_{Ci^*}(t) \geq 0)$$

where $i^* = \arg\min_i (n_C - K_{Ci}(0))$ is the index of link corresponding to minimum intact LIs. This hypothesis we call as MaxMin hypothesis (MaxMin distribution family can be introduced also).

Obviously, instead of calculation of cdf of rv N_{Cmn} it is more convenient to calculate cdf of rv $K_{Cmx} = n_C - N_{Cmn} = \max_{1 \leq i \leq K_L} (K_{C_i}(0))$ for which we have

$$F_{K_{Cmx}}(m) = \sum_{k=1}^{n_L} p_{K_L}(k) F^k_{K_C(0)}(m), \; m = 1, 2, ..., n_C - 1,$$

where $F_{K_C(0)}(m) = \sum_{k=1}^{m} p_{K_C}(k)$, $p_{K_C}(.)$ is a priori pmf of rv $K_C(0)$. Then for $N_{Cmn} = n_C - K_{Cmx}$ we have the cdf $F_{N_{Cmn}}(m) = P(n_C - K_{Cmx} < m) = P(n_C - m < K_{Cmx}) = 1 - F_{K_{Cmx}}(n_C - m)$

and pmf $p_{N_{Cmn}}(m) = F_{N_{Cmn}}(m) - F_{N_{Cmn}}(m-1), \; m = 2, ..., n_C - 1$, $p_{N_{C0}}(1) = F_{N_{C0}}(1)$.

But for $T_A = \max(t : N_{Cmn} - K_{Ci^*}(t) > 0)$ we have $F_{T_A}(t) = \sum_{m=1}^{n_C - 1} p_{N_{C0}}(m) F_{T_A | n_C = m}(t)$,

where $F_{T_A | n_C = m}(t)$ is cdf of T_A of one WMV for $n_C = m$ (see (1)).

Of course, we can reach the tensile failure of any specimens by increasing of load. So in every specimen there is at least one WMV and rv K_L is an integer which is larger or equal to one, more exactly: $1 \leq K_L \leq n_L$. Let rv K have a binomial or Poisson cdf (if $n_L \to \infty$). Then for rv K_L the conditional cdf of K under condition K >0 or definition K_L =1+ K can be used.

Recall that connections between T_A, tensile strength, X, and number of cycles up to fatigue failure, T_C, are defined by (2a) and (2b).

2.2.2. The initiation of WMVs takes place during the process of loading

For tensile test it can be assumed that the number of WMV depends on the load. So the parameter of the binomial distribution can be taken equal to $F(x)$ where $F(.)$ is cdf of tensile strength of one LI, x is load (see details in [4]).

For fatigue test in [5] it is supposed that WMVs do not originate simultaneously but in accordance with a Poisson process. Then the number of WMVs is a random function of time. It increases during fatigue loading with intervals X_i, $i = 1,2,3,\ldots$. So X_1, $X_1 + X_2$, $X_1 + X_2 + X_3$, ... are the time moments of initiation of new WMVs. Let us denote by T_j fatigue life of j-th specific WMV. Then the life of specimen

$$Y = \min(T_1, T_2 + X_1, T_3 + X_1 + X_2, \ldots).$$

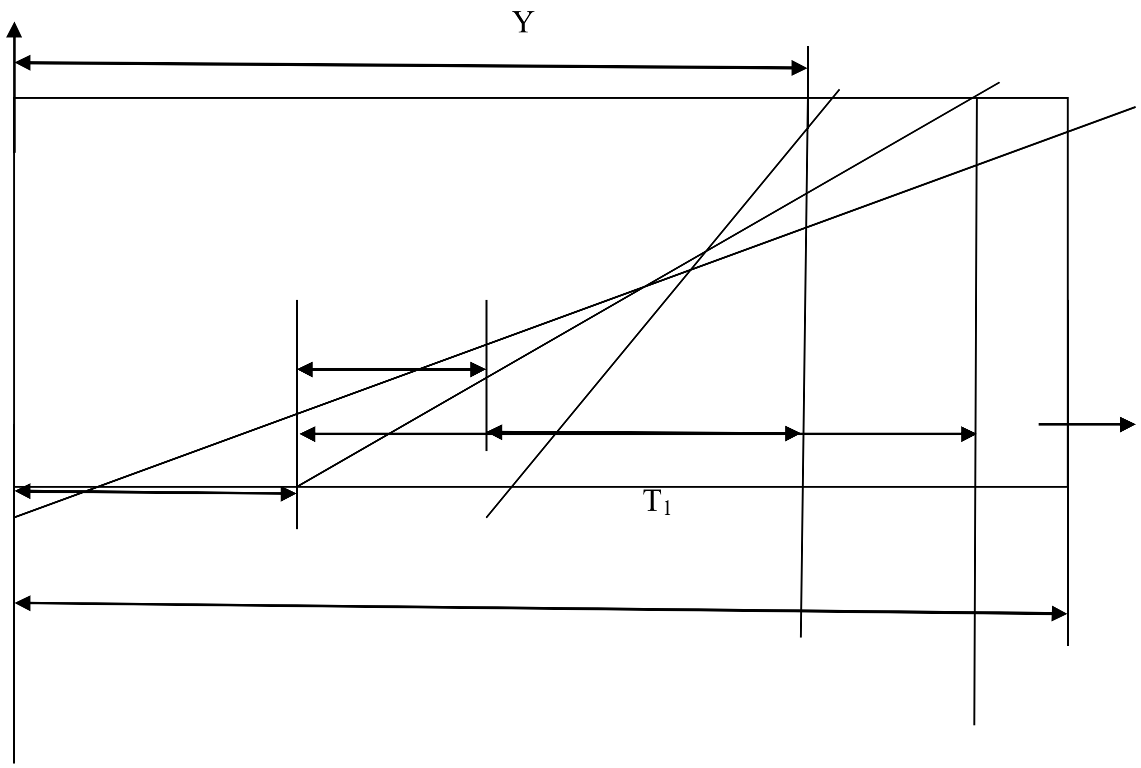

Fig. 3 . Definition of Y.

This equation can be written in the form $Y = \min(T_1, X_1 + Y_1)$, where $F_Y(y) = F_{Y_1}(y)$.

We have the following solution of this equation for the exponential distribution with a parameter μ of all independent random variables $X_1, X_2, X_3, \ldots$ (mean value of X is equal to $1/\mu$)

$$F_Y(y) = 1 - (1 - F_{T_C}(y)) \exp(-\mu \int_0^y F_{T_C}(t) dt).$$

where $F_{T_C}(t)$ is cdf of fatigue life of one WMV (see (2b), time unit is one cycle).

In [5] the example of using this approach for processing of test data is given.

Finally, we should mention that using results of fatigue test of glass fiber composite material example of the reasonably successful „ translation" of cycle with some positive assymetry into the pulsating cycle is also given in [5].

3. PROCESSING THE TEST DATA

In [4] there are examples of test data processing connected with MinMax approach. More exactly, there are examples of processing of results of fatigue tests (S-N curve and residual life) of carbon-fiber reinforced composite specimens using probability transition matrix of the form of Table 1 under assumption that there is only one WMV (see also reference in [4]). In [5] there is an example of processing glass-fiber reinforced composite specimen test data under assumption that different WMVs do not appear simultaneously but in accordance with the Poisson process. In both cases we have reasonable results of fitting test data and some prediction for different length and different stress ratio and examples of prediction of residual strength for two different preliminary loadings (S_i, n_i), $i = 1,2$: (292.53 MPa, 60 000), (390.05 MPa, 900).

In this paper a specific case of MaxMin approach is used for processing fatigue test and tensile strength test results of composite specimens made of unidirectional glass-fibre composite (Udo UD ES 500/300 - SGL epo GmbH c LH 160 of „Composites HAVEL"; [0/45/0]) for L= 60.

Because of the specific structure of specimens, for processing of tensile and fatigue data we use specific structure of matrix P_a (see [4]) with $r_R = 40$, r_Y=0 (there are only LIs); n_C=50. Lognormal distribution of LI tensile strength was assumed. For description of cdf of the rv $K_C(0)$ the conditional binomial cdf under condition that $K_C(0) < n_C$ (recall, equality $K_C(0) = n_C$ means failure of specimen) was used :

$$F_{K_{Ci}(0)}(k) = P(K_{Ci}(0) \le k \mid K_{Ci}(0) < n_C) =$$

$$\sum_{j=0}^{k} b(j, p_C, n_C) / (1 - b(n_C, p_C, n_C)),\ k = 1,...,n_C - 1,\ b(j, p_C, n_C) = p_C^i (1 - p_C)^{n_C - j} n_C! / j!(n_C - j)!\,.$$

For simplicity, only the case $K_L = n_L$ =100 was studied. So cdf

$F_{Cmx}(m) = \sum_{k=1}^{n_L} p_{K_L}(k) F_{K_C(0)}^{k}(m) = F_{K_C(0)}^{n_L}(m)$, $w = 1,2,...,n_C - 1$, was used.

The reasonable fitting of tensile (Fig. 4) and fatigue test results (Fig. 5) for k_m=0.150 was reached at $\theta_0 = 6.59$, $\theta_1 = 0.2$. These parameter do not differ too significantly from their estimates $\hat{\theta}_0 = 6.5869$, $\hat{\theta}_1 = 0.3008$ which are obtained processing tensile tests of strands (1200 fibers).

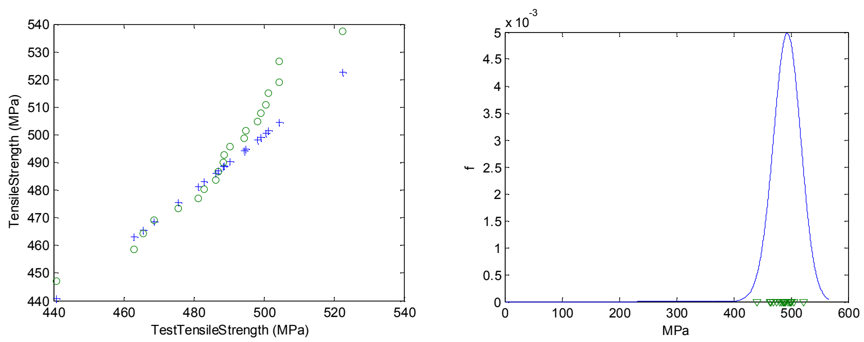

Fig. 4. Fitting of results of tensile strength test of specimens (+ and ∇) and predicted tensile strength pmf .

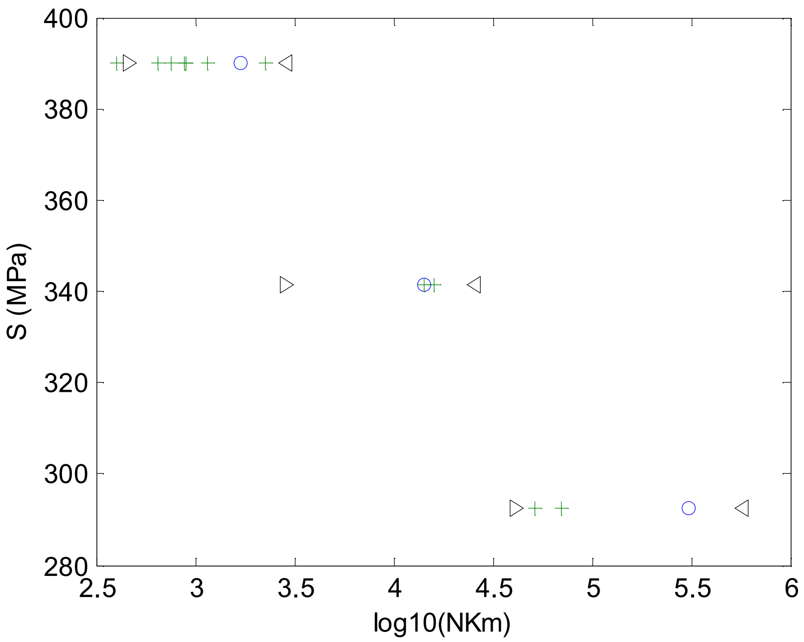

Fig. 5. Fitting of test fatigue life (+) (symbols > and < correspond to two sigma interval)

CONCLUSION

For different types of composite material the different versions of general model are applicable. As we see in Fig.3-Fig.5 and in conclusions of [4-7], by processing of test results it is shown that the considered versions of models, based on using MCh theory and both MinMax and MaxMin approaches (for modeling of the scale effect), can be used for nonlinear regression analysis in order to get fitting and some prediction of tensile and fatigue test results. For a general type of model a large number of parameters and specific type of a priori information should be known for corresponding numerical calculation. Clearly, for different types of composite material the different versions of general model are appropriate. Corresponding comparison analysis should be made and the best specific components of the general MCh model can be chosen for specific material (it is the subject of following papers). The components of the general MCh model are :

1.Type of cdf of mechanical characteristics (tensile strength, Young's modulus, …) and its parameters (for example, θ_0 and θ_1 for cdf with location and scale parameter) for LIs and matrix.

2. Parameters of composite structure (number of LIs in one WMVs, LIs part of cross section of WMV, …) .

3. Definition of failure of WMV (choice of $\bar{f}_C$, ε_{YC},…).

4. Definition of distribution of initial state of WMV.

5. Definition of distribution of number of WMV (links) in specimen as a series system.

6. Definition of process of loading $x1\infty$ for tensile or for fatigue test.

7. Criterion of quality of fitting and prediction of test data.

Some model parameters can be equated with (or can be taken approximately equal to) the parameters of cdf of the tensile strength of composite components and the parameters of composite structure (for example, relative total cross section area of LIs) or just can be chosen a priori (for example, the value of $\bar{f}_C$ in definition of failure of WMV) . Simultaneous fitting of results of both tensile and fatigue test of specimens allows estimation of other model parameters.

Of course, the considered models are too simple to describe the actual basic physical process. The influence of some model components (for example, details of description of tensile loading) should be studied carefully before final statistical analysis conclusion should be made. But the set of the model versions can be used as a wide field of study of composite material strength and fatigue life not only for graduate work of students but and for some engineering applications: for approximate prediction of the effect of not too drastic changes of mechanical characteristics of composite material components and type of load process $x1\infty$.

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