U- FUNCTION IN APPLICATIONS

Igor Ushakov
Sun Diego, California
e-mail: igusha22@gmail.com

The Method of Universal Generating Functions (U-functions) was introduced in [Ushakov, 1986]. Since then the method has been developed, in first order, by my friends and colleagues – Gregory Levitin and Anatoly Lisnianski. They actively and successfully apply the method of U-function to optimal resources allocation, to multi-state system analysis and other problems. Frankly, now I feel like a hen sat on duck eggs and then wanders how hatched chicks fearlessly swim so far from shore. ☺

I decided to remind you a Russian folk proverb: "new is well forgotten old". What is U-function? It is, first of all, generalization of a classical Generation Function (GF) permitting perform more general transforms. From technical side, this method represents a modification of the Kettelle's Algorithm conveniently arranged for calculations with the use of computer.

U-FUNCTION

One can represent any discrete distribution of random variable (r.v.) x_k as a set of pairs:

$$S_k = \left\{(x_k^{(1)}, p_k^{(1)}), (x_k^{(2)}, p_k^{(2)}),...\right\} \tag{1}$$

This distribution can be represented in the form of polynomial (common generating function)

$$\varphi_k(z) = \sum_{1 \le j < n_k} p_k^{(j)} z^{x_k^{(j)}} \quad . \tag{2}$$

where n_k, in principle, can be infinite.

The polynomial representation allows one to obtain the distribution of sums of random variables, using the convolution procedure.

If instead of the distribution of x_1+x_2 one wish to obtain the distribution of an arbitrary function $f(x_1, x_2)$, for any combination of realizations $x_1 = X_1^{(1)}$ and $x_2 = X_2^{(1)}$, the realization of the function $f(x_1, x_2)$ takes the value of $f(X_1^{(1)}, X_2^{(1)})$ with probability $p_1^{(1)} \cdot p_2^{(1)}$. Having the discrete distributions of two random variables x_1+x_2 in the form $S_1 = \left\{(X_1^{(1)}, p_1^{(1)}),...,(X_1^{(n_1)}, p_1^{(n_1)})\right\}$ and $S_2 = \left\{(X_2^{(1)}, p_2^{(1)}),...,(X_2^{(n_3)}, p_2^{(n_2)})\right\}$ one can obtain the discrete distribution S of the function

$f(x_1, x_2)$ as a "Descartes composition" of two sets S_1 and S_2 (here we use the term "interaction" because the operation reminds Descartes product but does not coincide with it) .

$$\begin{aligned}
&S_f = \{(f(X_1^{(1)}, X_2^{(1)}); p_1^{(1)} \cdot p_2^{(1)}),...,(f(X_1^{(n_1)}, X_2^{(n_2)}); p_1^{(1)} \cdot p_2^{(n_2)}),\\
&(f(X_1^{(2)}, X_2^{(1)}); p_1^{(2)} \cdot p_2^{(1)}),...,(f(X_1^{(2)}, X_2^{(n_2)}); p_1^{(2)} \cdot p_2^{(n_2)}),\\
&...\\
&(f(X_1^{(n_1)}, X_2^{(1)}); p_1^{(n_1)} \cdot p_2^{(1)}),...,(f(X_1^{(n_1)}, X_2^{(n_2)}); p_1^{(n_1)} \cdot p_{n^{(2)}}^{(n_2)})
\end{aligned} \tag{3}$$

This distribution can be conveniently obtained using the "composition procedure" over functions $\varphi_1(z)$ and $\varphi_2(z)$:

$$\varphi_f(z) = \underset{f}{\otimes}(\varphi_1(z), \varphi_2(z)) = \left(\sum_{1 \le j < n_1} p_1^{(j)} z^{X_1^{(j)}} \right) \underset{f}{\otimes} \left(\sum_{1 \le i < n_2} p_2^{(i)} z^{X_2^{(i)}} \right)$$

$$= \sum_{1 \le j < n_2} \sum_{1 \le i < n_2} p_1^{(j)} \cdot p_2^{(i)} \, z^{f(X_1^{(j)}, X_2^{(i)})}. \quad (4)$$

The composition operator $\underset{f}{\otimes}$ possesses commutative property, i.e.

$$\underset{f}{\otimes}(\varphi_1(z), \varphi_2(z)) = \underset{f}{\otimes}(\varphi_2(z), \varphi_1(z)) \quad (5)$$

and associative property, i.e.

$$\underset{f}{\otimes}(\varphi_1(z) \underset{f}{\otimes}(\varphi_2(z)), \varphi_3(z)) = \underset{f}{\otimes}((\varphi_2(z), \underset{f}{\otimes}((\varphi_1(z), (\varphi_3(z))) \underset{f}{\otimes}(\varphi_3(z), \underset{f}{\otimes}(\varphi_1(z), \varphi_2(z)). \quad (6)$$

if the function f possesses these properties. In the most applications this is the case, though some exceptions exist (see, for example, [Levitin, 2005]).

USING U-FUNCTION FOR SOLVING THE OPTIMAL REDUNDANCY PROBLEMS

Let us consider a series system consisting of n subsystems. Each of subsystem k contains at least one unit with probability of failure-free operation (PFFO) p_k and cost c_k . For increasing the system reliability, one can introduce redundancy in each subsystem. Denote the number of redundant units of subsystem k as x_k. All the possible PFFOs and costs of any subsystem k for different levels of redundancy can be represented by the set of triplets

$$S_k = \{(P_k(x_k), C_k(x_k), x_k), 1 \le x_k < \infty\}$$

where $C_k(x_k)$ is the total cost of x_k redundant units (usually, a linear function); and $P_k(x_k)$ PFFO or availability coefficient) of the subsystem when it contains x_k redundant units. It is well-known that for loaded redundancy (the so-called "hot standby") of group including one main and x_k identical redundant units takes the form

$$P_k(x_k) = 1 - (1 - p_k)^{x_k + 1};$$

and for an unloaded redundant (the so-called "cold standby") units takes the form

$$P_k(x_k) = \sum_{0 \le j \le x_k} \frac{(\lambda_k t)^j}{j!} \exp(-\lambda_k t);$$

The set of triplets S_k can be represented in the GF form as

$$\varphi_k(z) = \sum_{1 \le x_k < \infty} P_k(x_k) z^{C_h(x_k)} y^{x_k}. \quad (7)$$

Now consider a general procedure of optimal redundancy with the use of U-function. First take units 1 and 2 and arrange the Descartes interaction procedure between sets S_1 and S_2. To distinguish interaction procedure from common product of two generating function, let us introduce symbol $\otimes$.

$$\varphi_{1,2}(z,y)=\bigotimes_{+,\cup}\left(\varphi_1(z,y),\,\varphi_2(z,y)\right)=\bigotimes_{+,\cup}\left(\sum_{1\le x_1<\infty}P_1(x_1)\,z^{C_1(x_1)}y^{\{x_1\}},\ \sum_{1\le x_2<\infty}P_2(x_2)z^{C_2(x_2)}y^{\{x_2\}}\right)$$

$$=\sum_{\substack{1\le x_1<\infty\\1\le x_2<\infty}}P_1(x_1)\cdot P_2(x_2)\,z^{\otimes_+(C_1(x_1),C_2(x_2))}y^{\otimes_\cup(x_1,x_2)}=\sum_{\substack{1\le x_1<\infty\\1\le x_2<\infty}}P_1(x_1)\cdot P_2(x_2)\,z^{C_1(x_1)+C_1(x_1)}y^{\{x_1,x_2\}} \qquad (8)$$

$$=\sum_{1\le X_{1,2}<\infty}P_{1,2}(X_{1,2})z^{C_{1,2}(X_{1,2})}y^{X_{1,2}}.$$

From (8) clear that composition operator $\overset{\otimes}{}_{+}$ means summation corresponding components and operator $\overset{\otimes}{}_{\cup}$ means a collection of corresponding components.

Thus the obtained U-function $\varphi_{1,2}(z, y)$ represents the set of related PFFO, costs and numbers of redundant units for different configurations of the series connection of subsystems 1 and 2. The group of subsystems 1 and 2 now can be treated as an equivalent "aggregated" subsystem. Notice that solving optimal redundancy problem, one has to make some kind of "sifting" of function $\varphi_{1,2}(z, y)$. One has to order all terms of the final expression in (8) by increasing values of $C_{1,2}$ and exclude all terms of $\varphi_{1,2}(z, y)$ that have value of $P_{1,2}$ equal or smaller than previous terms. (If two terms have the same values C_{12} and $P_{1,2}$ one leaves an arbitrary single of them.). In the remaining terms, all $X_{1,2}$ are numbered by natural numbers in accordance with their order by increasing cost. This procedure is equivalent to deleting dominated terms

At the next step one obtains the UGF for the group of three subsystems 1, 2 and 3 applying the same convolution operator over U-function $\varphi_{1,2}(z)$ representing the aggregated subsystem and $\varphi_3(z)$ representing the subsystem 3:

$$\varphi_{1,2,3}(z,y)=\bigotimes_{+,\cup}\left(\varphi_{1,2}(z,y),\,\varphi_3(z,y)\right)=\bigotimes_{+,\cup}\left(\sum_{1\le X_{1,2}<\infty}P_{1,2}(X_{1,2})\cdot z^{C_{1,2}(X_{1,2})}y^{\{X_{1,2}\}},\ \sum_{1\le x_3<\infty}P_3(x_3)\,z^{C_3(x_3)}y^{\{x_3\}}\right)$$

$$=\sum_{\substack{1\le X_{1,2}<\infty\\1\le x_3<\infty}}P_{1,2}(X_{1,2})\cdot P_3(x_3)\,z^{\otimes_+(C_{1,2}(X_{12}),C_3(x_3))}y^{\otimes_\cup(X_{1,2},x_3)}=\sum_{\substack{1\le X_{1,2}<\infty\\1\le x_3<\infty}}P_{1,2}(X_{1,2})\cdot P_3(x_3)\,z^{C_{1,2}(X_{1,2})+C_3(x_3)}y^{\{X_{1,2},x_3\}}$$

$$=\sum_{1\le X_{1,2,3}<\infty}P_{1,2,3}(X_{1,2,3})z^{C_{1,2,3}(X_{1,2,3})}y^{X_{1,2,3}}.$$

(9)

This procedure continues until necessary final UGF representing the entire system is generated. Instead of further abstract presentation of the procedure, let us turn to a simple illustrative numerical example.

Consider a series system of two units. Let unit-1 and unit-2 are characterized by strings

$$S_1=\{M_1^{(1)},M_2^{(1)},...,M_{n_1}^{(1)}\}$$

and

$$S_2=\{M_1^{(2)},M_2^{(2)},...,M_{n_2}^{(2)}\}$$

Each multiplet M is a set of parameters $M_j^{(k)}=\{\alpha_{1j}^{(k)},\alpha_{2j}^{(k)},...,\alpha_N^{(k)}\}$ where N is the number of parameters in each multiplet.

"Interaction" of these two strings is an analogue of the Cartezian product whose memberts fill the cells of the following table:

Table 1.

	$M_1^{(1)}$	$M_2^{(1)}$	...	$M_{n_1}^{(1)}$
$M_1^{(2)}$	$M_1^{(1)} \otimes M_1^{(2)}$	$M_2^{(1)} \otimes M_1^{(2)}$	...	$M_{n_1}^{(1)} \otimes M_1^{(2)}$
$M_2^{(2)}$	$M_1^{(1)} \otimes M_2^{(2)}$	$M_2^{(1)} \otimes M_2^{(2)}$	...	$M_{n_1}^{(1)} \otimes M_2^{(2)}$
...	...	...	...	...
$M_{n_2}^{(2)}$	$M_1^{(1)} \otimes M_{n_2}^{(2)}$	$M_2^{(1)} \otimes M_{n_2}^{(2)}$	...	$M_{n_1}^{(1)} \otimes M_{n_2}^{(2)}$

Interaction of multiplets consists of iteractions of their similar parameters, for instance,

$$M_j^{(k)} \otimes M_i^{(h)} = \{(\alpha_{1j}^{(k)} \underset{f_1}{\otimes} \alpha_{1i}^{(h)}), (\alpha_{2j}^{(k)} \underset{f_2}{\otimes} \alpha_{2i}^{(h)}), ..., (\alpha_{Nj}^{(k)} \underset{f_N}{\otimes} \alpha_{Ni}^{(h)})\} \tag{3}$$

Operator $\otimes$, as well as each operator $\underset{f_s}{\otimes}$, in most natural practical cases possesses the commutativity property, i.e.

$$\underset{f}{\otimes}(a, b) = \underset{f}{\otimes}(b, a), \tag{4}$$

and the associativity property, i.e.

$$\underset{f}{\otimes}(a, b, c) = \underset{f}{\otimes}(a \underset{f}{\otimes}(b, c)) = \underset{f}{\otimes}((a \underset{f}{\otimes} b), c). \tag{5}$$

U-FUNCTION IN GENERAL CASE

Of course, operator $\underset{f_s}{\otimes}$ depends on the physical nature of parameter α_s and the type of structure, i.e. series or parallel.

Table 2.

Type of parameter	Type of structure	Result of interaction
A) α is unit's PFFO	series	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \alpha_{Aj}^{(k)} \times \alpha_{Ai}^{(h)}$
	parallel	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = 1 - (1 - \alpha_{Aj}^{(k)}) \times (1 - \alpha_{Ai}^{(h)})$
B) α is number of units in parallel	series	$\alpha_{Bj}^{(k)} \underset{f}{\otimes} \alpha_{Bi}^{(h)} = (\alpha_{Bj}^{(k)}; B_{Ri}^{(h)})$
	parallel	$\alpha_{Bj}^{(k)} \underset{f}{\otimes} \alpha_{Bi}^{(h)} = (\alpha_{Bj}^{(k)}; B_{Ri}^{(h)})$
C) α is unit's cost (weight)	series	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \alpha_{Aj}^{(k)} + \alpha_{Ai}^{(h)}$
	parallel	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \alpha_{Aj}^{(k)} + \alpha_{Ai}^{(h)}$
D) α is unit's ohmic resistance	series	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \alpha_{Aj}^{(k)} + \alpha_{Ai}^{(h)}$
	parallel	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \left[(\alpha_{Aj}^{(k)})^{-1} + (\alpha_{Ai}^{(h)})^{-1}\right]^{-1}$

Type of parameter	Type of structure	Result of interaction
E) α is unit's capacitance	series	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \left[(\alpha_{Aj}^{(k)})^{-1} + (\alpha_{Ai}^{(h)})^{-1}\right]^{-1}$
	parallel	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \alpha_{Aj}^{(k)} + \alpha_{Ai}^{(h)}$
F) α is pipeline unit's capacitance	series	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \min\left\{\alpha_{Aj}^{(k)}, \alpha_{Ai}^{(h)}\right\}$
	parallel	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \alpha_{Aj}^{(k)} + \alpha_{Ai}^{(h)}$
G) α is unit's random time to failure	series	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \min\left\{\alpha_{Aj}^{(k)}, \alpha_{Ai}^{(h)}\right\}$
	parallel	$\alpha_{Aj}^{(k)} \underset{f}{\otimes} \alpha_{Ai}^{(h)} = \max\left\{\alpha_{Aj}^{(k)}, \alpha_{Ai}^{(h)}\right\}$

In the problem of optimal redundancy, one deals with triplet of type "*Probability-Cost-Number of units*" for each redundant group: $M_j = \{\alpha_{1j}, \alpha_{2j}, \alpha_{3j}\}$. If there is a system of *n* series subsystems (single elements or redundant groups) , one has to use a procedure almost completely coincided with the procedure of compiling the dominating sequences at the Kettelle's algorithm.

In accordance with the description given above, the block diagram of the using U-functions, for example, for four subsystems can be presented as follows (see Figures 1 and 2).

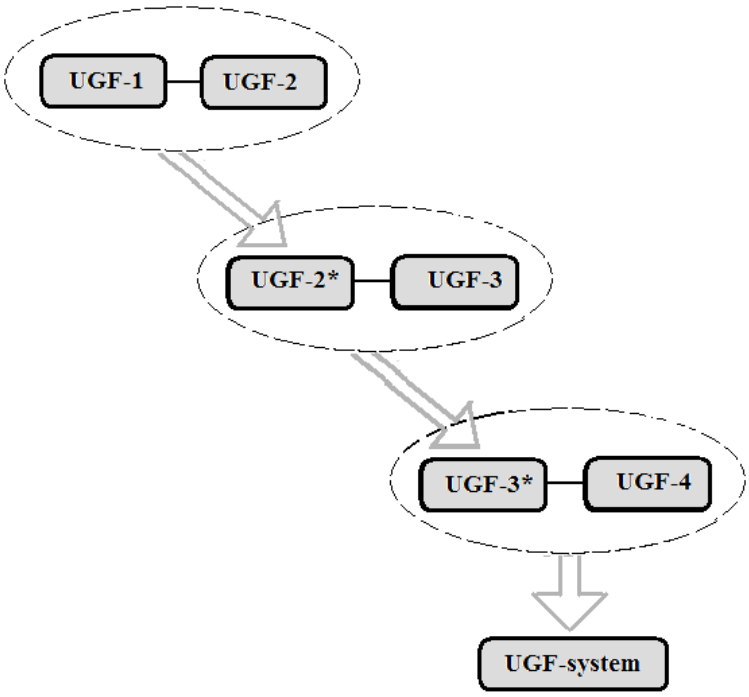

Figure 1. Block-diagram of the sequential procedure of solving.

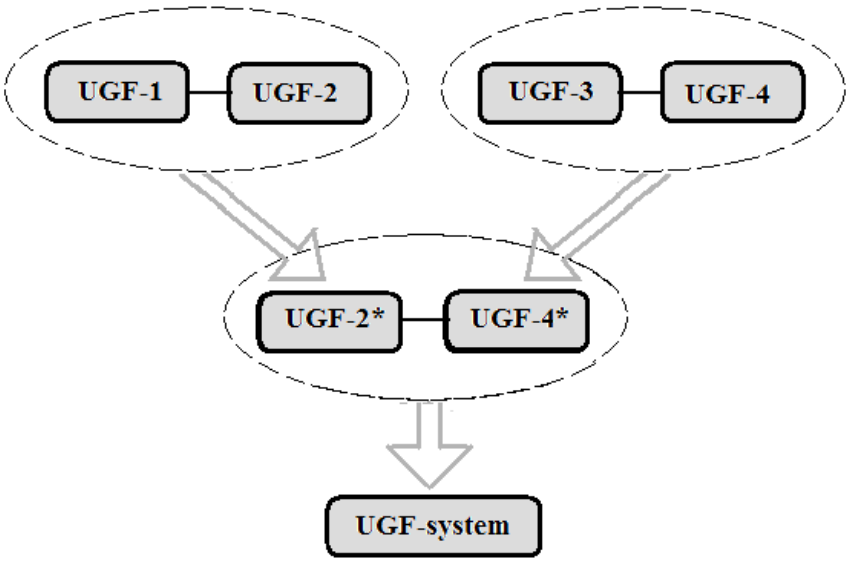

Figure 2. Block-diagram of the dichotomy procedure of solving.

CONCLUSION

The method of U-function is one of the methods of directed enumerating. It showed its effectiveness for solving a number of practical problems concerning with optimal resources allocation and analysis of multi-state systems and system consisting of multi-state elements.

REFERENCES (in chronological order)

(1986) Ushakov, I.A. Universal Generating Function (in Russian). Engineering Cybernetics, No. 5.
(1987) Ushakov, I.A. Universal generating function. Soviet Journal Computer Systems Science, No.3.
(1987) Ushakov, I.A. Optimal standby problem and a universal generating function. Soviet Journal Computer and System Science, No.4.
(1987) Ushakov, I.A. Solution of multi-criteria discrete optimization problems using a universal generating function. Soviet Journal of Computer and System Sciences , No. 5.
(1988) Ushakov, I.A. Solving of optimal redundancy problem by means of a generalized generating function. Elektronische Informationsverarbeitung und Kybernetik, No.4-5
(1995) Gnedenko, B.V., and Ushakov I.A. Probabilistic Reliability Engineering. John Wiley & Sons.
(1998) Levitin, G., Lisnianski, A., Ben Haim, H., and Elmakis, D. Redundancy optimization for series-parallel multi-state systems, IEEE Transactions on Reliability, , No. 2.
(1999) Levitin, G., and Lisnianski, A. Importance and sensitivity analysis of multi-state systems using universal generating functions method, Reliability Engineering & System Safety, No. 65.
(2000) Ushakov, I.A. The method of generating sequences. European Journal of Operational Research, No. 2.
(2000) Levitin, G., and Lisnianski, A. Optimal Replacement Scheduling in Multi-state Series-parallel Systems. Quality and Reliability Engineering International, , No. 16.
(2001) Levitin, G. Redundancy optimization for multi-state system with fixed resource-requirements and unreliable sources. IEEE Transactions on Reliability, No. 50.

(2001) Levitin, G., and Lisnianski, A. A new approach to solving problems of multi-state system reliability optimization, Quality and Reliability Engineering International, No. 47.
(2002) Levitin, G. Optimal allocation of multi-state elements in linear consecutively-connected systems with delays. International Journal of Reliability Quality and Safety Engineering, No. 9.
(2003) Levitin, G. Optimal allocation of multi-state elements in linear consecutively-connected systems. IEEE Transactions on Reliability, No. 2.
(2005) Levitin, G. The Universal Generating Function in Reliability Analysis and Optimization. Springer.