OPTIMAL REDUNDANCY IN SYSTEMS WITH MULTI-LEVEL UNITS

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ABSTRACT

Method of Universal Generating Function (UGF) was introduced in [1]-[5] and got further fundamental developing in [7]-[8]. Here we give an example how method of UGF can be implemented to solution problems of optimal redundancy for systems consisting of multi-level units.

The Method of Universal Generating Functions (U-functions) was introduced in [1]-[5]. Before detailed consideration of this method, let us remark that applying to the optimal redundancy problem this method represents a modification of the Kettelle's Algorithm [1] conveniently arranged for calculations with the use of computer.

Detailed description of the UGF method can be found in [1]-[5] and [5].

For the Reader's convenience, we begin with a numerical example that can explain the idea of the problem solution more transparently than general arguing. The final description of the algorithm is given at the end.

<u>Example</u>. Consider a simplest series system of two units (see figure below).

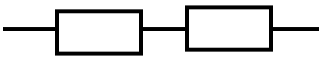

Figure 1. Series system consisting of two units.

However, each unit itself is not a simple binary element but multistate element that is characterized by several levels of performance. Performance may be measured various physical values. Effectiveness of such system operation depends on levels of performance of Unit-1 and Unit-2.

Let units are characterized by the following parameters:

Unit-1

Level of performance (W_1)	Probability p_1	Cost of a single unit
100%	p_{11}=Pr{ W_1=100%}=0.9	c_1=1
70%	p_{12}=Pr{ W_1=100%}=0.05	
40%	p_{13}=Pr{ W_1=100%}=0.04	
0%	p_{14}=Pr{ W_1=100%}=0.01	

Unit-2

Level of performance (W_1)	Probability p_2	Cost of a single unit
100%	P_{21}=Pr{ W_2=100%}=0.8	c_2=2
80%	P_{22}=Pr{ W_2=80%}=0.18	
20%	P_{23}=Pr{ W_2=20%}=0.01	
0%	P_{24}=Pr{ W_2=0%}=0.01	

Assume that performance effectiveness of each unit can be improved by using simple redundancy and that each moment of time unit performance is equal to the performance of the best component of the redundant group. Thus, behavior of Unit-1, consisting of the main component and single redundant element, can be depicted as in Figure 2.

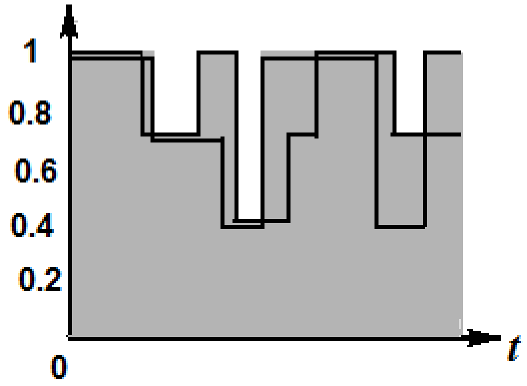

Figure 2. A realization of stochastic behavior of Unit-1, consisting of two elements, main and redundant. The shadowed area denotes the behavior of the Unit-1.

For Unit-2 analogous process is presented in Figure 3.

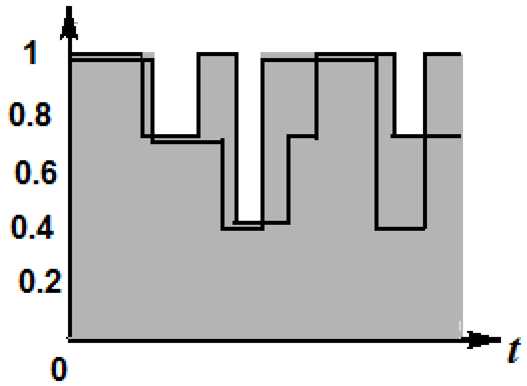

Figure 3. A realization of stochastic behavior of Unit-2, consisting of two elements, main and redundant. The shadowed area denotes the behavior of the Unit-2.

Further, assume that the entire system (series connection of Unit-1 and Unit-2) is characterized by the worst level of effectiveness of its units at each moment of time. In Figure 4, one can see the system behavior for the case when both units consist of a single main element.

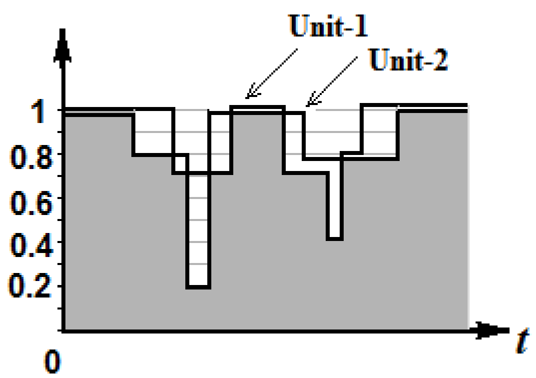

Figure 4. A realization of stochastic behavior of the entire system when both its units consist of a single main element. The shadowed area denotes the behavior of the system.

Let the problem is to find optimal redundant elements allocation for solving two optimal redundancy problems:

(1) Direct problem: Find such an allocation of redundant elements than delivers average level of the system performance not less than W^0 with minimum possible cost of redundant elements;
(2) Inverse problem: Find such an allocation of redundant elements than delivers maximum possible level of system performance under condition that the total expenses on redundant elements do not exceed C^0 units of cost.

Now consider construction of dominating sequence during the optimization process. (For details about dominating sequence, see [1] or [2].) In principle, one has to construct a table of type that presented below and choose members of dominating sequence.

Table 1. Construction of dominating sequence.

		Number of redundant elements for Unit-1			
		0	**1**	**2**	**…**
	0	X=(0, 0) P(0, 0) W(0,0) C(0, 0)	X=(1, 0) P(1, 0) W(1, 0) C(1, 0)	X=(2, 0) P(2, 0) W(2, 0) C(2, 0)	…
	1	X=(0, 1) P(0, 1) W(0,1) C(0, 1)	X=(1, 1) P(1, 1) W(1, 1) C(1, 1)	X=(2, 1) P(2, 1) W(2, 1) C(2, 1)	…
Number of redundant elements for Unit-2	**2**	X=(0, 2) P(0, 2) W(0, 2) C(0, 2)	X=(1, 2) P(1, 2) W(1, 2) C(1, 2)	X=(2, 2) P(2, 1) W(2, 2) C(2, 2)	…
	…	…	…	…	…

Further discussion will be provided in terms of Universal Generating Functions. As one sees, in this case we deal with quadruplets of type:

{**Vector of units' variants; Discrete distribution of performance levels; System cost**}.

The problem complicates due to necessity of calculations because Probabilities of performance levels and Performance levels are not numbers but vectors that needed special calculations. This aspect will be demonstrated below. Here we would like to note that there is no necessity to calculate quadruplets for all cells of Table 1. Fortunately, we can use the property of Kettelle Algorithm: members of dominating sequences are located around table's diagonal and corresponding cells form simply connected area. It allows to use "dichotomy tree" procedure, i.e. avoid unnecessary calculations by cutting non-perspective branches (see Figure 5).

Indeed, consider bordering cells around simple connected area (they marked with sign "x".). There is no dominating cells in area located upper the right border, and there is no dominating cells in area located lower the left border.

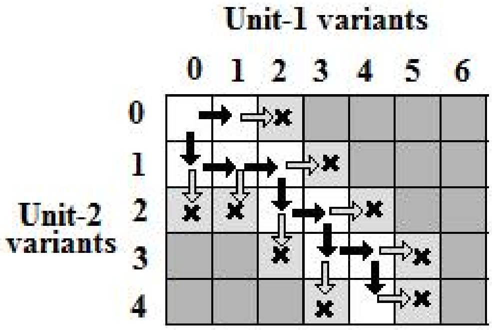

Figure 5. Example of excluding non-perspective branches. Black arrows are members of dominating sequence, Grey arrows are trial test that led to non-perspective variants marked by "x". All cells marked with dark grey cannot contain dominating quadruplets.

Thus, in this case calculations occur to be sufficiently compact. However, as we mentioned above some special calculations for each redundant group have to be done.

Let us consider a numerical example.

In accordance with described above calculating procedure, one has to consider first variant (0, 0), i.e just Unit-1 and Unit-2 with no redundancy at all, and find quadruple, In this case resulting solution will be:

$\{0;\ [(p_{11}, W_{11}), (p_{12}, W_{12}), (p_{13}, W_{13}), (p_{14}, W_{14})]; c_1\} \otimes \{0;\ [(p_{21}, W_{21}), (p_{22}, W_{22}), (p_{23}, W_{23}), (p_{24}, W_{24})]; c_2\} = \{0 \underset{\rightarrow}{\otimes} 0;\ [(p_{11}, W_{11}), (p_{12}, W_{12}), (p_{13}, W_{13}), (p_{14}, W_{14})] \underset{UGF}{\otimes} \{0;\ [(p_{21},W_{21}), (p_{22}, W_{22}), (p_{23}, W_{23}), (p_{24}, W_{24})];\ c_1 \underset{+}{\otimes} c_2\}$.

Here we use the following operators:

$\underset{\rightarrow}{\otimes}$ is an operator of forming a vector, i.e. $j \underset{\rightarrow}{\otimes} k = (j,k)$;

$\underset{UGF}{\otimes}$ is an operator equivalent to the U-function, i.e. $\left(\sum_{j\in A} p_j z^{W_j}\right) \underset{UGF}{\otimes} \left(\sum_{k\in B} p_k z^{W_k}\right) = \sum_{\forall j,\forall k} p_j \cdot p_k z^{W_j \underset{\min}{\otimes} W_k}$,

where, in turn, $W_j \underset{\min}{\otimes} W_k = \min(W_j, W_k)$; $c_1 \underset{+}{\otimes} c_2$ is operator of summation, i.e. $c_1 \underset{+}{\otimes} c_2 = c_1 + c_2$.

Numerical results are presented in Table 2.

This leads to the following final result:

$P^{(0.0)}(W_{syst}=100\%) = 0.72$;
$P^{(0.0)}(W_{syst}=80\%) = 0.171$;
$P^{(0.0)}(W_{syst}=70\%) = 0.04+0.0095=0.0495$;
$P^{(0.0)}(W_{syst}=40\% = 0.032+0.0076=0.0396$;
$P^{(0.0)}(W_{syst}=20\%) = 0.009+0.0005+0.004=0.0099$;
$P^{(0.0)}(W_{syst}=0\%) = 0.008+0.0019+0.0001+0.0001+0.0009+0.0005+0.0004= 0.0201$.

Cost of additional units in this case equals 0. As one can easily calculate, the average level of the system performance is equal to

$W_{syst}^{(0,0)} = 0.72 + 0.171 \cdot 0.8 + 0.0497 \cdot 0.7 + 0.0396 \cdot 0.5 + 0.0095 \cdot 0.2 \approx 0.9092.$

Table 2. Step 1 of the process of optimization

(0, 0)		Unit-2					
		p_{21}=0.8 $W_{21}^{(0)}$=100%	p_{22}=0.18 $W_{22}^{(0)}$=80%			p_{23}=0.01 $W_{23}^{(0)}$=20%	p_{24}=0.01 $W_{24}^{(0)}$=0%
Unit-1	p_{11}=0.9 $W_{11}^{(0)}$ =100%	$p_{21}\cdot p_{11}$= 0.72 $\min(W_{21}^{(0)}, W_{11}^{(0)})$ =100%	$p_{22}\cdot p_{11}$= 0.171 $\min(W_{22}^{(0)}, W_{11}^{(0)})$ =80%			$p_{23}\cdot p_{14}$= 0.009 $\min(W_{23}^{(0)}, W_{11}^{(0)})$ =20%	$p_{24}\cdot p_{14}$= 0.009 $\min(W_{21}^{(0)}, W_{11}^{(0)})$ =0%
	p_{12}=0.05 $W_{12}^{(0)}$ =70%	$p_{21}\cdot p_{12}$= 0.04 $\min(W_{21}^{(0)}, W_{12}^{(0)})$ =70%	$p_{22}\cdot p_{12}$=0.0095 $\min(W_{22}^{(0)}, W_{12}^{(0)})$ =70%			$p_{23}\cdot p_{14}$= 0.0005 $\min(W_{23}^{(0)}, W_{12}^{(0)})$ =20%	$p_{24}\cdot p_{14}$= 0.0005 $\min(W_{21}^{(0)}, W_{11}^{(0)})$ =0%
	p_{13}=0.04 $W_{13}^{(0)}$ =40%	$p_{21}\cdot p_{13}$= 0.032 $\min(W_{21}^{(0)}, W_{13}^{(0)})$=40%	$p_{22}\cdot p_{13}$=0.0076 $\min(W_{22}^{(0)}, W_{13}^{(0)})$ =40%			$p_{23}\cdot p_{14}$= 0.0004 $\min(W_{23}^{(0)}, W_{13}^{(0)})$ =20%	$p_{24}\cdot p_{14}$= 0.0004 $\min(W_{21}^{(0)}, W_{11}^{(0)})$ =0%
	p_{14}=0.01 $W_{14}^{(0)}$ =0%	$p_{21}\cdot p_{14}$= 0.008 $\min(W_{21}^{(0)}, W_{14}^{(0)})$ =0%	$p_{22}\cdot p_{14}$= 0.0019 $\min(W_{22}^{(0)}, W_{14}^{(0)})$ =0%			$p_{23}\cdot p_{14}$= 0.0001 $\min(W_{23}^{(0)}, W_{14}^{(0)})$ = 0%	$p_{24}\cdot p_{14}$= 0.0001 $\min(W_{23}^{(0)}, W_{14}^{(0)})$ = 0%

Now let's make trial steps to the neighbor cells: check cells (1, 0) and (0, 1). Let us start with cell (1, 0) in accordance with Figure 4. First find performance levels distribution for Unit-1 consisting of two elements, main and redundant.

Table 3. Forehand calculation of performance levels distribution for Unit-1, consisting of two elements, main and redundant.

		Element-1			
Element-1		p_{11}=0.9 $W_{11}^{(0)}$=100%	p_{12}=0.05 $W_{12}^{(0)}$=70%	p_{13}=0.04 $W_{13}^{(0)}$)=40%	p_{14}=0.01 $W_{14}^{(0)}$=0%
	p_{11}=0.9 $W_{11}^{(0)}$ =100%	$(p_{11})^2$=0.81 $W_{11}^{(0)}$=100%	$p_{12}\cdot p_{11}$=0.045 $\max(W_{12}^{(0)}, W_{11}^{(0)})$ =100%	$p_{13}\cdot p_{11}$=0.036 $\max(W_{13}^{(0)}, W_{11}^{(0)})$ =100%	$p_{14}\cdot p_{11}$=0.009 $\max(W_{14}^{(0)}, W_{11}^{(0)})$ =100%
	p_{12}=0.05 $W_{12}^{(0)}$ =70%	$p_{11}\cdot p_{12}$=0.045 $\max(W_{11}^{(0)}, W_{12}^{(0)})$ =100%	$(p_{12})^2$=0.025 $\max(W_{12}^{(0)}, W_{12}^{(0)})$ =70%	$p_{13}\cdot p_{12}$=0.002 $\max(W_{13}^{(0)}, W_{12}^{(0)})$ =70%	$p_{14}\cdot p_{12}$=0.0005 $\max(W_{14}^{(0)}, W_{12}^{(0)}))$ =70%
	p_{13}=0.04 $W_{32}^{(0)}$ =40%	$p_{11}\cdot p_{13}$=0.036 $\max(W_{11}^{(0)}, W_{32}^{(0)})$ =100%	$p_{12}\cdot p_{13}$=0.002 $\max(W_{12}^{(0)}, W_{32}^{(0)})$ =70%	$(p_{13})^2$=0.0016 $W_{32}^{(0)}$=40%	$p_{14}\cdot p_{13}$=0.0004 $\max(W_{14}^{(0)}, W_{32}^{(0)})$ =40%
	p_{14}=0.01 $W_{14}^{(0)}$=0%	$p_{11}\cdot p_{14}$=0.009 $\max(W_{11}^{(0)}, W_{14}^{(0)})$ =100%	$p_{12}\cdot p_{14}$=0.0005 $\max(W_{12}^{(0)}, W_{14}^{(0)})$ =70%	$p_{13}\cdot p_{14}$=0.0004 $\max(W_{13}^{(0)}, W_{14}^{(0)})$ =40%	$(p_{14})^2$=0.0001 $W_{14}^{(0)}$=0%

On the basis of this table, one gets for Unit-1 the following distribution

$\Pr\{W_1^{(1)} = 100\%\} = P_{11}^{(1)} = (p_{11})^2 + 2p_{11} \cdot (p_{12} + p_{13} + p_{14}) = 0.81 + 2 \cdot (0.045+0.036+0.009) = 0.99$;

$\Pr\{W_1^{(1)} = 70\%\} = P_{12}^{(1)} = (p_{12})^2 + 2 \cdot p_{12} \cdot (p_{13} + p_{14}) = 0.025 + 2 \cdot 0.025\ (0.002+0.0005) = 0.0075$;

$\Pr\{W_1^{(1)} = 40\%\} = P_{13}^{(1)} = (p_{13})^2 + 2p_{13} \cdot p_{14} = 0.0016 + 2 \cdot 0.0016 \cdot 0.0004 \approx 0.0016$:

$\Pr\{W_1^{(1)} = 0\%\} = P_{14}^{(1)} = (p_{14})^2 = 0.0001$.

Using these results, one can compile Table 4 that gives performance levels distribution for the system characterized by vector of redundant elements $X = (1, 0)$.

Table 4. Step 3 of the optimization process.

(1, 0) **C$_{system}$**$=c_1 = 1$		**Unit-2**					
		$\boldsymbol{p_{21}=0.8}$ $\boldsymbol{W_{21}^{(0)}=100\%}$	$\boldsymbol{p_{22}=0.19}$ $\boldsymbol{W_{22}^{(0)}=80\%}$			$\boldsymbol{p_{23}=0.01}$ $\boldsymbol{W_{23}^{(0)}=20\%}$	$\boldsymbol{p_{24}=0.01}$ $\boldsymbol{W_{24}^{(0)}=0\%}$
Unit -1	$\boldsymbol{P_{11}^{(1)}=0.99}$ $\boldsymbol{W_{11}^{(1)}=100\%}$	$p_{21} \cdot P_{11}^{(1)} = 0.792$ $\min(W_{21}^{(0)}, W_{11}^{(1)}) = 100\%$	$p_{22} \cdot P_{11}^{(1)} = 0.188$ $\min(W_{22}^{(0)}, W_{11}^{(1)}) = 80\%$			$p_{23} \cdot P_{11}^{(1)} \approx 0.01$ $\min(W_{23}^{(0)}, W_{11}^{(1)}) = 20\%$	$p_{24} \cdot P_{11}^{(1)} \approx 0.01$ $\min(W_{24}^{(0)}, W_{11}^{(1)}) = 0\%$
	$\boldsymbol{P_{12}^{(1)}=0.0075}$ $\boldsymbol{W_{12}^{(1)}=70\%}$	$p_{21} \cdot P_{12}^{(1)} = 0.006$ $\min(W_{21}^{(0)}, W_{12}^{(1)}) = 70\%$	$p_{22} \cdot P_{12}^{(1)} \approx 0.0014$ $\min(W_{22}^{(0)}, W_{12}^{(1)}) = 70\%$			$p_{23} \cdot P_{12}^{(1)} \approx 0.0001$ $\min(W_{23}^{(0)}, W_{12}^{(1)}) = 20\%$	$p_{24} \cdot P_{12}^{(1)} = 0.0001$ $\min(W_{24}^{(0)}, W_{12}^{(1)}) = 0\%$
	$\boldsymbol{P_{13}^{(1)}=0.0016}$ $\boldsymbol{W_{13}^{(1)}=40\%}$	$p_{21} \cdot P_{13}^{(1)} 0.0013$ $\min(W_{21}^{(0)}, W_{13}^{(1)}) = 40\%$	$p_{22} \cdot P_{13}^{(1)} \approx 0.0003$ $\min(W_{22}^{(0)}, W_{13}^{(1)}) = 40\%$			$p_{23} \cdot P_{13}^{(1)} \approx 0$ $\min(W_{23}^{(0)}, W_{13}^{(1)}) = 20\%$	$p_{24} \cdot P_{13}^{(1)} \approx 0$ $\min(W_{24}^{(0)}, W_{13}^{(1)}) = 0\%$
	$\boldsymbol{P_{14}^{(1)}=0.0001}$ $\boldsymbol{W_{14}^{(1)}=0\%}$	$p_{21} \cdot P_{14}^{(1)} \approx 0.0001$ $\min(W_{21}^{(0)}, W_{14}^{(1)}) = 0\%$	$p_{22} \cdot P_{14}^{(1)} \approx 0$ $\min(W_{22}^{(0)}, W_{14}^{(1)}) = 0\%$			$p_{23} \cdot P_{14}^{(1)} \approx 0$ $\min(W_{23}^{(0)}, W_{14}^{(1)}) = 0\%$	$p_{24} \cdot P_{14}^{(1)} \approx 0$ $\min(W_{24}^{(0)}, W_{14}^{(1)}) = 0\%$

This leads to the following final result:

$P^{(1.0)}(W_{syst}=100\%) = 0.792$;

$P^{(0.0)}(W_{syst}=80\%) = 0.188$;

$P^{(0.0)}(W_{syst}=70\%) = 0.006+0.0014=0.0074$;

$P^{(0.0)}(W_{syst}=40\% = 0.0013+0.0003=0.0016$;

$P^{(0.0)}(W_{syst}=20\%) = 0.01+0.0001=0.0101$;

$P^{(0.0)}(W_{syst}=0\%) = 0.008+0.0019+0.0001+0.0001+0.0009+0.0005+0.0004= 0.0201$.

Cost of additional units in this case equals 1. Average system's performance level equals

$W_{syst}^{(1,0)} = 0.792 + 0.188 \cdot 0.8 + 0.0497 \cdot 0.7 + 0.0396 \cdot 0.4 + 0.0095 \cdot 0.2 \approx 0.9502$.

Then try another neighbor cell, namely (0, 1). Beforehand, one has to perform an additional calculation of performance levels distribution for Unit-2 consisting of two elements, main and redundant.

Table 5. Forehand calculation of performance levels distribution for Unit-2, consisting of two elements, main and redundant.

		Element-2			
		p_{21}=0.8 $W_{21}^{(0)}$=100%	p_{22}=0.19 $W_{22}^{(0)}$=80%	p_{23}=0.01 $W_{23}^{(0)}$=20%	p_{24}=0.01 $W_{24}^{(0)}$=0%
Element-2	p_{21}=0.8 $W_{21}^{(0)}$=100%	$(p_{21})^2$=0.64 $W_{21}^{(0)}$=100%	$p_{22}\cdot p_{21}$=0.045 max $(W_{22}^{(0)}, W_{21}^{(0)})$ =100%	$p_{23}\cdot p_{21}$=0.036 max $(W_{23}^{(0)}, W_{21}^{(0)})$ =100%	$p_{24}\cdot p_{21}$=0.008 max $(W_{24}^{(0)}, W_{21}^{(0)})$ =100%
	p_{22}=0.19 $W_{21}^{(0)}$=80%	$p_{21}\cdot p_{22}$=0.152 max $(W_{21}^{(0)}, W_{21}^{(0)})$ =100%	$(p_{22})^2$=0.0361 $W_{22}^{(0)}$=80%	$p_{23}\cdot p_{22}$=0.0002 max $(W_{23}^{(0)}, W_{21}^{(0)})$ =80%	$p_{24}\cdot p_{22}$=0.0002 max $(W_{24}^{(0)}, W_{21}^{(0)})$ =80%
	p_{23}=0.01 $W_{23}^{(0)}$=20%	$p_{21}\cdot p_{23}$=0.008 max $(W_{21}^{(0)}, W_{23}^{(0)})$ =100%	$p_{22}\cdot p_{23}$=0.0002 max $(W_{22}^{(0)}, W_{23}^{(0)})$ =80%	$(p_{23})^2$=0.0001 $W_{23}^{(0)}$=20%	$p_{24}\cdot p_{23}$=0.0001 max $(W_{24}^{(0)}, W_{23}^{(0)})$ =20%
	p_{24}=0.01 $W_{24}^{(0)}$=0%	$p_{21}\cdot p_{24}$=0.008 max $(W_{21}^{(0)}, W_{24}^{(0)})$ =100%	$p_{222}\cdot p_{24}$=0.0002 max $(W_{22}^{(0)}, W_{24}^{(0)})$ =70%	p $p_{23}\cdot p_{24}$=0.0001 max $(W_{23}^{(0)}, W_{24}^{(0)})$ =20%	$(p_{24})^2$=0.0001 $W_{24}^{(0)}$=0%

On the basis of this table, one gets for Unit-2, consisting of two elements, the following distribution

$\Pr\{W_2^{(1)} = 100\%\} = P_{21}^{(1)} = (p_{21})^2 + 2p_{21}\cdot(p_{22}+p_{23}+p_{34}) = 0.64+2\cdot 0.8\cdot(0.045+0.036+0.008) \approx 0.7709;$

$\Pr\{W_2^{(1)} = 80\%\} = P_{22}^{(1)} = (p_{22})^2 + 2\cdot p_{22}\cdot(p_{23}+p_{24}) = 0.0361+2\cdot 0.0361\,(0.0002+0.0002) \approx 0.0361;$

$\Pr\{W_2^{(1)} = 20\%\} = P_{23}^{(1)} = (p_{13})^2 + 2p_{13}\cdot p_{14} = 0.0001+0.0001+0.0001 = 0{,}0003:$

$\Pr\{W_2^{(1)} = 0\%\} = P_{24}^{(1)} = (p_{14})^2 = 0.0001.$

After such preparations, one can construct a table with system's performance levels distribution for the system configuration characterized by vector of redundant elements $X = (0, 1)$. This leads to the following final result:

$P^{(1.0)}(W_{syst}=100\%) = 0.6038;$
$P^{(0.0)}(W_{syst}=80\%) = 0.0325;$
$P^{(0.0)}(W_{syst}=70\%) = 0.0386+0.0018=0.0404;$
$P^{(0.0)}(W_{syst}=40\% = 0.0308+0.0014=0.0322;$
$P^{(0.0)}(W_{syst}=20\%) \approx 0.0003;$
$P^{(0.0)}(W_{syst}=0\%) = 0.0077+0.0004+0.0001 \approx 0.0082.$

Cost of additional units in this case equals 2 units of cost. Average system's performance level equals

$$W_{syst}^{(0,1)} = 0.6038 + 0.0.0325\cdot 0.8 + 0.0404\cdot 0.7 + 0.0322\cdot 0.4 + 0.0003\cdot 0.2 \approx 0.671.$$

Table 6. Step 4 of the optimization process.

(0, 1) **C**$_{\text{system}}$$=c_2 = 2$		**Unit-2**					
		$P_{21}^{(1)}$**=0.7709** $W_{21}^{(1)}$**=100%**	$P_{22}^{(1)}$**=0.0361** $W_{22}^{(1)}$**=80%**			$P_{23}^{(1)}$**=0.0003** $W_{23}^{(1)}$**=20%**	$P_{24}^{(1)}$**=0.0001** $W_{24}^{(1)}$**=0%**
Unit-1	p_{11}**=0.9** $W_{11}^{(0)}$ **=100%**	$P_{21}^{(1)} \cdot p_{11} \approx 0.6038$ min $(W_{21}^{(1)}, W_{11}^{('0)})$ =100%	$P_{22}^{(1)} \cdot p_{11} \approx 0.0325$ min$(W_{22}^{(0)}, W_{11}^{('1)})$ =80%			$P_{23}^{(1)} \cdot p_{11} \approx 0.0003$ min$(W_{23}^{(0)}, W_{11}^{('1)})$ =20%	$P_{24}^{(1)} \cdot p_{11} \approx 0.0001$ min$(W_{24}^{(0)}, W_{11}^{('1)})$ =0%
	p_{12}**=0.05** $W_{12}^{(0)}$**=70%**	$P_{21}^{(1)} \cdot p_{12}$ $p_{12}p_{12} \approx 0.0386$ min$(W_{21}^{(0)}, W_{12}^{('1)})$ =70%	$P_{22}^{(1)} \cdot p_{12} \approx 0.0018$ min$(W_{22}^{(0)}, W_{12}^{('1)})$ =70%			$P_{23}^{(1)} \cdot p_{12} \approx 0$ min$(W_{23}^{(0)}, W_{12}^{('1)})$ =20%	$P_{24}^{(1)} \cdot p_{12} \approx 0$ min$(W_{24}^{(0)}, W_{12}^{('1)})$ =0%
	p_{13}**=0.04** $W_{32}^{(0)}$**=40%**	$P_{21}^{(1)} \cdot p_{13}p_{12} \approx 0.0308$ min$(W_{21}^{(0)}, W_{13}^{('1)})$ =40%	$P_{22}^{(1)} \cdot p_{13} \approx 0.0014$ min$(W_{22}^{(0)}, W_{13}^{('1)})$ =40%			$P_{23}^{(1)} \cdot p_{13} \approx 0$ min$(W_{23}^{(0)}, W_{13}^{('1)})$ =20%	$P_{24}^{(1)} \cdot p_{13} \approx 0$ min$(W_{24}^{(0)}, W_{13}^{('1)})$ =0%
	p_{14}**=0.01** $W_{14}^{(0)}$**=0%**	$P_{21}^{(1)} \cdot p_{14} \approx 0.00771$ min$(W_{21}^{(0)}, W_{14}^{('1)})$ =0%	$P_{22}^{(1)} \cdot p_{14} \approx 0.0004$ min$(W_{22}^{(0)}, W_{14}^{('1)})$ =0%			$P_{23}^{(1)} \cdot p_{14} \approx 0$ min$(W_{23}^{(0)}, W_{14}^{('1)})$ =0%	$P_{24}^{(1)} \cdot p_{14} \approx 0$ min$(W_{24}^{(0)}, W_{14}^{('1)})$ =0%

Thus, for vector (1, 0) one has additional cost equal 1 and $W_{syst}^{(1,0)} \approx 0.9502$ and for vector (0, 1) corresponding values equal to 2 and 0.671, so system configuration (1, 0) is dominating over configuration (0, 1), since higher average performance level delivers with less expenses. It means that all vectors of type (0, *k*) are excluded from further analysis.

The next cells, for which current trials have to be done, are cells (1, 1) and (2, 0) in accordance with self-explanatory Figure 6.

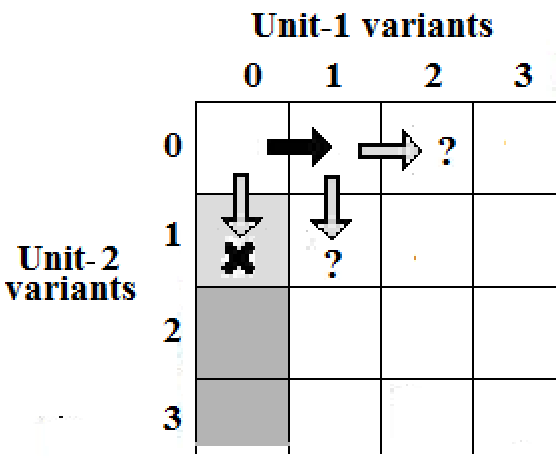

Figure 6. Directions of further analysis of cells

The next cells under investigation are (2, 0) and (1, 1), one can see from Figure 7.

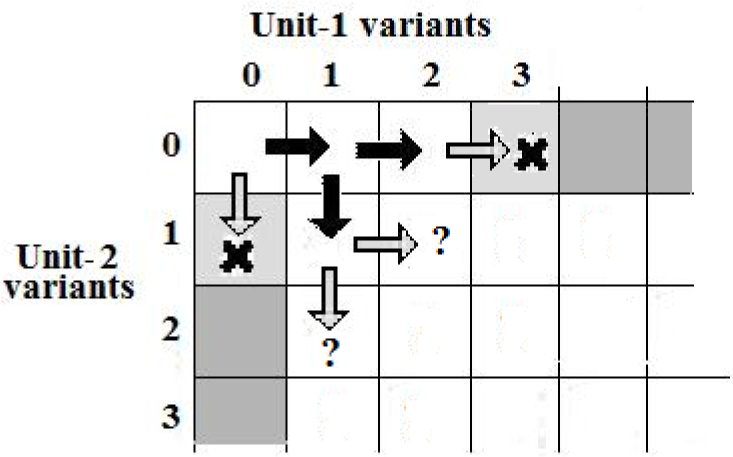

Figure 7. Further development of checking cells

Avoiding simple, however cumbersome calculations, let us present only final results (see Table 5).

Table 7. Costs and levels of performance for different vectors of redundant elements.

		Unit-1: Number of redundant elements						
		0	1	2	3	4	5	6
Unit-2: Number of redundant elements	0	C=0 W= 90.16	C=1 W= 94.26	C=2 W= 94.57				…
	1	C=2 W= 94.68	C=3 W= 99.16	C=4 W= 99.50	C=5 W= 99.53			…
	2	C=4 W= 95.03	C=5 W= 99.54	C=6 W= 99.89	C=7 W= 99.92	**?**		…
	3		C=7 W= 99.61	C=8 W= 99.95	**?**			….
	4							….
	…	…	….	…	…	…	…	…
	Legend: light grey color – dominated cells, dark grey color – non-prospective variants.							

Probably, the last table needs some explanations. System without redundant elements initially has average level of performance (W) equals 90.16%. Next phase of calculation is checking neighbor cells to cell (0, 0), i.e. (1, 0) and (0, 1). After adding a redundant element of the 1st type, one gets W= 94.26% and after adding a redundant element of the 2nd type, one gets W= 94.68%. Both cells contain dominating vectors of redundant elements. Next phase of trials are vectors (2, 0), (1, 1) and (0, 2). Vectors (1, 1) gives W=94.57 with total cost of redundant elements C=3. Vector (2, 0) is dominated by vector (0, 1) since possesses lower value with the same expenses for redundant elements. Therefore all vectors of type (3, 0), (4, 0) and so on, are excluded from further trials. Vector (1, 1) is dominating.

Next phase is trial of neighbor cells to the currently existing cells with dominating vectors, These cells are (2, 1), (1, 2) and (0, 3) (Remind that vector (3, 0) is excluded as dominated one.) As one can see from Table 5, vector (2, 1) dominates over vector (0, 2), so all vectors of type (0, 3), (0, 4) and soon are excluded from further trials. Vectors (1, 2) and (2, 1) belong to the dominating sequence of vectors.

Such trials and selection of dominating vectors continued until appearance of first vector with the average level of performance higher than required value of W^o for the direct problem of optimal redundancy, or until total expense of all redundant elements are nor exceed given value C^o for the inverse problem. These comments become absolutely transparent if one take a look on Figure 8.

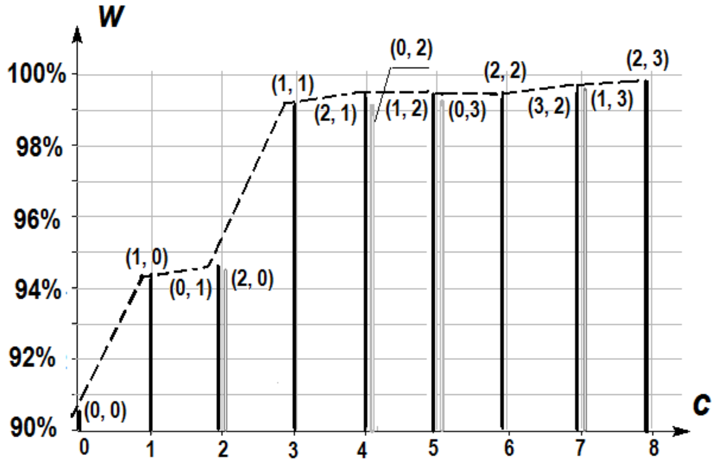

Figure 8. Depiction of the process of compiling the dominating sequence

From Table 5, one can see that optimal solution for requirement that the average level of system performance is not less than W^o =0.999 is delivered by vector (3,2), and the total expenses of redundant elements is 7 cost units. For the total expenses on redundant elements limited by $C^o \leq 4$ cost units, one gets maximum possible solution as vector (1, 2) that characterizes by W=99.54%.

It is interesting what happens with the optimal solution if one changes costs of elements> Let us assume that for the same system cost of a single redundant element of the 1st type is c_1=2 and the cost am element of the 2nd type c_2=1.

Table 8. Costs and levels of performance for different vectors of redundant elements for new elements' costs.

		Unit-1: Number of redundant elements						
		0	1	2	3	4	5	6
Unit-2: Number of redundant elements	0	C=0 W=90.156	C=2 W= 94.26456					…
	1	C=1 W= 94.68072	C=3 W= 99.16318	C=5 W= 99.50462				…
	2	C=2 W= 95.02683	C=4 W= 99.54058	C=6 W= 99.88507	C=8 W= 99.9156			…
	3	C=3 W= 95.08558	C=5 W= 99.60514	C=7 W= 99.9502	?			… .
	4		C=6 W= 99.61784	?				… .
	…	…	… .	…	…	…	…	…
	Legend: light grey color – dominated cells, dark grey color – non-prospective variants.							

In this case optimal solutions found from Table 6 are: For the direct problem vector (2, 3), for which W=99.95% and total expenses on redundant elements are equal to 7 cost units, and for inverse problem the solution is (1, 2), for which W=99.54% and total expenses C=4. For inverse problem, the solutions coincide with each other in both cases.

Solution of optimal redundancy problems for system consisting of several multilevel units seems a bit cumbersome. However, let us note that all enumerative methods like dynamic programming practically unsolvable without computerizing calculations. Numerical example above was solved with the help of a simple programs using Microsoft Excel.

For complex systems consisting of *n* multiple multistate units, one can compile a simple program for a mainframe computer. The algorithm should include the following steps.

i. FIRST STEP

1. Take an *n*-dimensional vector of redundant elements $X^{(0)} = (x_1^{(0)} = 0, x_2^{(0)} = 0, ..., x_n^{(0)} = 0)$.
2. Perform calculations to get initial pair of values $(W_{syst}^{(0)}, C_{syst}^{(0)})$ (see Table.2).
3. Put calculated pair $(W_{syst}^{(0)}, C_{syst}^{(0)})$ into list of dominating solutions,

ii. SECOND STEP

4. Generate vectors $X_i^{(1)}$ such that each of them distinguishes from $X^{(0)}$ by changing number of elements of Unit-*i* on one, i.e. $X_i^{(1)} = (x_1^{(0)} = 0, x_2^{(0)} = 0, ..., x_i^{(0)} = 1, ..., x_n^{(0)} = 0)$.

iii. THIRD STEP

5. For each $X_i^{(1)}, \overline{i = 1, n}$, calculate new values of $P_{ik_i}^{(1)}$, for all k_i where k_i is the number of performance levels of Uniy-*I* (see Tables 3 and 5).
6. Perform corresponding calculations for getting *n* pairs $(W_1^{(1)}, C_1^{(1)}), (W_2^{(1)}, C_2^{(1)}), ..., (W_n^{(1)}, C_n^{(1)})$, for all vectors (see Tables 4 and 6).
7. Analyze all pairs obtained in previous point to form a set $G^{(1)}$ that includes only dominating vectors $X_i^{(1)}$.
8. Return to the 3rd step, using vectors belonging to set $G^{(1)}$

Stopping rules:

(a) For direct optimization problem, choose such a vector $X_i^{(k)}$ among $G^{(k)}$ that was obtained at the *k*-th step of the optimization process that delivers $\min_{\forall i \in G^{(k)}} C_i^{(1)}$ for all $W_i^{(k)} \geq W^0$.

(b) For inverse optimization problem, choose such a vector $X_i^{(k)}$ among $G^{(k)}$ that was obtained at the *k*-th step of the optimization process that delivers $\max_{\forall i \in G^{(k)}} W_i^{(1)}$ for all $C_i^{(k)} \leq C^0$..

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