MEAN RESIDUAL LIFE CRITERIA OF FIRST PASSAGE TIME OF SEMI-MARKOV PROCESS BASED ON TOTAL TIME ON TEST TRANSFORMS

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ABSTRACT

Mean residual life criteria of first passage time of semi-Markov process is considered. Properties of transition probability functions when using scaled Total Time on Test (TTT) transform for some criteria of mean residual life are discussed. Application to Multistate reliability system is also addressed.

1 INTRODUCTION

First passage times of appropriate stochastic process have often been used to represent times to failure of devices or systems which are subject to shocks and wear, random repair time and random interruptions during their operations. The life distribution properties of these processes have therefore been widely investigated in Multistate system reliability and maintenance literature. The life distributions involved in devices or systems have several interesting properties such as increasing mean residual life (IMRL), decreasing mean residual life (DMRL), etc. The total time on test (TTT) transform is used as a tool for identification of failure distribution model in binary system. Marshell and Shaked (1983), (1986) and Shantikumar (1984) considered processes with new better than used (NBU) first passage times. Belzunce et al. (2002) derived, for the uniformizible, continuous time Markov process, conditions in terms of discrete uniformized Markov chain for the second order NBU and NBU based on laplace transformation classes.

Karasu and Ozekici (1989) studied NBUE and new worse than used in expectation (NWUE) properties of increasing Markov processes and Markov Chains. Lam (1992) considered the NBUE and NWUE properties of Markov renewal processes.

Use of TTT transform for the identification of failure rate models is discussed by Barlow and Campo (1975). Later, Klefsjo (1982) presented some relationship between the TTT transform and other ageing properties (with their duals) of random variable, eg. decreasing mean residual life (DMRL), NBU, NBUE, harmonically new better than used in expectation (HNBUE) and heavy tailedness. Abouammoh and Khalique (1987) further discussed properties of scaled TTT transform for some criteria of the mean residual life such as decreasing mean residual life average (DMRLA), decreasing harmonic mean residual life average (DHMRLA), new better than used harmonic mean residual life average (NBUHMRLA), and new better than used mean residual life average (NBUMRLA).

But when we consider a complex system whose performance process is Markov or semi-Markov, we need the knowledge of DMRLA properties or other relevant ageing properties for applying suitable maintenance and repair/replacement policies. The identification of failure rate model of a system whose performance process is Markov/semi-Markov will be helpful to the engineers and designers for applying suitable maintenance and repair or replacement policies, since identification of failure rate model using TTT describes new methods for analyzing nonnegative observations. Chacko et.al (2010) discussed use of TTT transform in identifying failure rate model of semi-Markov reliability system.

In this paper, we consider a semi-Markov process whose first passage time distribution is DMRL or DMRLA or NBUHMRL or NBUMRL (with their duals). We consider the reliability function based on the transition probability function in the upstates.

This paper is arranged as follows. Section 2 describes various ageing properties of a lifetime random variable. Section 3 recall the existing results for identification of failure rate model of random variables based on TTT. In Section 4, we introduce some sufficient conditions for the MRL criteria of the semi-Markov process based on TTT built from transition probability function. An illustrative example for multistate system is given in section 5. Conclusions are given at the last section.

2. AGEING OF A LIFETIME RANDOM VARIABLE

The concept of ageing is very important in reliability theory. 'No ageing' means the age of a component has no effect on the distribution of residual life time. 'Positive ageing' describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of the component. On the other hand, 'Negative ageing' has an opposite effect on the residual lifetime.

Let R(t) = 1 − F(t) be the survival or reliability function of a lifetime random variable. Let X_t be the random variable representing the residual life time of a unit which has attained the age t. Then the respective distribution function and survival function are $F_x(t)$ and $R_x(t)$. Next it is seen that $R_x(t) = R(t+x) / R(t)$. This is the conditional probability that the unit survived up to time t, will not fail before additional x units of time. Further $R_0(t) = R(t)$.

By positive ageing we mean the phenomenon where by an older system has shorter remaining life time in some statistical sense than a newer or younger one.

That is $R_x(t) = R(t+x)/ R(t) < R_0(t)$ or
$R_x(t) = R(t+x)/R(t)$ is decreasing in t.
Similarly, for negative ageing, $R_x(t) = R(t+x)/R(t)$ is increasing in t.
Obliviously, any study of the phenomenon of ageing is to be based on $R_x(t)$ and functions related.

2.1 Failure Rate Function

The conditional failure rate or failure rate at time t is defined as

$$\lambda(t) = \lim_{x\to 0} \frac{F(t+x)-F(t)}{xR(t)},$$

so that, $\lambda(t) = \frac{f(t)}{R(t)}$ when F is absolutely continuous and f(t) is the probability density function of F(t). The failure rate function has been extensively studied in literature since it is very important parameter in reliability theory.

Another important order is mean residual life order. The definition of mean residual life is given below.

2.2 Mean Residual Life

Let T_D be the lite time. The mean residual life is defined as

$$\mu(t) = E(T_D - t | T_D > t) = \frac{\int_t^\infty R(x)dx}{R(t)}$$

if $t < \infty$ and zero otherwise.

The failure rate function $\lambda(t)$ will be continuous and twice differentiable for all t > 0 with the exception of the exponential distribution.

3. TOTAL TIME ON TEST TRANSFORM

Total time on test (TTT) transform is a fundamental tool in reliability investigation.

Let X has distribution F. Given a sample of size n from the non-negative random variables X, let $X_{(1)} \le X_{(2)} \le ... \le X_{(k)} \le ... \le X_{(n)}$ be the sample. TTT to the rth failure from distributions F and is, $T(X_{(r)}) = nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + ... + (n-r+1)(X_{(r)} - X_{(r-1)}) = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}$

Define

$H_n^{-1}(r/n) = \frac{1}{n} T(X_{(r)})$ and $H_n^{-1}(r/n) = \int_0^{F_n^{-1}(r/n)} (1 - F_n(u))du$,

where $F_n(u) = \begin{cases} 0 & u < X_{(i)} \\ i/n & X_{(i)} \le u < X_{(i+1)} \\ 1 & X_{(n)} > u \end{cases}$ $F_n^{-1}(x) = \inf\{x : F_n(x) \ge u\}$,

The fact that $F_n(u) \to F(x)$ a.s. implies, by Glivenko Cantelli Theorem, $\lim_{n\to\infty, r/n \to t} \int_0^{F_0^{-1}(r/n)} (1 - F_n(u))du = \int_0^{F^{-1}(t)} (1 - F(u))du \quad t \in [0,1].$

We define TTT transform of F as $H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(u))du \ t \in [0,1].$

3.1. Model Identification

TTT is a very important index in reliability for the model identification of lifetime data. TTT test plots are useful for analyzing non-negative data. Using these plots incomplete data can be analyzed and there is a theoretical basis for such an analysis. We can define ageing properties in terms of TTT transforms. TTT transforms permits us to classify distributions according to their failure rate. Total time on test plots also permits the comparison of distribution functions with respect to their failure rate and they can be used to find a model for the data under study. Thus in many aspects TTT is very important part in the study of reliability. But the existing results are limited to the case of random variables. We extend the study to the case of semi-Markov process in the next section, for the first passage time random variable.

Let $G(x) = 1 - \exp(-x/\theta), \quad x, \theta \ge 0$ be the exponential distribution with mean μ. Then

$H_G^{-1}(t) = \int_0^{G^{-1}(t)} e^{-x/\theta} dx = \int_0^{G^{-1}(t)} \theta dG(x) = \theta t$ and scaled TTT,

$$\phi(t) = \frac{H_G^{-1}(t)}{H_G^{-1}(1)} = t, \quad t \in [0,1]. \tag{3.1}$$

The scaled TTT of the Exponential distribution is a 45^0 line on $[0,1]$. The normalized total time on test is the boundary between the corresponding transforms of IFR and DFR distributions. TTT that permits to classify distributions according to their failure rate is that its slope evaluated at *t* = *F*(*x*) is the reciprocal of the failure rate at *X*.

$$\frac{d}{dt}H_F^{-1}(t)|_{t=F(x)} = \frac{(1-t)}{f[F^{-1}(t)]}|_{t=F(x)} = \frac{1-F(x)}{f(x)} = \frac{1}{\lambda(x)}, \tag{3.2}$$

where λ is the failure rate of *F*.

Now we consider the following results in Barlow and Campo (1975).

Proposition 3.1 *F IFR(DFR))* $\Rightarrow \frac{H_F^{-1}(t)}{H_F^{-1}(1)}$ *is concave or (convex) in* $t \in [0,1]$.

Proposition 3.2 *F IFRA(DFRA) implies* $\frac{H_F^{-1}(t)}{tH_F^{-1}(1)} \uparrow (\downarrow)$ *in* $t \in [0,1]$.

Proposition 3.3 *F NBU(NWU))* $\Rightarrow$ *slope of* $\frac{H_F^{-1}(t)}{H_F^{-1}(1)}$ *is larger(smaller) at the origin than at any other t,* $0 < t < 1$.

For *F* NWU reverse the direction of inequalities. Now we consider a characterization for NBUE (NWUE) and DMRL (IMRL), see Klefsjo (1982).

Theorem 3.1 *A life distribution F is NBUE(NWUE) if and only if* $\phi(t) \le (\ge) t$ *for* $0 \le t \le 1$.
Theorem 3.2 *A life distribution F is DMRL(IMRL) if and only if* $Q(t) = (1-\phi(t))/(1-t)$ *is decreasing (increasing) for* $0 \le t \le 1$.

The following are some important results in model identification of a univariate lifetime random variable.

Theorem 3.3 (Abouammoh and Khalique (1987)) Let F and $\phi_F(t)$ be as in above theorem, then we have the following

1. F is DMRL (IMRL) if and only if $1-\phi_F(t)-(1-t)\phi(t) \le (\ge) 0, 0 < t \le 1$
2. F is NBUMRL(NWUMRL) if and only if $\phi_F(t) \ge (\le) t, 0 \le t \le 1$.
3. F is DMRLA (IMRLA) if and only if $1/t\int_0^t (1-\phi_F(x))/(1-x)dx$ is decreasing (increasing) for 0 < t < 1.
4. F is NBAMRL (NWAMRL) if and only if $\int_0^t (1-\phi_F(x))/(1-x)dx \le (\ge) t$ for $0 \le t < 1$..
5. F is DHMRLA (IHMRLA) if and only if $1/t\int_0^t (1-x)/(1-\phi_F(x))dx$ is increasing (decreasing) in t for 0 < t < 1.
6. F is NBUHMRL(NWUHMRL) if and only if $\int_0^t (1-x)/(1-\phi_F(x))dx \ge (\le) t$ for $0 \le t < 1$.

But the existing results are limited to the case of random variables. We extend the study to the case of semi-Markov process in the next section, for the first passage time random variable.

4. MEAN RESIDUAL LIFE CRITERIA OF A SEMI-MARKOV SYSTEM

We are concerned with a multistate system (MSS) having $M + 1$ states 0,1,…,M where '0' is the best state and '*M*' is the worst state, see Barlow and Wu (1978) for details of MSSs. At time zero the system begins at its best state and as time passes the system begins to deteriorate. It is assumed that the time spent by the system in each state is random with arbitrary sojourn time distribution. The system stays in some acceptable states for some time and then it moves to unacceptable (down) state. The first time at which the MSS enters the down state after spending a random amount of time in acceptable states is termed as the first passage time (failure time) to the down state of the MSS. We study the aging properties of the first passage time distribution of the MSS modeled by the semi-Markov process $\{Y_t, t \geq 0\}$. In the MSS with states *{0,1,…,k,k+1,…,M}* where *{0,1,…,k}* is the acceptable states, the sojourn time between state '*i*' to state '*j*' is assumed to be distributed with arbitrary distribution F_{ij}.

4.1 First Passage time and Reliability Function

Let $E = \{0,1,\ldots,M\}$ be a set representing the state of the MSS and probability space with probability function *P*, on which we define a bivariate time homogeneous Markov chain $(X,T) = \{X_n, T_n, n \in \{0,1,2,\ldots\}\}$, X_n takes values of *E* and T_n on the half real line $R^+ = [0,\infty)$, with $0 \leq T_1 \leq T_2 \leq \ldots \leq T_n \leq \ldots$ Put $U_n = T_n - T_{n-1}$ for all $n \geq 1$. This Markov process is called a Markov renewal process (MRP) with transition function, the semi-Markov kernel, Q = [Q_{ij}], where $Q_{ij}(t) = P[X_{n+1} = j, U_n \leq t \mid X_n = i], i, j \in E, t \geq 0$ and $Q_{ii}(t) = 0, i \in E, t \geq 0$.

Now we consider the semi-Markov process (SMP), as defined in Pyke (1961). It is the generalization of Markov process with countable state space. SMP is a stochastic process which moves from one state to another of a countable number of states with successive states visiting form a Markov chain, and that the process stays in a given state a random length of time, the distribution of which may depend on this state as well as on the one to be visited in the next. Define $Z_t = X_{N_t}, N_t = \sup\{n, T_n = U_1 + U_2 + \ldots + U_n \leq t\}$, it is the semi-Markov process associated with the MRP defined above. In terms of *Z*, the times $T_1, T_2, \ldots$ are successive times of transitions for *Z*, and $X_0, X_1, \ldots$ are successive states visited.

If Q has the form $Q_{ij}(t) = P[X_{n+1} = j \mid X_n = i][1 - e^{-\lambda(i)t}], i, j \in E, t \geq 0,$ for some function λ(*i*), $j \in E$ then the process Z_t is a Markov process. That is, in a Markov process, the distributions of the sojourn times are all exponential independent of the next state. The word *semi*-Markov comes from the somewhat limited Markov property which *Z* enjoys, namely, that the future of *Z* is independent of its past given the present state provided the "present" is the time of jump. Let I_{ij} =indicator function of $\{i = j\}$. Define the transition probability that system occupied state $j \in E$ at time $t > 0$, given that it is started at state *i* at time zero, as, $i, j \in E, t \geq 0$

$$p_{ij}(t) = P[Z_t = j \mid Z_0 = i] = P[X_{N_t} = j \mid X_0 = i] = h_i(t) I_{ij} + Q * P(t)(i,j),$$

where $h_i(t) = 1 - \sum_k Q_{ik}(t)$, $P(t) = [p_{ij}(t)]$ and $Q * P(t)(i,j) = \sum_{k \in E} \int_0^t Q_{ik}(dx) p_{kj}(t-x)$

To obtain the reliability function of the semi-Markov system described above, we must define a new process, *Y* with state space $U \cup \nabla$, where *U* denotes set of all up states *{*0*;* 1*; :::; k}* and ∇ is the absorbing state in which all the states *{k +* 1*, …;M}* of the system is united. Let T_D denote the time of first entry to the down states of *Z* process.

That is, $Y_t = Z_t(\omega)$ if $t < T_D(\omega)$ and $Y_t = \nabla$ if $t \geq T_D(\omega)$.

Let $1=(1,1,...,1)^1$, a unit row vector with appropriate dimension. The process Y_t is a semi-Markov process with semi-Markov kernel

$$\begin{bmatrix} \overbrace{Q_{11}(t)}^{Up} & \overbrace{Q_{12}(t)}^{Down} \\ 0 & 0 \end{bmatrix}$$

We denote $\alpha=(\overbrace{\alpha(0),...,\alpha(k)}^{Up,\alpha_1},\overbrace{\alpha(k+1),...,\alpha(M)}^{Down,\alpha_2})$ where $\alpha(i)=P(Y_0=i)$.

The reliability function is

$$R(t)=P[\forall u\in[0,t],Z_u\in U]=P[Y_t\in U]=\sum_{j\in U}P[Y_t=j]$$

$$=\sum_{i\in U}\sum_{j\in U}P[Y_t=j,Y_0=i]$$

$$=\sum_{i\in U}\sum_{j\in U}p_{ij}(t)\alpha(i).$$

4.2. Model Identification

In order to identify the failure rate behavior of a semi-Markov system based on the transition probability function, we define the TTT based on transition probability function in up states as follows. Let F be the first passage time distribution of a semi-Markov system, define

$$H_{p_{ij}}^{-1}(t)=\int_0^{F^{-1}(t)}p_{ij}(u)du,\forall i,j\in U,t\in[0,1]$$

where

$$F^{-1}(t)=\inf\{(1-R(t))\geq t\}$$

$$=\inf\left\{x:\left(1-\sum_{i\in U}\sum_{j\in U}p_{ij}(x)\alpha(i)\right)\geq t\right\}$$

$$=\inf\left\{x:\left(\sum_{i\in U}\sum_{j\in U}p_{ij}(x)\alpha(i)\right)\leq 1-t\right\}$$

But

$$H_F^{-1}(t)=\int_0^{F^{-1}(t)}\sum_{i\in U}\sum_{j\in U}p_{ij}(u)du,\forall i,j\in U,t\in[0,1]$$

$$=\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)}p_{ij}(x)dx$$

$$=\sum_{i\in U}\sum_{j\in U}\alpha(i)H_{p_{ij}}^{-1}(t)$$

Then

$$H_F^{-1}(1)=\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)}p_{ij}(x)dx=\sum_{i\in U}\sum_{j\in U}\alpha(i)H_{p_{ij}}^{-1}(1)\quad \frac{H_F^{-1}(t)}{H_F^{-1}(1)}=\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)}p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)}p_{ij}(x)dx},\quad t\in[0,1].$$

Chacko and Manoharan (2009) proved the following if $p_{ij}(x)$ is monotonic increasing or decreasing.

Proposition 4.1 *The first passage time distribution of a semi-Markov system F is IFR if* $\frac{H_F^{-1}(t)}{H_F^{-1}(1)} \le t$ *and concave in* $t \in [0,1], \forall i, j \in U$.

Proposition 4.2 *The first passage time distribution of a semi-Markov system F is DFR if* $\frac{H_F^{-1}(t)}{H_F^{-1}(1)} \ge t$ *and* $\frac{H_{p_{ij}}^{-1}(t)}{H_{p_{ij}}^{-1}(1)}$ *convex in* $t \in [0,1], \forall i, j \in U$.

Remark 4.1 *The constant failure rate model arises when it is both IFR and DFR. Therefore we must have* $\frac{H_{p_{ij}}^{-1}(t)}{H_{p_{ij}}^{-1}(1)} = t, \quad \forall i, j \in U$.

Proposition 4.3 *The first passage time distribution of a semi-Markov system F IFRA implies* $\sum_{i\in U}\sum_{j\in U}\frac{d}{dt}H_{p_{ij}}^{-1}(t) \ge 1, \quad t \in [0,1].$

Proposition 4.4 *The first passage time distribution of a semi-Markov system F DFRA implies* $\sum_{i\in U}\sum_{j\in U}\frac{d}{dt}H_{p_{ij}}^{-1}(t) \le 1, \quad t \in [0,1].$

Proposition 4.5 *F NBU(NWU)) implies*

$$\left(\sum_{i\in U}\sum_{j\in U}\frac{d}{dt}H_{p_{ij}}^{-1}(t)\,|_{t=0} - \sum_{i\in U}\sum_{j\in U}\frac{d}{dt}H_{p_{ij}}^{-1}(t)\right) \ge (\le) 0, \quad t \in [0,1], \quad i, j \in U.$$

Proposition 4.6 *The first passage time distribution of a semi-Markov system F is NBUE* $\frac{H_{p_{ij}}^{-1}(t)}{H_{p_{ij}}^{-1}(1)} \le t, \quad \forall i, j \in U, \quad t \in [0,1].$

Proposition 4.7 *The first passage time distribution of a semi-Markov system F is NWUE if* $\frac{H_{p_{ij}}^{-1}(t)}{H_{p_{ij}}^{-1}(1)} \ge t, \quad \forall i, j \in U, \quad t \in [0,1].$

Proposition 4.8 *The first passage time distribution of a semi-Markov system F is DMRL(IMRL) if*

$$\frac{dH_{p_{ij}}^{-1}(t)}{dt} \ge (\le) 1, \quad \forall i, j \in U, \quad t \in [0,1].$$

We prove the following.

Theorem 4.1. Let F and $\phi_F(t)$ be as in above theorem, then we have the following

1. F is DMRL (IMRL) if

$$\left(\int_0^{F^{-1}(1)} p_{ij}(x)dx - \int_0^{F^{-1}(t)} p_{ij}(x)dx\right)/(1-t) \le (\ge) \frac{d\int_0^{F^{-1}(t)} p_{ij}(x)dx}{dt}, \quad \forall i, j \in U, \quad t \in [0,1].$$

Proof:

$$\left(\int_0^{F^{-1}(1)} p_{ij}(x)dx - \int_0^{F^{-1}(t)} p_{ij}(x)dx\right)/(1-t) \le (\ge) \frac{d\int_0^{F^{-1}(t)} p_{ij}(x)dx}{dt}, \quad \forall i, j \in U, \quad t \in [0,1].$$

It implies, for

$$\left(\int_0^{F^{-1}(1)} p_{ij}(x)dx - \int_0^{F^{-1}(t)} p_{ij}(x)dx\right) - (1-t)\frac{d\int_0^{F^{-1}(t)} p_{ij}(x)dx}{dt} \le (\ge) 0, \quad \forall i, j \in U, \quad t \in [0,1].$$

Taking over U and dividing both sides by $\dfrac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{}$, we get

$$\left(1-\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}\right)-(1-t)\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)d\int_0^{F^{-1}(t)} p_{ij}(x)dx}{dt\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}\le(\ge)0,\quad \forall i,j\in U,\quad t\in[0,1].$$

By theorem (3.3), we proved the result.

2. F is NBUMRL (NWUMRL) if $\left(\dfrac{\int_0^{F^{-1}(t)} p_{ij}(x)dx}{\int_0^{F^{-1}(1)} p_{ij}(x)dx}\right)\ge(\le)t, i,j\in U,\quad t\in[0,1]$.

Proof: Suppose $\left(\dfrac{\int_0^{F^{-1}(t)} p_{ij}(x)dx}{\int_0^{F^{-1}(1)} p_{ij}(x)dx}\right)\ge(\le)t, i,j\in U,\quad t\in[0,1]$

$$\Rightarrow\left(\int_0^{F^{-1}(1)} p_{ij}(x)dx-t\int_0^{F^{-1}(t)} p_{ij}(x)dx\right)\ge(\le)0\quad \forall i,j\in U,\quad t\in[0,1]$$

$$\Rightarrow\left(\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx-t\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx\right)\ge(\le)0,\quad t\in[0,1]$$

$$\Rightarrow\left(\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx}\right)\ge(\le)t,\quad t\in[0,1]$$

By theorem (3.3) we proved the result.

3. F is DMRL (IMRL) if $\left(\dfrac{\int_0^{F^{-1}(1)} p_{ij}(x)dx}{t}\right)$, $t\in[0,1]$ is decreasing (increasing) in t, $i,j\in U,\quad t\in[0,1]$.

Proof: Suppose if $\left(\dfrac{\int_0^{F^{-1}(1)} p_{ij}(x)dx}{t}\right)$, $t\in[0,1]$ is decreasing (increasing) in t, $i,j\in U,\quad t\in[0,1]$.

Then $\left(\dfrac{\int_0^{F^{-1}(t)} p_{ij}(x)dx}{\int_0^{F^{-1}(1)} p_{ij}(x)dx}\right)\ge(\le)t, i,j\in U,\quad t\in[0,1]$

$$\Rightarrow\left(\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx-t\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx\right)\ge(\le)0,\quad t\in[0,1]$$

$$\Rightarrow \left(1 - \frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx} \right) \le (\ge) 1-t, \quad t\in[0,1]$$

$$\Rightarrow \tfrac{1}{1-t}\left(1 - \frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx} \right) \le (\ge) 1, \quad t\in[0,1].$$

Rate of increase of $\Rightarrow \tfrac{1}{1-t}\left(1 - \frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx} \right), \quad t\in[0,1]$ is smaller (larger) than that of t.

By theorem (3.3) we proved the result.

4. F is NBAMRL (NWAMRL) if

$$\left(1/t\int_0^{F^{-1}(1)} p_{ij}(x)dx \ge (\le) \int_0^{F^{-1}(t)} p_{ij}(x)dx \right) \quad \forall i,j\in U, \quad t\in[0,1]$$

Proof: Suppose, $\left(1/t\int_0^{F^{-1}(1)} p_{ij}(x)dx \ge (\le) \int_0^{F^{-1}(t)} p_{ij}(x)dx \right) \quad \forall i,j\in U, \quad t\in[0,1]$

$$\Rightarrow \left(\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx} \right) \ge (\le) t, \quad t\in[0,1]$$

$$\Rightarrow \tfrac{1}{1-t}\left(1 - \frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx} \right) \le (\ge) 1, \quad t\in[0,1]$$

By theorem (3.3) we proved result.

5. F is DHMRLA (IHMRLA) if $\left(\frac{\int_0^{F^{-1}(1)} p_{ij}(x)dx}{t} \right), \quad t\in[0,1]$, is increasing (decreasing) in t, $i,j\in U, \quad t\in[0,1]$.

Proof: Given**,** $\left(1/t\int_0^{F^{-1}(1)} p_{ij}(x)dx \ge (\le) \int_0^{F^{-1}(t)} p_{ij}(x)dx \right) \quad \forall i,j\in U, \quad t\in[0,1]$

$$\Rightarrow \tfrac{1}{1-t}\left(1 - \frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx} \right) \le (\ge) 1, \quad t\in[0,1]$$

$$\Rightarrow 1 \le (\ge) \frac{1-t}{\left(1-\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx}\right)}, \quad t\in[0,1]$$

The rate of increase of $\int_0^t \frac{1-u}{\left(1-\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(u)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx}\right)}, \quad t\in[0,1]$ is larger than that of t.

$$\Rightarrow 1/t\int_0^t \frac{1-u}{\left(1-\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(u)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx}\right)}, \quad t\in[0,1] \text{ is increasing in t.}$$

6. F is NBUHMRL (NWUHMRL) if

$\left(1/t\int_0^{F^{-1}(1)} p_{ij}(x)dx \ge (\le)\int_0^{F^{-1}(t)} p_{ij}(x)dx\right) \quad \forall i,j\in U, \quad t\in[0,1]$.

Proof: $\left(1/t\int_0^{F^{-1}(1)} p_{ij}(x)dx \ge (\le)\int_0^{F^{-1}(t)} p_{ij}(x)dx\right) \quad \forall i,j\in U, \quad t\in[0,1]$

$$\Rightarrow \int_0^t \frac{1-u}{\left(1-\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(u)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx}\right)} \ge (\le)\int_0^t dt, \quad t\in[0,1].$$

Hence by theorem (3.3), we proved the result.

5 APPLICATION AND ILLUSTRATIVE EXAMPLE

We are concerned with a multistate system (MSS) having M+1 states 0, 1, ...,M where '0' is the best state and 'M' is the worst state, see Chacko and Manoharan (2009) for details of MSSs. At time zero the system begins at its best state and as time passes the system begins to deteriorate. It is assumed that the time spent by the system in each state is random with arbitrary sojourn time distribution. The system stays in some acceptable states for some time and then it moves to unacceptable (down) state. The first time at which the MSS enters the down state after spending a random amount of time in acceptable states is termed as the first passage time (failure time) to the down state of the MSS. Major application of the above results is in maintenance and repair of complex systems such as age and block replacement policies. A variety of applications in maintenance and replacement policies of a binary system can be seen in Barlow and Proschan (1996).

Example 5.1 *Consider a Markov process in continuous time and discrete state space {1,2,…,M} given in Doob (1953), p.241.* The system starts in state '1' at time zero and as it enters 'M', it remains there. Consider the intensity matrix, $Q=[q_{ij}]$ with entries

$q_{ij} = 0, \quad i \in \{1,2,...,M-1\}, j \neq i+1, q_{ii+1} = q, \quad q_{iM} = 0.$

The Kolmogorov's system of differential equation becomes,
for $p_{ij}(t-u) = P[Y_t = j \mid Y_u = i], 0 \leq u < t$ and we take $u = 0$,

$p^1_{ik}(t) = -qp_{ik}(t) + qp_{i+1k}(t), i < M, \quad p_{Mk}(t) = 0$

with initial conditions, $p_{ik}(0) = \delta_{ik}$, the indicator of *{i=k}*. Then,

$p_{Mk}(t) = 0, \quad k \neq M, p_{MM}(t) = 1$

and it is easily verified that the solution is

$$p_{ik}(t) = \begin{cases} 0, \quad k < i \\ \dfrac{(qt)^{k-i} e^{-qt}}{\Gamma(k)}, \quad i \leq k < M \\ e^{-qt}[e^{qt} - 1 - qt - ... - \dfrac{(qt)^{M-i-1}}{\Gamma(M-i)}], \quad k = M \end{cases}$$

Here the process is of monotone paths. Now consider $\forall i, j \in \{0,1,...,M-1\}$

$$H^{-1}_{p_{ij}}(1) \quad = \int_0^\infty \frac{(qt)^{k-i} e^{-qt}}{\Gamma(k)} dt = \frac{q^{k-i}}{\Gamma(k)} \int_0^\infty t^{k-i} e^{-qt} dt = \frac{\Gamma(k-i+1)}{\Gamma(k)q}$$

Therefore

$$\frac{H^{-1}_{p_{ij}}(t)}{H^{-1}_{p_{ij}}(1)} = \frac{\Gamma(k)q}{\Gamma(k-i+1)} \int_0^{F^{-1}(t)} \frac{(qu)^{k-i} e^{-qu}}{\Gamma(k)} du$$

$$= \frac{q^{k-i+1}}{\Gamma(k-i+1)} \int_0^{F^{-1}(t)} u^{k-i} e^{-qu} du$$

This is an increasing function of *t* and bounded by 1. Therefore $\dfrac{dH^{-1}_{p_{ij}}(t)/dt}{H^{-1}_{p_{ij}}(1)}$ increases *in t* $\forall i, j \in U$.

6. CONCLUSIONS

The identification of the failure rate model of first passage time distribution of a semi-Markov process is discussed. The results are applicable to systems like power generation system whose performance is measured in terms of productivity or capacity and having more than two levels of performance. Preventive or corrective maintenance can be applied to the MSS if we have the knowledge regarding its failure behavior, since type of the failure rate is an important parameter for the maintenance and replacement policies.

REFERENCES

1. Barlow, R. E. and Proschan, F. (1975) Statistical Theory of Reliability and Life Testing, Holt, Rinehart and Winston, New York.
2. R. E. Barlow and Campo, R. A. (1975) Total Time On Test Processes And Application To Failure Data Analysis , reserch report No. University of California.
3. Belzunce, F., Ortega, E. M. and Ruizon, J. M. (2002) Ageing Properties of First-Passage Times of Increasing Markov Processes, Adv. Appl. Prob., 34, 241-259.

4. Klefsjo, B. (1981) On Ageing Properties And Total Time on Test Transforms, *Scand. Jr. Statistics*., 9, 37-41
5. Chacko, V. M. and Manoharan, M. (2009) Ageing properties of first passage time distribution of multistate reliability system, ProbStat Forum: e-journal, 2, 22-30.
6. Chacko, V. M., Praveena, P. C. and M. Manoharan (2010) Ageing properties of a semi-Markov system and Total Time on test transforms, In. Trans. Mathematical Science and Computers, 14 Vol 3, 257-275. (Presented at ICM2010, Hyderbad, India).
7. Doob, J. L. (1953) Stochastic Process, John Wiley and sons, New York.
8. Karasu, I. and Ozekici, S. (1989) NBUE and NWUE properties of increasing Markov processes,J. Appl. Prob., 27, 827-834.
9. Lam, C. Y. T. (1992) New Better than used in expectation processes, J. Appl. Prob., 29, 1116-1128.
10. Marshell and Shaked (1983) New better than used processes, Adv. Appl. Prob., 15, 601-615.
11. Marshell and Shaked (1986) NBU processes with general state space, Math. Oper. Res., 11, 95-105.
12. Pyke, R. (1961) Markov renewal processes: definitions and preliminary properties, Ann. Math.Stat., 32, 1231-1242.
13. Shantikumar (1984) Processes with NBU first passage times, Adv. Appl. Prob., 16, 667-686.