COMPLEX DELTA - FUNCTION

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ABSTRACT

Smagin V.A. the brief review of a history of introduction of delta - function on a complex plane. The proof of the mathematical form of complex delta - function is given. The example of application of complex delta - function for a presence of stationary value alternation casual process of accumulation with the information income and the charge is given.

Introduction

From a history of mathematics, V.M.Kalinin, professor SPBSU in his book " my formulas " [1] 1996, referring to N.A.Lebedev, the professor of academy him A.F.Mozhajskiy, the outstanding expert in the theory of a complex variable, writes the following (so-called citations and formulas of the author of the book further are resulted):

« Delta - function $\delta(x)$ is invented not by mathematics. Her has opened for the first time, probably, Oliver Heaviside, but wide and almost at physics and engineering she has received a general recognition after her has entered into practice Pol Dirak, without the reference to compatriot Heaviside. Mathematics have recognized its right on existence only as functional, making functions its value in zero. The big and deep theory of the generalized functions has appeared. At a statement of the classical mathematical analysis delta - function (equal to zero everywhere except for a point the area under the diagram of this function is considered zero where she is equal $+\infty$, and to equal unit), - is usually ignored. The antipathy which « functions » had to similar classical mathematics is quite clear. For example, my generation mathematics and physics has grown on V.I.Smirnov's textbooks and G.M.Fihtengol'ts where delta - function is not mentioned at all ».

Now delta - function in the modern analysis is used widely. She is defined on a material axis and connected to concepts of function of distribution of probabilities and characteristic function. By the way, function of distribution to present [2]:

$$F(x) = F_1(x) + F_2(x) + F_3(x), \tag{1}$$

where $F_1(x) -$ the function of jumps growing only in points of accounting set of points of breaks of function $F(x), F_2(x) -$ continuous function, possible which points of growth form set of zero measure Lebeg, and its increment on this set is equal to an increment $F(x)$ on it. $F_2(x)$ is not absolutely continuous, the derivative of her is equal all points or to zero, or infinity. $F_3(x) -$ absolutely continuous function, with usual derivative - density of probability $f(x)$. For absolutely continuous functions characteristic function is used:

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x) = \int_{-\infty}^{+\infty} e^{itx} f(x)dx\,. \tag{2}$$

Let step Heaviside, a constant a with probability 1 is given. Its characteristic function

$$\varphi(t) = e^{iat}\,. \tag{3}$$

To it corresponds derivative - delta - function $\delta(t-a)$.

Whether there is an analogue of function Heaviside and delta - function on a complex plane? The given question can seem inappropriate, but the answer to him appears positive [1].

« As all of us know, frequently, while remain in material area, the true mathematical nature of any phenomenon remains latent. As a classical example similarity of properties of trigonometrical and hyperbolic functions serves. On a complex plane all at once becomes simple and clear: actually it is the same functions which have been written down in a little bit changed system of coordinates».

« The similar phenomenon is found out and with delta - function. If her to define on a complex plane for analytical function $f(x)$ by the formula

$$f(z) = \int_S f(\varsigma)\delta(\varsigma - z)d\varsigma, \ \forall z \in D. \tag{4}$$

For a contour S laying in the field of D analyticity of function $f(z)$ and bypassing a point z, that, obviously $\delta(z) = \dfrac{1}{2\pi iz}$, and any mystery or a paradoxically in it is not present. The unnatural kind she gets at attempt to drive her from a complex plane on a material axis, thus there are all well-known approximations of delta – function as narrow language or the extended bell. In material area it is possible to enter delta – function axiomatic. Advantage of such approach that logically is not required to define this concept, having attributed it to initial, initial, indefinable. Its properties, for example, $\delta(x) = \dfrac{d}{dx}\varepsilon(x)$ for individual step Heaviside are set only.

Its decomposition in a number and in integral Fourier will give

$$\delta(x) = \frac{1}{\pi}(\frac{1}{2} + \sum_{k=1}^{\infty}\cos kx), \ \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{tx}dt. \tag{5}$$

Divergence of a lines and integral does not interfere with their use as divergence асимптотических numbers does not prevent their applications. Introduction thus deltas - functions in the mathematical analysis transforms a class piece smooth functions into a class of differentiable functions, and for them the formula of integration in parts takes place».

«The first consequence from such approach for me was rather unexpected. All of us have got used to that numbers Fourier explosive piece smooth functions badly converge, and they cannot be differentiated term by term. It appeared incorrect. Numbers Fourier at term by term differentiation automatically allocate deltas - functions as their numbers in points of break, and after their reduction true formulas turn out. It is the easiest to illustrate it the simple example frequently included in textbooks for an interval $(-\pi, \pi)$:

$$\frac{\pi \cos ax}{2 \sin a\pi} = \frac{1}{2a} + \sum_{n=1}^{\infty}(-1)^n \frac{n \cos nx}{a^2 - n^2},$$

$$\frac{\pi \sin ax}{2 \sin a\pi} = \frac{1}{2a} + \sum_{n=1}^{\infty}(-1)^n \frac{n \sin nx}{a^2 - n^2}. \tag{6}$$

The second formula turns out from the first differentiation on x. But also the first turns out differentiation from the second if to take into account, that from jumps in points $(2k+1)$ appears at the left composed

$$-\pi\sum_{k=-\infty}^{\infty}\delta(x - (2k+1)\pi) = -(\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n \cos nx), -\infty < x < \infty. \tag{7}$$

It is necessary to reduce superfluous composed.

Accurate application of delta - function also allows to remove (take off) vain, erected on Heaviside: it is considered, that a source of many mistakes at formal application of operational method Heaviside for the decision of the linear differential equations with constant factors is that fact, that operators of differentiation p and p-1 do not switch integration. The authorship of this statement takes up G.Dzheffris. It repeats and in R.Kurant's well-known textbook and D.Gil'bert " Methods of mathematical physics " from which I shall bring the citation: " the Basis of a method is made with introduction of operators of differentiation and integration, p and p^{-1}, as mutual - return operations. We shall enter into consideration for functions of time t at $t > 0$ operators of integration p^{-1} and differentiation p by equality

$$p^{-1}f(t) = g(t) = \int_0^t f(\tau)d\tau, \qquad pg(t) = f(t) = \frac{dg}{dt} \quad . \tag{8}$$

For construction of calculation with the rules appropriate to rules of algebra, importance represents that fact, that operators p and p^{-1} are mutual - are return or, symbolically, that

$$pp^{-1} = p^{-1}p = 1 \, . \tag{9}$$

To provide this parity, we should enter the following restriction: the operator p can be applied only to such functions $g(t)$, for which $g(0) = 0$. Otherwise we would have:

$$p^{-1}pg = \int_0^t \frac{dg(\tau)}{d\tau} = g(t) - g(0), \qquad pp^{-1}g = \frac{d}{dt}\int_0^t g(\tau)d\tau = g(t) \quad , \tag{10}$$

hence $$p^{-1}pg \neq pp^{-1}g'' \, . \tag{11}$$

Actually Heaviside all has made that operators p and p^{-1} switched: it has invented delta - function which I write down in the designations accepted now, with properties

$$\frac{d}{dt}\varepsilon(t) = \delta(t) \quad , \quad \sum_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau = g(0) \quad ,$$

Also has defined a class of so-called originals, considering, that for all elements $g(t)$ of this class $g(0) = 0$ at $t < 0$. Thus for switching operators p also p^{-1} are not present necessity to demand, that $g(+0) = 0$: switching takes place and without this condition. Only it is necessary to take into account, that because of gallop in zero of function $g(t)$ at differentiation appears delta - function:

$$pg(t) = g'(t) + g(+0)\delta(t),$$

где $$g'(t) = \begin{cases} \dfrac{dg}{dt}, t > 0, \\ 0, t < 0, \end{cases}$$

therefore $$p^{-1}pg = \int_0^t [g`(\tau) + g(+0)\delta(\tau)]d\tau = g(t) = pp^{-1}g \text{ ».} \tag{12}$$

However, use of the complex delta - function offered by V.M.Kalinin in the applied analysis inconveniently enough.

Function Heavisaide and delta - function on a complex plane. It is proved, that delta - function is equal to a derivative from individual function. Thus, function Heavisaide is integrated for delta - function Dirak:

$$U(t) = \int_{-\infty}^{t} \delta(z)dz \, . \tag{13}$$

In the book [2] constant with probability unit equal a, is submitted as characteristic function $\varphi(t) = e^{iat}$. Hence, the appropriate delta - function will be equal $\delta(t-a)$.

Characteristic function of a normal amount with average m and a deviation σ is equal:

$$\varphi_1(t) = e^{-\frac{\sigma_1^2 t^2}{2} + im_1 t}. \tag{14}$$

Function (14) is submitted depending on a material variable t that is marked at it by an index 1. We shall present similar function depending on an imaginary variable. Sizes of a population mean will become im_2, a deviation $-\ i\sigma_2$, and characteristic function-

$$\varphi_2(t) = e^{\frac{\sigma_2^2 t^2}{2} - mt}. \tag{15}$$

The sum of two independent normal amounts has normal distribution with average, equal to the sum of average, and a dispersion equal to the sum of dispersions. Really,

$$e^{-\frac{\sigma_1^2 t^2}{2} + im_1 t} e^{\frac{\sigma_2^2 t^2}{2} - mt} = e^{-\frac{\sigma_1^2 - \sigma_2^2}{2} t^2 + i(m_1 + im_2)t}. \tag{16}$$

Now let absolute values $|\sigma_1| = |\sigma_2|$ then the right part (16) will be equal to size

$$e^{i(m_1 + im_2)t}, \tag{17}$$

which represents characteristic function of individual angular step Heaviside on a complex plane with indexes of axes $(1,i)$. Hence, the appropriate delta - function on a complex plane in a point (x_0, y_0) can be submitted as

$$\delta_C(z) = \delta_C(z - x_0 - iy_0). \tag{18}$$

The sum of casual deviations from this point (x_0, y_0) represents a casual vector with the specified distribution (16) and characteristic function (17) under condition of $\sigma_1, \sigma_2 \to 0$. Private acknowledgement of it is that fact, that the density of one-dimensional normal distribution at it $\sigma \to 0$ represents delta - function in one-dimensional material distribution to axes.

Example. We shall consider an example 2 of [3] with elimination of discrepancies and in more detail. On a plane $(1,i),\ i = \sqrt{-1}$ it is observed alternating process of restoration with incomes and charges. Duration of serviceability and restoration of full serviceability we shall define in density of probabilities $f_X(t), f_Y(t)$, and density of probabilities of sizes of the income and the charge- $g_R(t), g_K(t)$, all of them are concentrated on $[0 \le t < \infty)$. All random variables are in pairs independent.

Realization of process has the following interpretation. From the beginning of coordinates the object starts to function, after the expiration of time it(he) refuses and instantly acts on restoration. During casual time of serviceability it accumulates a random variable of the income. This income can represent quantity of the advanced, saved up information, cost etc. During casual time of restoration the object can accumulate a random variable of the charge, alternating measuring the same dimension, as dimension of the income. Thus the income can be both positive, and negative. After end of the first cycle of functioning, process renews in the second cycle and so on can proceed indefinitely. So, we deal with process of restoration together with process of accumulation [6]. In figure 1 provisional realization of considered processes submitted.

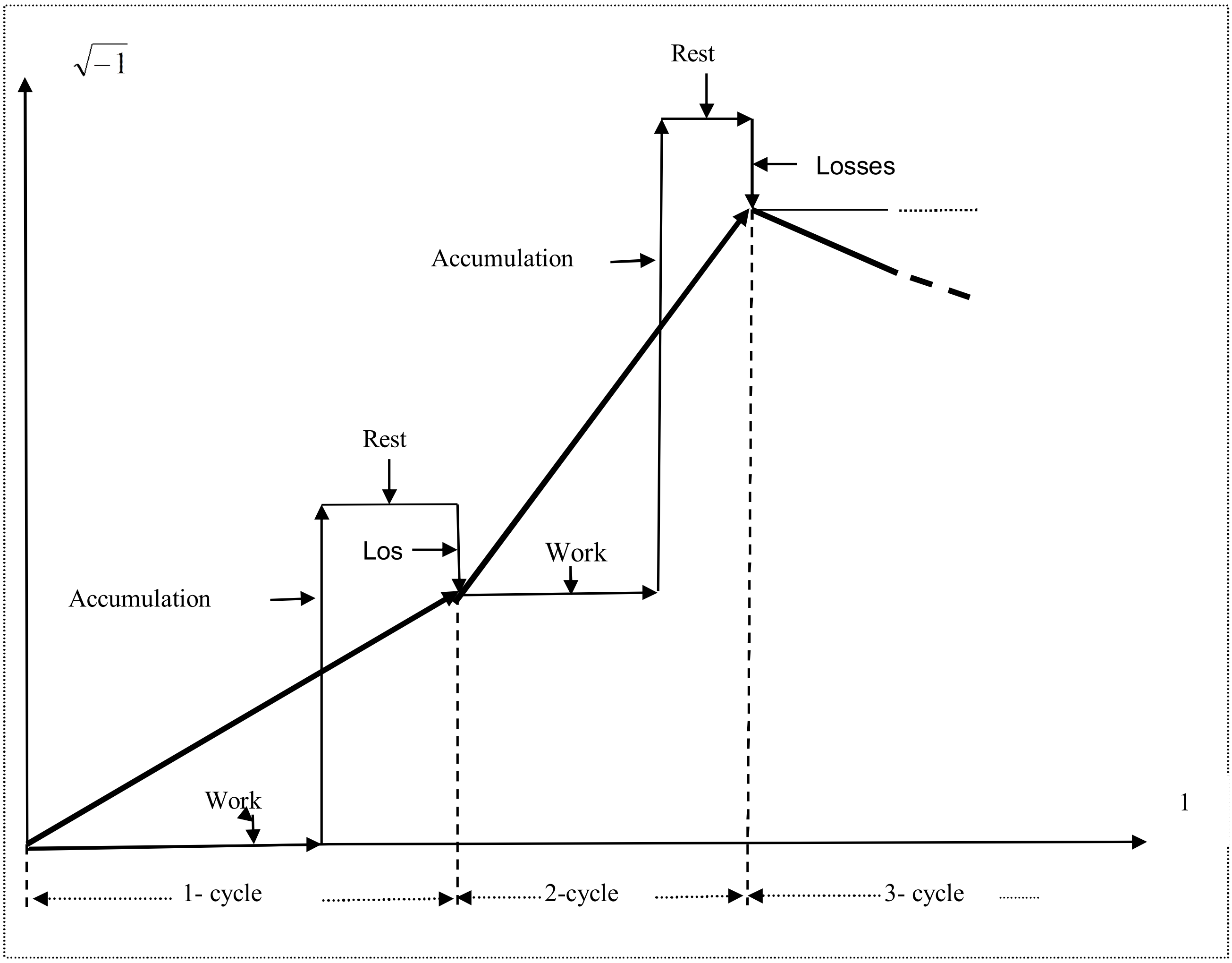

Figure 1. Realization alternating process with accumulation and the charge

Our problem is to find the stationary decision for complex alternating casual process with accumulation.

Let two density of probabilities of material variables $a_X(x), b_Y(y)$, with x, y, concentrated are given on $[0,t)$. Using them and (18) we shall construct the following expression (3):

$$\varphi_Z(z) = \int_0^\infty\int_0^\infty \delta(z - x - iy)a_X(x)b_Y(y)dxdy = \int_0^{z/i} a_X(x - iy)b_Y(y)dy = -i\int_0^z a_X(x)b_Y(i(x - z))dx\,. \quad (21)$$

Under the semantic contents (21) it is density of probability of a vector $x + iy$ on planes $(1, \sqrt{-1})$, analogue « complex density ». Having applied to her transformation Laplace, we shall receive:

$$\varphi_Z^*(z) = \int_0^\infty e^{-us}\varphi_Z(u)du = \varphi_X^*(s)\varphi_Y^*(is), \quad (22)$$

where $*, s -$ a symbol and a variable of transformation Laplace. Expression (22) can be received, using as well characteristic functions.

In conditions of our example we shall have:

$$\varphi_X^*(s) = f_X^*(s)f_R^*(is), \quad \varphi_Y^*(s) = f_Y^*(s)f_K^*(is)\,. \quad (23)$$

For a presence of the stationary decision it is applicable known expression from the theory of restoration for alternating process [6]:

$$K_{\Gamma}^{*}(s) = \frac{1-\varphi_X^{*}(s)}{s[1-\varphi_X^{*}(s)\varphi_Y^{*}(s)]}. \tag{24}$$

This decision of known integrated equation Voltera for function of readiness of the theory of reliability in transformation Laplace.

The stationary decision (24), at $t \to \infty$ we shall find, having applied one of theorems Taubere type:

$$K_{\Gamma} = \lim_{t\to\infty} K_{\Gamma}(t) = \lim_{s\to 0} sK_{\Gamma}^{*}(s). \tag{25}$$

As a result of the decision we shall receive:

$$K_{\Gamma} = \frac{\nu_X + i\nu_R}{(\nu_X + \nu_Y) + i(\nu_R + \nu_K)}. \tag{26}$$

Where ν_X – average time of non-failure operation of object, ν_Y – average time of restoration of object after his refusal, ν_R – average size of target accumulation by object during his efficient condition in a cycle, ν_K – average size of the charge of the received accumulation in a cycle.

For an illustration of the numerical decision we shall accept the following initial sizes: $\nu_X = 100\,h., \nu_Y = 10\,h., \nu_R = 20\,units\,\inf ormation, \nu_K = 10\,units\,\inf ormation$. From (26) it is received $\mathrm{Re}(K_{\Gamma}) = 0{,}892; \mathrm{Im}(K_{\Gamma}) = -0{,}062;$ size of the module $M = \sqrt{(\mathrm{Re}(K_{\Gamma}))^2 + (\mathrm{Im}(K_{\Gamma}))^2} = 0{,}894$. We shall find value of a phase

$$tg(\psi) = \frac{\nu_Y\nu_R - \nu_X\nu_K}{\nu_X(\nu_X + \nu_R) + \nu_R(\nu_R + \nu_K)}, \psi = a\tan(-\frac{2}{29}) = -0{,}069\,rad. = -3{,}945\deg.$$

Under condition of when the income and the charge are absent $\nu_R = \nu_K = 0$, value $K_{\Gamma} = 0{,}909$. The relative mistake of calculation will make < 2 %. If to put $\nu_X = 0, \nu_Y = 0,$ then $K_{\Gamma} = \nu_R/(\nu_R + \nu_K) = 0{,}667$. It means, that the share of the income under the attitude to the sum of the income and the charge will make $\approx 67\%$. Thus, alongside with an estimation of readiness of object on an axis 1, it is possible to receive an estimation of profitability of use of object. It justifies application of the complex approach in research of more complex model in comparison with models [6].

Let's make one remark concerning the given example. At the decision of the practical problems connected to information processes, it is natural to believe, that the size of information work should be directly connected to time of serviceability and restoration of object. In our example it is possible to confirm this statement the following. If average time before refusal of object equally ν_X, the average size of the income can be expressed as $\nu_R = I_R\nu_X$, and average size of the charge- $\nu_K = I_K\nu_Y$, where I_R, I_K sizes of the income and the charge of object in unit of time. We shall put, for example, $I_R = 5$ units of the information in one hour, and $I_K = 2$ unit of the information in one hour. Then sizes of average values of the income and the charge will be equal: $\nu_R = 5\cdot 100 = 500\,un., \nu_K = 2\cdot 10 = 20\,un.$, and relative private receptions of the net profit – $\nu_R/(\nu_R + \nu_K) = 0{,}962$. Thus, dependence of a share of the general profit of object can be taken into account due to communication of an operating time and restoration with information productivity of both processes. This remark does not exclude an opportunity of application and other, more complex models "income - charge".

Conclusion

In given article acknowledgement of expression for delta - function on the complex density, entered by the author [3] is given. Examples of her use in the analysis in the same place are given. Also it is necessary to note, that for the first time the term « the complex probability » was entered by D.R. Cox for the decision not Markov problems of the theory of reliability [4]. However, problem to remain a question on introduction complex probabilistic measures in the analysis and a graphic representation of the complex functions determined on a complex variable. Expansion of complex numbers is the field quaternion's Hamilton, forming not to switch algebra with division above a field of real numbers. Thus everyone quaternion can be submitted as

$$a = a_0 + ia_1 + ja_2 + ka_3 = (a_0 + ia_1) + (a_2 + ia_3)j, \tag{19}$$

Where a_0, a_1, a_2, a_3 real numbers, and $i, j, k -$ special quaternions, forming together with the valid unit basis of four-dimensional space and satisfying the following system of equality:

$$\begin{cases} i^2 = j^2 = k^2 = -1, \\ jk = -kj = i,\ ki = -ik = j,\ ij = -ji = k \end{cases}. \tag{20}$$

Use of complex numbers and quaternions in probability theory for the description of casual multivariate processes and decisions of the appropriate scientific and technical problems, in our opinion, has the big prospects and demands deeper studying. By the way, now opportunities of performance of mathematical operations even with complex numbers on a computer, unfortunately, are extremely limited [5]. The example of the decision of a problem given in article by definition of stationary value of function of readiness of object with accumulation of quantity of the information evidently enough shows advantages of application of the complex analysis in probability theory.

Reference

1. Kalinin V.M. « My formulas ». – Publ. SPBSU him. M.V.Lomonosov, 1966.
2. Dugue D. Theoretical and applied statistics. – Transl. from French V.M.Kalinin. – M: the Science. – 1972. – 383 p.
3. Smagin V.A. Probabilistic the analysis of a complex variable. – AVT. – 1999, 2. – pp. 3-13.
4. Cox D.R. A use of complex probabilities in the theory of stochastic processes. – Proc. Soc., V. 51. – 1955.
5. Gatsenko O.J., Smagin V.A. Elements probabilistic the analysis of a complex variable. – Saint Petersburg, VCA him. A.F.Mozhajskogo. – 1998. – 18 p.
6. Cox D.R. Renewal theory. - London: Methuen and Co Ltd, New York: John Wiley and Sons Inc. - 1961. - 300 p. (the Russian edition in B.V.Gnedenko's translation).