APPLICATION OF CHEBYSHEV- AND MARKOV-TYPE INEQUALITIES IN STRUCTURAL ENGINEERING

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ABSTRACT

This paper aims at bringing out the usefulness of Chebyshev- and Markov- type inequalities in structural engineering design decision making. By examining whether the bounds arising from Chebyshev - type inequality (associated with these are weak upper bound probabilities) enclose the respective experimental values for deflections of six ferrocement I-beams and web shear fatigue life of a steel plate girder it is inferred that the bounds and the associated probabilities estimated are realistic and hence can be used in structural engineering design decision making. The paper also presents some recent developments in application of Markov type inequalities (which are due to Steliga and Szynal (2010)) for estimation of bounds on probability of an event sought. The importance of such bounds in structural engineering applications is brought out. It is shown from the results of Monte Carlo simulation that the bounds on probability of an event, estimated using the method presented by Steliga and Szynal, are sharp. One of the important advantages of the bounds presented by Steliga and Szynal (2010) is that the original (hidden/internal) random variable need not have well defined moments. Possible engineering applications are also pointed out.

Keywords: Chebyshev inequality, Markov inequality, deflection, fatigue life

1 CHEBYSHEV INEQUALITY- SOME PRELIMINARIES

Let X be a random variable representing an action or response quantity. Example of action quantity can be load (or loading intensity), external bending moment or external traction force. The response quantity can be deflection, rotation, warping, strain, crack width. In most engineering applications we may not be knowing the actual probability density function (pdf) of X; yet, we will be asked to answer questions like $P[g(x) \geq r]$=?. It may be noted that $g(X)$ is a function of random variable and r is a specified value. Such decision making probabilities are required in limit state design of structural components (viz. Bolotin, 1969).

In the face of non-availability of pdf of X can we make probabilistic inferences about $P[g(X) \geq r]$. It can be shown that (viz. Gnedenko, 1976)

$$P[g(X) \geq r] \leq \frac{E[g(X)]}{r} \tag{1}$$

Let $g(x) = [x - \mu]^2$; $r = z^2\sigma^2$ where μ and σ are mean and standard deviation of X. According to Chebyschev's inequality this probability computed from (Gnedenko, 1976)

$$P[(X-\mu)^2 \geq z^2\sigma^2] \leq \frac{E[(X-\mu)^2]}{z^2\sigma^2}$$

$$\text{(or) } P[(X-\mu)^2 \geq z^2\sigma^2] \leq \frac{1}{z^2} \quad (2)$$

Therefore, Chebyshev's inequality gives weak upper bound on the desired probability. We note from Eq.(2) that z should be greater than or equal to 1 since if $z < 1$ (though positive) the interpretation as the value of probability will not be proper. Also, since we are not having any information on type of pdf of X the bounds will be weak. Efforts have been made in the literature to sharpen the bounds and to determine two-sided bounds and also, to determine bounds for multivariate case (with- and without- correlation effect).

While Eq. (2) gives the one-sided bound, let us consider two-sided case. When the distribution is symmetrical about the mean, the symmetrical bounds around the mean are given by (Steliga and Szynal, 2010),

$$P[k_1 < \times < k_2] \geq 1 - \frac{4\sigma^2}{(k_2-k_1)^2} \quad \forall(k_2-k_1) > 2\sigma \quad (3)$$

When the pdf of the random variable is not symmetrical about the mean, the bounds are given by,

$$P[k_1 < \times < k_2] \geq \frac{4[(\mu-k_1)(k_2-\mu)-\sigma^2]}{(k_2-k_1)^2} \quad (4)$$

Where $k_1 < \mu < k_2; \quad (\mu-k_1)(k_2-\mu) > \sigma^2$

2 APPLICATIONS

In this section two example problems demonstrating the use of Chebyshev inequalities in determining the weak upper bound probabilities, those required for engineering decision making, are presented. One of the highlights of these examples is, to infuse confidence in engineering applications, to compare the results with the respective experimental values.

Example 1: In this example an attempt has been made to estimate the weak upper bound probabilities on random central deflection of ferrocement I-beams used for roofing in low-cost housing. This example is considered since the test data on central deflection, at different stages of loading, was available for six specimens. These specimens were tested at the structural engineering laboratory of Indian Institute of Science, Bangalore, in 1980s. The details of tests and the test results are available in (Prakash Desayi and Balaji Rao (1988), Prakash Desayi and Balaji Rao (1993), Balaji Rao (1990)). Also, an effort was made to determine statistical properties of deflections using Monte Carlo simulation technique. More details about basic random variables considered and details of simulation are presented in Balaji Rao (1990). The final results of simulation (viz. mean and standard deviations of deflection) for six specimens considered here, at different stages of loading, are presented in Table 1. Also presented in this table are experimental values of central deflections. The weak upper bound probabilities associated with bounds of lengths 2.25σ, 2.5σ, 2.75σ and 3σ are computed using Chebyshev inequalities. These probabilities are computed for two conditions : (a) assuming that the pdf of deflection, at different stages of loading, are symmetrical about the mean, and, (b) assuming that the pdf of deflection is unknown or unsymmetrical about the mean. The values of the bounds and their corresponding probabilities are presented in Table 1 typically for first two interval lengths. Since a bound of length 3σ is very often used in engineering decision making, the same are compared for the cases (a) and (b) in Figs.

1 – 6 for the specimens considered. Also shown in these figures are the experimentally observed deflections. From these figures it is observed that at almost all stages of loading, the estimated bounds contain the observed deflection suggesting that the estimated weak upper bound probabilities are acceptable and can be used in engineering decision making. If it is felt that the length of interval of 3σ is high, from Table 1 it is noted that, at higher stages of loading, even though the lengths of interval are small, the bounds enclose the experimentally observed deflections. It may be noted that the weak upper bound probabilities for the two intervals presented in Table 1 vary between 20 to 21% (which is small though). *These observations suggest that the Chebyshev's inequalities can be used for engineering decision making.*

Table 1. Bounds of different interval lengths and their comparison with experimental results

Specimen designation	Applied load (kN)	Exp. deflection (mm)	Results of Monte Carlo simulation, Balaji Rao (1990)		Bounds – symmetrical (length of interval =2.25 σ)[1]		Bounds – unsymmetrical (length of interval = 2.25σ)[2]	
			Mean, μ (mm)	Standard deviation, σ (mm)	Lower bound (μ – 1.125σ) (mm)	Upper bound (μ + 1.125σ) (mm)	Lower bound (μ – σ) (mm)	Upper bound (μ + 1.25σ) (mm)
MI1	2	0.038	0.015	0.004	0.011	0.019	0.012	0.019
	4	0.069	0.033	0.020	0.010	0.056	0.012	0.058
	8	0.216	0.317	0.157	0.140	0.493	0.160	0.513
	10	0.407	0.555	0.184	0.348	0.762	0.371	0.785
	15	0.9	1.154	0.246	0.878	1.431	0.908	1.462
MI2	1	0.035	0.024	0.005	0.018	0.030	0.018	0.035
	1.5	0.075	0.036	0.009	0.026	0.046	0.027	0.054
	8	1.273	1.542	0.378	1.117	1.967	1.164	2.297
	10.41	1.965	2.336	0.460	1.818	2.854	1.876	3.257
	13	2.809	3.190	0.549	2.370	4.013	2.641	4.287
MI3	2	0.021	0.015	0.003	0.011	0.019	0.011	0.020
	4.5	0.065	0.035	0.018	0.015	0.055	0.018	0.058
	10	0.261	0.422	0.204	0.192	0.651	0.217	0.677
	15	0.728	1.047	0.270	0.743	1.350	0.777	1.384
	17.62	1.02	1.375	0.304	1.033	1.717	1.071	1.755
MI4	1	0.036	0.020	0.005	0.015	0.025	0.015	0.026
	1.5	0.05	0.030	0.007	0.023	0.038	0.023	0.039
	8	1.4	1.083	0.363	0.675	1.491	0.720	1.536
	9.9	1.45	1.645	0.419	1.173	2.116	1.225	2.169
	10	1.63	1.673	0.423	1.196	2.148	1.249	2.201
MI5	3	0.04	0.020	0.005	0.015	0.025	0.016	0.026
	4	0.062	0.027	0.007	0.012	0.034	0.020	0.035
	15	0.645	0.699	0.286	0.378	1.021	0.413	1.057
	16	0.701	0.820	0.299	0.483	1.156	0.520	1.193
	17.79	0.825	1.035	0.321	0.674	1.396	0.714	1.436
MI6	2	0.077	0.035	0.008	0.026	0.045	0.027	0.045
	3	0.12	0.053	0.013	0.038	0.068	0.040	0.070
	9	0.73	0.731	0.478	0.193	1.268	0.253	1.328
	9.6	0.932	0.915	0.516	0.334	1.496	0.399	1.560
	14	2.8	2.356	0.678	1.593	3.119	1.677	3.204

Note: 1,2 – associated weak upper bound probabilities are 0.21 and 0.20, respectively

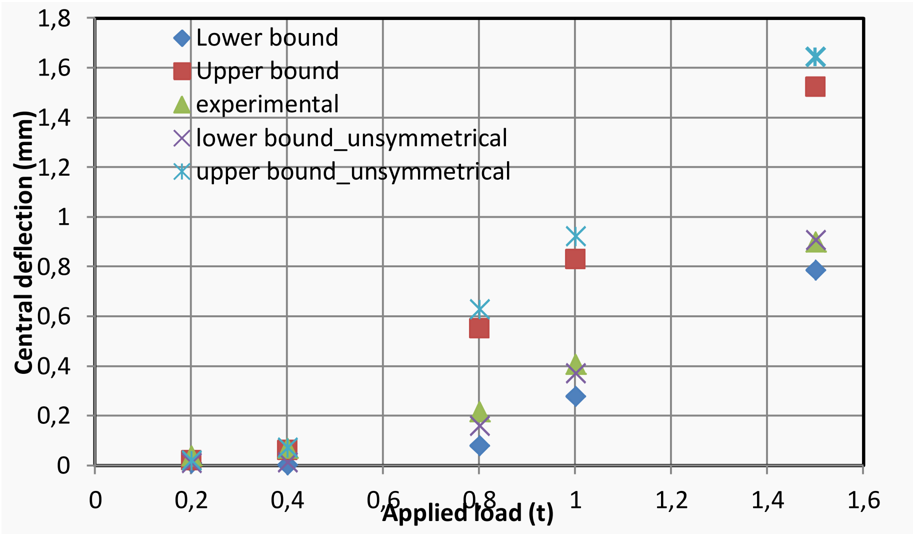

Figure 1. Load versus deflection plot for specimen MI1 with (μ -1.5σ, μ +1.5 σ) symmetrical bounds (associated minimum probability 0.56) and (μ - σ, μ + 2.0 σ) for unsymmetrical bounds (associated minimum probability 0.44) with experimental values

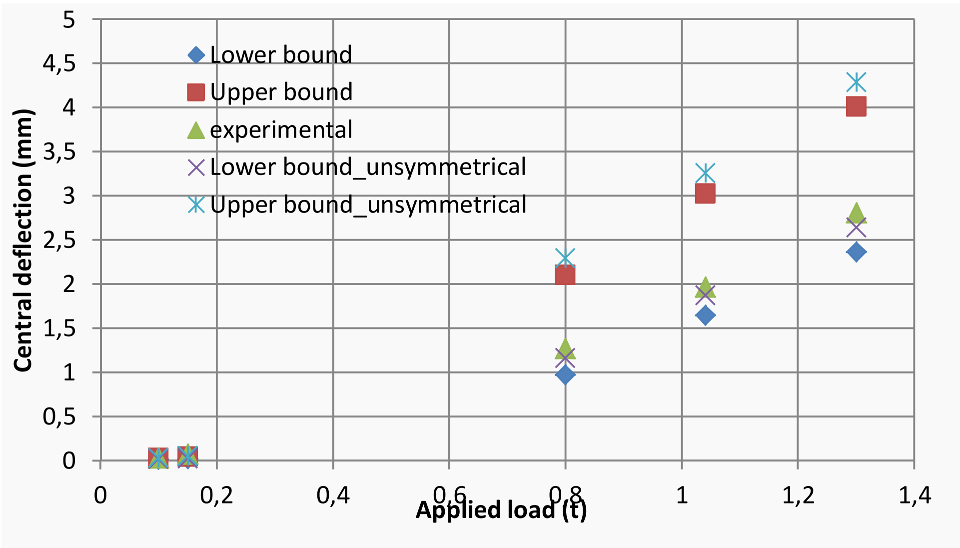

Figure 2. Load versus deflection plot for specimen MI2 with (μ -1.5σ, μ +1.5 σ) symmetrical bounds (associated minimum probability 0.56) and (μ - σ, μ + 2.0 σ) for unsymmetrical bounds (associated minimum probability 0.44) with experimental values

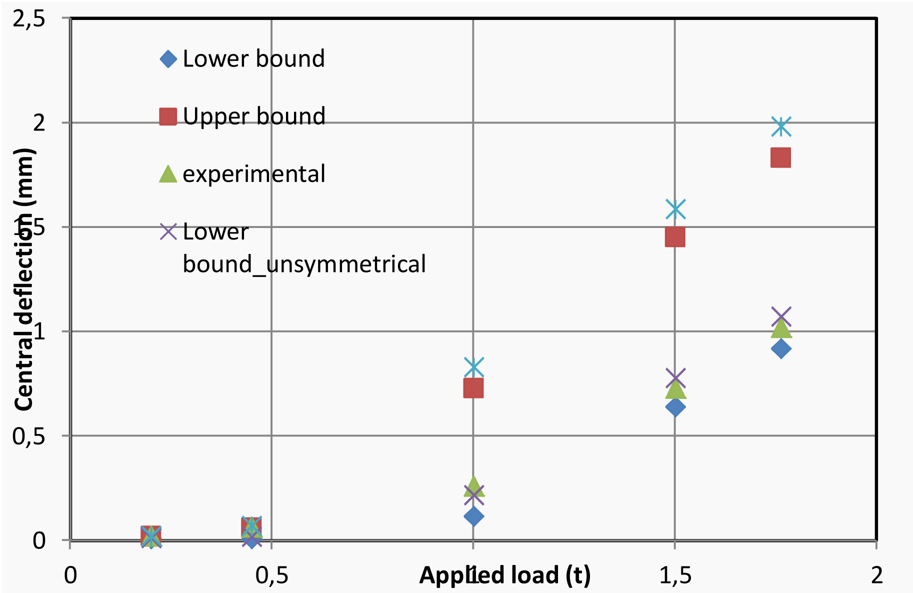

Figure 3. Load versus deflection plot for specimen MI3 with (μ -1.5σ, μ +1.5 σ) symmetrical bounds (associated minimum probability 0.56) and (μ - σ, μ + 2.0 σ) for unsymmetrical bounds (associated minimum probability 0.44) with experimental values

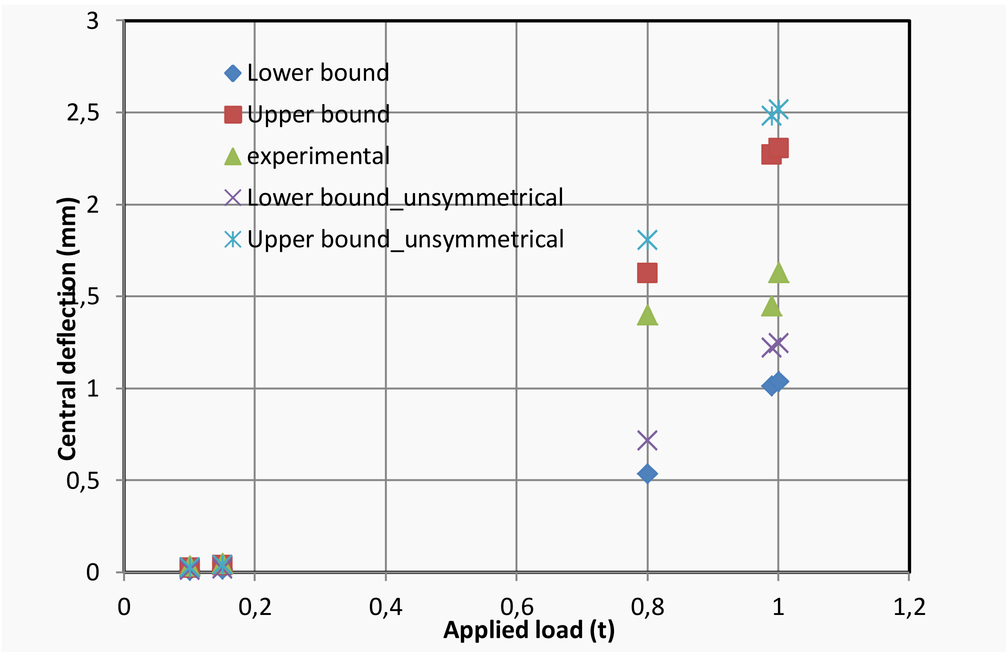

Figure 4. Load versus deflection plot for specimen MI4 with (μ -1.5σ, μ +1.5 σ) symmetrical bounds (associated minimum probability 0.56) and (μ - σ, μ + 2.0 σ) for unsymmetrical bounds (associated minimum probability 0.44) with experimental values

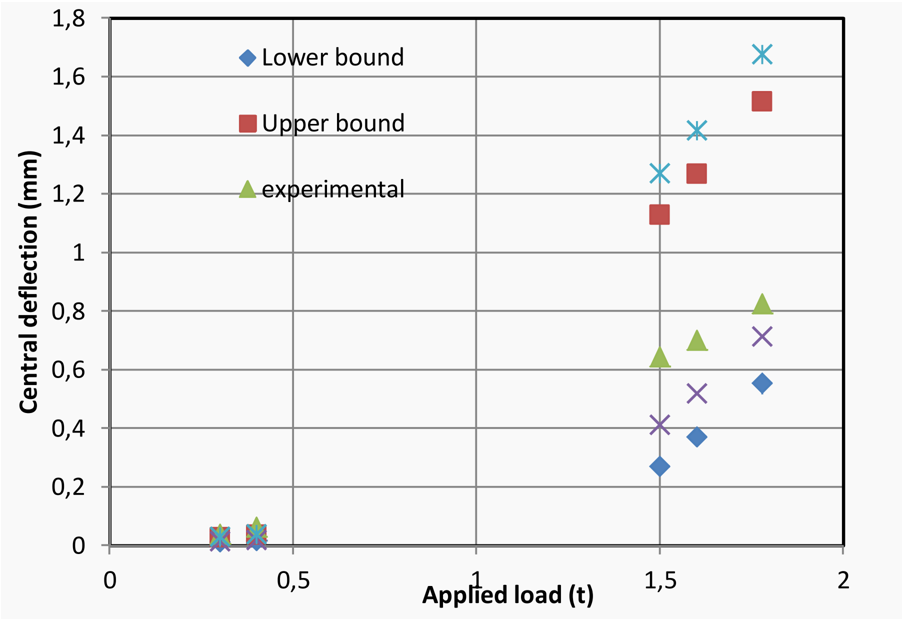

Figure 5. Load versus deflection plot for specimen MI5 with (μ -1.5σ, μ+1.5 σ) symmetrical bounds (associated minimum probability 0.56) and (μ - σ, μ + 2.0 σ) for unsymmetrical bounds (associated minimum probability 0.44) with experimental values

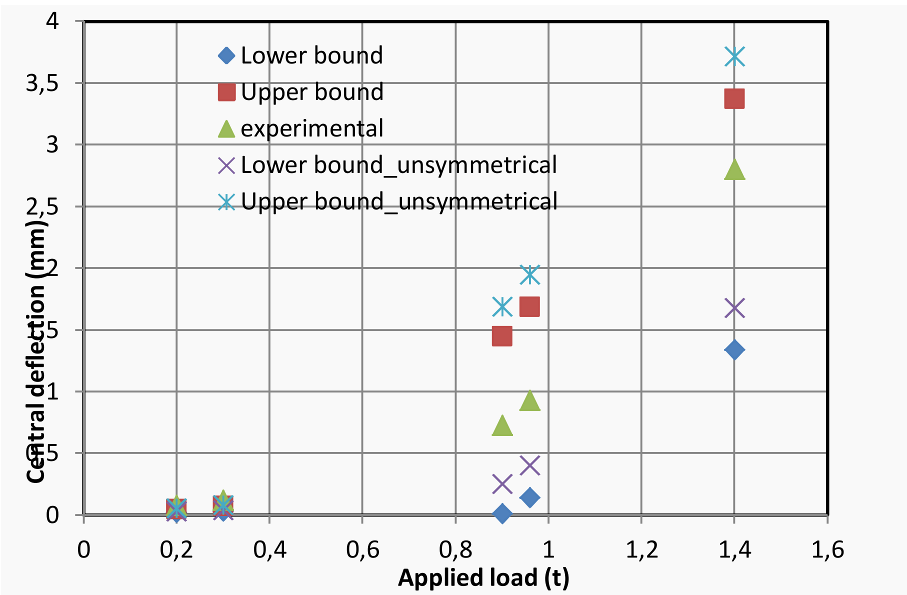

Figure 6. Load versus deflection plot for specimen MI6 with (μ -1.5σ, μ+1.5 σ) symmetrical bounds (associated minimum probability 0.56) and (μ - σ, μ + 2.0 σ) for unsymmetrical bounds (associated minimum probability 0.44) with experimental values

Example 2: This example shows how Chebyshev's inequality will help in fatigue resistant design of steel plate girders of a plate girder bridge. More details of this problem can be found in Balaji Rao and Anoop (2013).

The basic equation used in predicting the fatigue life using S – N approach is given by,

$$N_f = C\left(\Delta\sigma\right)^{-b} \tag{5}$$

where N_f is the number of load cycles to the fatigue limit and $\Delta\sigma$ is the applied stress range, C and b are the material parameters, known as the fatigue strength coefficient and the fatigue strength exponent, respectively.

It is known that the number of cycles to failure (i.e. fatigue life, N_f), at a given applied stress range is a random variable. A typical plot showing the same is presented in Fig. 7. It may be noted that the nature of pdf and the statistical properties of N_f may depend on stress range. While it is desirable to establish the nature of these probability distributions using fatigue tests, it is expensive and time consuming. The median and the 5% and 95% fractiles of fatigue life computed using the transformation of variable technique, at different applied stress ranges are shown in the figure. More details of the probabilistic analysis of the fatigue life of the plate girder are presented in Balaji Rao et.al. (2013). Also shown in this figure are experimental fatigue lives reported in literature. Except in few cases, experimental scatter is enclosed by the estimated bounds. Let us apply the Chebyshev's inequality to determine the bounds on fatigue life. At the applied stress range of 270 MPa the mean (μ) and standard deviation (σ) of N_f are respectively 7.471E+05 and 3.447E+05. Assuming the bounds to be symmetrical and $(k_2$-$k_1) = 3\sigma$, the probability that the fatigue life will be between (2.3005E+05, 12.6415E+05) is equal to or greater than 0.556. On the other hand if the distribution is unsymmetrical about the mean, even though we may keep $(k_2$-$k_1) = 3\sigma$, assuming the bounds to be (4.024E+05, 14.365E+05), that satisfies the conditions associated with Eq. (4), then the probability that the bound will contain actual life will be equal to greater than 0.444. This value of probability is less than the case when the bounds are symmetrical for the reason that in order to assume that the bounds are symmetrical we should have had more justification/confidence and this gets embedded in our predictions.

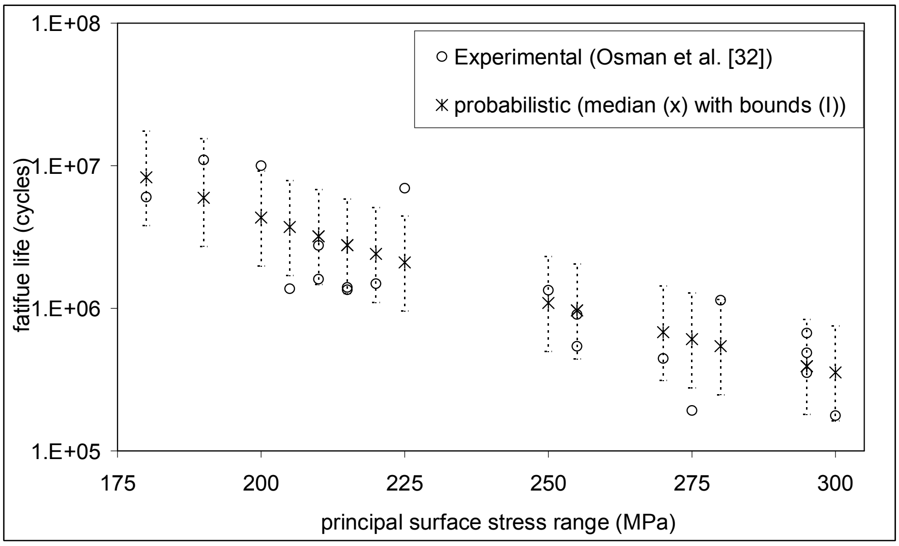

Figure 7. Comparison of results of probabilistic analysis with experimentally observed values of fatigue life for plate girders reported in literature

2 SOME RECENT DEVELOPMENTS

Extension of Markov-type inequalities to a class of random variables without moment condition requirement was proposed by Steliga and Szynal (2010). This is a flexible formulation that enables the computation of bounds on probabilities for the random variable which does not have well defined moments. Some of the examples of random variables that belong to this category are alpha stable random variables. These random variables have fractional moments and the existence of the order of moments depends on the value of the exponent. Recent R&D at CSIR-SERC has revealed that for describing the variations in some of the engineering quantities, alpha-stable distributions are more appropriate (Balaji Rao and Anoop, 2012). Hence, the results of Steliga and Szynal (2010) are important and the same are considered further.

Steliga and Szynal (2010) assumed the following conditions: (1) X is a positive random variables ($X \geq 0$ a.s.), (2) $\boldsymbol{G}$ is a class of all positive, strictly increasing functions g with $g(0) = 0$. Let $\boldsymbol{N}$ be set of integers. Making use of Markov inequalities, the following inequality has been proved by them.

For a given $\varepsilon > 0$ and k in $\boldsymbol{N}$

$$
\begin{aligned}
E\left[\frac{g(kX)}{g(kX)+g((k-1)X+\varepsilon)}I[X \geq \varepsilon]\right] \leq P[X \geq \varepsilon] &\leq 2E\left[\frac{g(kX)}{g(kX)+g((k-1)X+\varepsilon)}I[X \geq \varepsilon]\right] \\
&\leq 2E\left[\frac{g(kX)}{g(kX)+g((k-1)X+\varepsilon)}\right]
\end{aligned}
\tag{6}
$$

Where $I[.]$ denotes the indicator function. The above bounds are valid only when X is a positive random variable. The computation of bounds does not require moments of X. However, we should know the functional form (strictly increasing) of $g(.)$ and an idea about the realizations of X. We should also be in a position to determine the expected values of $g(.)$. The strictly increasing function g(.) can be formulated based on phenomenological modeling involving X.

Possible structural engineering application: For example, the average (smoothed) roof load-displacement curve of a moment resisting frame subjected to lateral loads, obtained using pushover analysis, over the range of engineering design interest, can be considered as strictly increasing function of load. In a gravity controlled experiment, the variations observed in roof displacements of nominally similar frames can be attributed to the variations in dimensions and strengths of materials. These variations can be aggregated into overall rigidity/compliance of the frame (X, which is always positive, and thus satisfying the condition required to estimate the bounds using the above equation). We can always assign an acceptable probability distribution to the deflection, generate random values of deflection following the assigned pdf and indirectly estimate the rigidity and examine the indicator function. This exercise would circumvent us from directly generating random realizations of X (as already indicated this random variable may not have well defined moments). Still we will be able to compute the bounds on the required probability with regard to X. This also suggests that the formulations presented by Steliga and Szynal (2010) can be used in an inverse problem to characterize, probabilistically, the internal variable. However, the condition $k \in N$ may perhaps need to relaxed through proper formulations.

Computation of bounds on probability of required event using Eq. (6) requires X to be positive. Let X be the random variable which is real valued. Then, the bounds presented above for positive random variables can be used by substituting $|X|$ in the place of X. Let us consider some special cases wherein does X take on negative values whose moments may or may not exist and a strictly increasing function of X, $g(X)$, is observable and whose distribution is known (in such a case X can be considered as a hidden/internal variable whose value is inferred based on a physical relationship $g(.)$ and X).

The following functional form is assumed for ε >0. We will now consider special cases and provide necessary bounds for $P(X \geq \varepsilon)$.

$$(a)\quad g(|x|) = |x|^m, \quad m \in N$$

$$(b)\quad g(|x|) = |x|^r, \quad 0 < r < \infty$$

In order to compute denominator in Eq. (6), we need the following bounds (Gut, 2005), whenever $x > 0$ and $y > 0$,

$$(x+y)^r \leq \begin{cases} x^r + y^r & for\ 0 \leq r \leq 1 \\ 2^{r-1}(x^r + y^r) & for\ r \geq 1 \end{cases} \tag{7}$$

The following are the bounds derived by Steliga and Szynal (2010). These would be useful in engineering applications some of which are pointed out in the next section.

$$LB_k^I(r;\varepsilon) = 2E\left[\frac{(k|X|)^r}{(k|X|)^r + [(k-1)|X| + \varepsilon]^r} I(|X| \geq \varepsilon)\right], \quad k \in N, \quad r > 0$$

$$LB_k^I(m;\varepsilon) = 2E\left[\frac{(k|X|)^m}{(k|X|)^m + [(k-1)|X| + \varepsilon]^m} I(|X| \geq \varepsilon)\right], \quad k, m \in N$$

$$MB_k^I(r;\varepsilon) = 2E\left[\frac{(k|X|)}{(k|X|) + [(k-1)|X| + \varepsilon]} I(|X| \geq \varepsilon)\right]^r, \quad k \in N, \quad 0 < r \leq 1$$

$$MB_k^I(r;\varepsilon) = 2^r\, E\left[\frac{(k|X|)}{(k|X|) + [(k-1)|X| + \varepsilon]} I(|X| \geq \varepsilon)\right]^r, \quad k \in N, \quad r \geq 1$$

The predictive power of the above equations, characterised by sharpness of bounds, for $g(x) = |x|^r$, when the underlying random variable is following normal and lognormal distributions are presented in Tables 1 and 2, respectively, for different values of k and r. To estimate the lower and upper bounds, Monte Carlo simulation technique involving 10^5 simulation cycles are used. The actual probabilities whose bounds are being estimated are presented in foot note of these tables. *From the results presented in these tables it is inferred that the bounds developed by Steliga and Szynal are tight and can be used in the engineering applications for making probability statements about the internal variable which is responsible for generating the random observable g(X).* However, it should be noted that the assumptions made in deriving the bounds should be satisfied. As stated earlier, one of the limitations, perhaps, in engineering application would be the need imposed by $k \in N$.

Possible structural engineering application: In many engineering problems, X may take negative values and also moments of X may not exist. For example, recent studies by Balaji Rao and Anoop (2012), at CSIR-SERC, have shown that the description of evolution of surface strain field of a reinforced concrete flexural member follows a Levy process. Accordingly, at any stage of loading, the fluctuations in surface strains may be described using an alpha-stable distribution. It is known that handling such random variables can be difficult and may be desirable to make probabilistic inferences of these variables based on the probabilistic variations in observables such as deflections and/or crackwidths which are functions of internal variables such as strains. This study is being furthered at CSIR-SERC.

Table 2. Results of simulation (N = 10^5 cycles) for the bounds on P[$|X|$≥2]* for *X* being normally distributed with mean = 0.0 and standard deviation = 1.0; $g(x) = |x|^r$

k	*r*	¾	½	¼	1/9	1/16
1	LBI(k;r;ε) MBI(k;r; ε)	0.04852617 0.05764368	0.04759212 0.067207573	0.04665248 0.078375767	0.04612905 0.085371748	0.04594573 0.087966528
2	LBI(k;r; ε) MBI(k;r; ε)	0.04702721 0.05591246	0.04658857 0.065866515	0.04614941 0.077596523	0.04590531 0.08499524	0.04581986 0.087748455
4	LBI(k;r;∈) MBI(k;r; ε)	0.04634894 0.05511564	0.04613601 0.0652417	0.04592302 0.077229053	0.04580468 0.084816504	0.04576326 0.087644691
9	LBI(k;r; ε) MBI(k;r; ε)	0.04598943 0.05469043	0.04589629 0.064906321	0.04580315 0.07703066	0.0457514 0.084719696	0.04573329 0.087588427
16	LBI(k;r; ε) UBI(k;r; ε)	0.04586632 0.05454438	0.04581421 0.064790817	0.04576211 0.076962149	0.04573316 0.084686215	0.04572303 0.087568958
25	LBI(k;r; ε) MBI(k;r; ε)	0.04580979 0.05447725	0.04577653 0.064737677	0.04574326 0.076930597	0.04572478 0.084670787	0.04571832 0.087559985
36	LBI(k;r; ε) MBI(k;r; ε)	0.0457792 0.05444092	0.04575613 0.064708895	0.04573307 0.076913499	0.04572025 0.084662424	0.04571577 0.08755512

Note: * Actual probability value = 0.0455

Table 3. Results of simulation (N = 10^5 cycles) for the bounds on P[*X*≥3]* for *X* being lognormally distributed with mean = 2.0 and standard deviation = 0.50**; $g(x) = |x|^r$

k	r	¾	½	¼	1/9	1/16
1	LBI(k;r; ε) MBI(k;r; ε)	0.03954956 0.04701278	0.0390874 0.055247321	0.03862401 0.064931017	0.03836626 0.071029536	0.03827603 0.07329724
2	LBI(k;r; ε) MBI(k;r; ε)	0.03882517 0.04616696	0.03860355 0.054587084	0.03838181 0.064544453	0.03825858 0.070841976	0.03821545 0.073188447
4	LBI(k;r; ε) MBI(k;r; ε)	0.03848589 0.04576672	0.03837727 0.054272124	0.03826864 0.064358563	0.03820829 0.070751383	0.03818716 0.073135818
9	LBI(k;r; ε) MBI(k;r; ε)	0.03830327 0.04555033	0.03825551 0.054101171	0.03820776 0.06425727	0.03818123 0.070701911	0.03817194 0.073107056
16	LBI(k;r; ε) MBI(k;r; ε)	0.03824028 0.04547556	0.03821352 0.054041991	0.03818676 0.06422214	0.03817189 0.070684736	0.03816669 0.073097067
25	LBI(k;r; ε) MBI(k;r; ε)	0.03821129 0.04544111	0.0381942 0.054014712	0.0381771 0.064205936	0.0381676 0.07067681	0.03816427 0.073092457
36	LBI(k;r; ε) MBI(k;r; ε)	0.03819559 0.04542245	0.03818372 0.053999923	0.03817186 0.064197148	0.03816527 0.070672511	0.03816297 0.073089956

Note : *Actual probability value = 0.0382; **parameters of lognormal: - lamda = 0.66261654; exi = 0.246068276;

3 SUMMARY

This paper aims at bringing out the usefulness of Chebyshev- and Markov- type inequalities in structural engineering design decision making. By examining whether the bounds arising from Chebyshev - type inequality (associated with these are weak upper bound probabilities) encloses the respective experimental values for: (a) prediction of central deflection of six ferrocement I-beams, and, (b) fatigue life of a steel plate girder of a plate-girder bridge, against the limit state of web shear buckling, it is inferred that the bounds and the associated probabilities estimated are realistic and hence can be used in structural engineering design decision making. The paper also presents recent developments in determination of inequalities of the type of Markov, which are due to

Steliga and Szynal (2010). The importance of such bounds in structural engineering applications is brought out. It is shown from the results of Monte Carlo simulation that the bounds on probability of an event sought, estimated using the method presented by Steliga and Szynal, are sharp. One of the important advantages of the bounds presented by Steliga and Szynal (2010) is that the original (hidden/internal) random variable need not have well defined moments. Possible engineering applications are also pointed out.

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