CHI-SQUARED GOODNESS OF FIT TEST FOR GENERALIZED BIRNBAUM-SAUNDERS MODELS FOR RIGHT CENSORED DATA AND ITS RELIABILITY APPLICATIONS

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ABSTRACT

Generalized Birnbaum-Saunders (GBS) distributions are proposed by Díaz-García *et al.* ([15], [16]) based on the family of elliptically contoured univariate distributions. This model is well-known as the highly flexible lifetime model by the difference in the degrees of kurtosis and asymmetry and processes uni-modality and bimodality. In this paper, a modifier Chi-squared goodness-of-fit test based on Nikulin-Rao-Robson statistics $\boldsymbol{Y}_n^2$ is developed for the family of GBS distributions for the right censored data with unknown parameters by using the maximum likelihood estimation (MLE). Some applications of this model in survival analysis discuss also in the section of real study.

Keywords and phrases: Birnbaum-Saunders distribution, Breast cancer, Carcinoma data, Chi-squared test, Censoring sample, GBS distributions, Goodness-of fit test, NRR test, Survival analysis.

1 Introduction

In 1969, Birnbaum and Saunders [9] have been proposed a model with two shape and scale parameters that is well known as Birnbaum-Saunders (BS) distribution. After their work, there was a lot of research work on this model and its applications in reliability and survival analysis. It must be mentioned as the work of Desmond [14] who strengthened the physical justification for the use of this distribution by relaxing some assumptions early bade Birnbaum-Saunders. Based on this distribution, Leiva *et al.* [23] has worked to model survival times of patients with multiple myeloma by using prognostic variables with censored data. A chi-squared test for this model is analyzed by Tahir [37] in 2012. In addition, in the recent research of Nikulin *et al.* ([30], [2], [29]) considered these applications of this model in the accelerated lifetimes (AFT) models and redundant systems. Nowadays, the BS distribution has known as cumulative damage distributions and it is a very useful in fatigue, reliability and survival analysis. However, its field of application has been extending beyond the original context of material fatigue and reliability analysis.

Therefore, studies to expand of the BS distribution have been looking for researchers in recent years, such as: Owen ([32], [33], [31]) proposed a three parameter Birnbaum-Saunders distribution, in 2000. Later, Volodin and Dzhungurova [38] developed a general family of fatigue life distributions denominated the crack distribution, which includes the Birnbaum-Saunders distribution as a particular case. In particular, we should be mention a generalized family of life distribution which is suggested by Díaz-García *et al.* [15] in their technical report in 2002, is called as the Generalized Birnbaum-Saunders (GBS) distributions. In his works, Díaz-García was obtained a distribution of the Birnbaum-Saunders type with different degrees of kurtosis, uni-modality, bimodality and absence of moments by basing on the family of elliptically contoured univariate distributions (which known as standard symmetrical distributions in R. A complete review about the

GBS distributions can be found in Sanhueza, Leiva, and Balakrishnan [36]. The purpose of this paper, we analyze a Nikulin-Rao-Robson $\boldsymbol{Y}_n^2$ goodness-of-fit tests for these distributions in the case of right censoring observations. We also demonstrate the applications of this model by applying it to reliability and survival data.

2 Generalized Birnbaum-Saunders distributions

As is already known, a random variable T following the BS distribution allows the stochastic representation

$$T = \beta\left[\alpha\frac{Z}{4} + \sqrt{\alpha^2\frac{Z}{2} + 1}\right]^2 \approx BS(\alpha,\beta), \qquad \alpha > 0, \beta > 0, \tag{1}$$

where, $Z \approx N(0,1)$. Then the random variable Z may be stochastically represented in the form

$$Z = \frac{1}{\alpha}\left[\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right] \approx N(0,1). \tag{2}$$

In 2002, Díaz-García *et al.* [15] were developed the BS distribution becomes GBS distributions which are related to standard symmetrical distributions in R, also known as elliptically contoured or simple Elliptic distribution ([1], [8], [21], [19], [12]).

If a random variable Z follows an Elliptic distributions which correspond to all the symmetric distribution in R, denoted by $Z \approx EC(\mu,\ \sigma^2;\ g)$, the probability density function $f_Z(z)$ and cumulative distribution function $F_Z(z)$ of Z given by,

$$f_Z(z;\ \mu,\sigma^2) = cg\left(\frac{(z-\mu)^2}{\sigma^2}\right), \qquad F_Z(z;\ \mu,\sigma^2) = \int_{-\infty}^{z} f_Z(u;\ \mu,\sigma^2)du\,, z \in R, |\mu| < \infty, \sigma > 0\,.$$

respectively, where, $g(\cdot)$ is the kernel of the probability density function of Z, c is the positive normalization constant, such that $\frac{1}{c} = \int_{-\infty}^{+\infty} g(u^2)du$. The families Elliptic distributions include three sub-models: Kotz Type (KT), Pearson type VII (PVII) and type-III generalized logistic (LIII) distributions, for more details on these distributions is given by Anderson [1] Balakrishnan [8], Fang [19], Gupta [21], Cambanis [12] and others. The Normal, Cauchy, Laplace, Logistic, Power Exponential and $t(\nu)$-Student distributions are particular cases of these symmetric sub-classes in R. In table 1 below, we recall some results for kernel function $g(\cdot)$, the constant c corresponding with standard symmetric distribution $EC(0,\ 1;\ g)$ in R.

Distribution	Notation	c	$g(z^2), z \in R$
Normal	$N(0,1)$	$\frac{1}{\sqrt{2\pi}}$	$exp\left(-\frac{z^2}{2}\right)$
$t(\nu)$-Student	$t(\nu)$	$\frac{\Gamma\left\{\frac{\nu+1}{2}\right\}}{\Gamma(\nu+2)\sqrt{\nu\pi}}$	$\left\{1+\frac{z^2}{\nu}\right\}^{-\frac{\nu+1}{2}}$
Laplace	$L(0,1)$	0.5	$e^{-\lvert z\rvert}$
Logistic	$Log(0,1)$	1	$\frac{e^{-z}}{(1+e^{-z})^2}$
Cauchy	$C(0,1)$	$\frac{1}{\pi}$	$\frac{1}{1+z^2}$
Power exponential	$PE(\nu)$	$\frac{\nu}{(2\nu)^{\frac{1}{2\nu}}\Gamma\left(\frac{1}{2\nu}\right)}$	$exp\left(-\frac{1}{2\nu}\lvert z\rvert^{2\nu}\right)$
LIII	$LIII(q)$	$\frac{\Gamma(2s)}{\Gamma^2(q)}$	$\frac{e^{qz}}{(1+e^z)^{2q}}, q > 0$

Distribution	Notation	c	$g(z^2), z \in R$
Pearson VII	$PVII(q,r)$	$\frac{\Gamma(q)}{\Gamma(\frac{q-1}{2})\sqrt{r\pi}}$	$\left[1+\frac{z^2}{r}\right]^{-q}, q > \frac{1}{2}, \nu > 0$
Kotz type	$KT(q,r,s)$	$\frac{sr^{(2q-1)/2s}}{\Gamma(\frac{2q-1}{2s})}$	$z^{2(q-1)}e^{-rz^{2s}}, q > \frac{1}{2}, r > 0, s > 0$

Table 1: Kernel $g(\cdot)$ and normalization constants c for some indicated distributions.

Following Díaz-García *et al.* the random variable T in (1) allows the GBS distributions, denoted by $T \approx GBS(\alpha, \beta; g)$,

$$T = \beta\left[\alpha\frac{Z}{4} + \sqrt{\alpha^2\frac{Z}{2} + 1}\right]^2 \approx GBS(\alpha,\beta,g), \alpha > 0, \beta > 0,$$

iff the random variable Z which is given by the expression

$$Z = \frac{1}{\alpha}\left[\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right] \approx EC(0,1;\ g).$$

So, the probability density function of T can be written as

$$f_T(t,\alpha,\beta) = \frac{c}{2\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right\} g\left(\frac{1}{\alpha^2}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right]^2\right), t > 0, \alpha > 0, \beta > 0, \quad (3)$$

the cumulative distribution function of $T \approx$ GBS(α, β ; g) is expressed by

$$F_T(t,\alpha,\beta) = F_Z\left\{\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right]\right\}, t > 0, \alpha > 0, \beta > 0, \quad (4)$$

the GBS hazard rate, survival and cumulative hazard functions are

$$\lambda_T(t,\alpha,\beta) = \frac{f_Z\big(a_t(\alpha,\beta)\big)A_t(\alpha,\beta)}{1 - F_Z\big(a_t(\alpha,\beta)\big)},$$
$$S_T(t,\alpha,\beta) = 1 - F_Z\big(a_t(\alpha,\beta)\big), \text{and } \Lambda_T(t,\alpha,\beta) = -ln\{S_T(t,\alpha,\beta)\}, \quad (5)$$

respectively, where

$$a_t(\alpha,\beta) = \frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right];\ A_t(\alpha,\beta) = \frac{1}{2\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right].$$

It is clear that the properties of GBS distributions depends on the kernel function $g(\cdot)$ and the unknown parameter $\boldsymbol{\theta} = (\alpha,\beta)^T$. The statistical theory and methodology of the GBS distributions, also some results for this flexible family of distributions mainly related to transformations, the hazard failure and censored data type II which can be found in the works of Sanhueza, Leiva et al.[36].

Table 2 below shown some probability density function of $T \approx GBS(\alpha, \beta; g)$, corresponding the specific symmetric distribution $EC(0,1;\ g)$ in Table 1.

The Figure 1, 2, 3 and 4 below illustrates some curve of the probability densities and hazard rate functions of $T \approx GBS(\alpha, \beta; g)$, allows with the kernel indicative.

Kernel $g(\cdot)$	Distribution	Probability density function of $T \approx GBS(\alpha, \beta; g)$, $f(t,\alpha,\beta;g), (t > 0, \alpha > 0, \beta > 0)$
N(0, 1)	GBS-Normal (BS)	$\frac{1}{2\alpha\beta\sqrt{2\pi}}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\exp\left\{-\frac{1}{\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right\}$

Kernel	Distribution	Probability density function of ,
	GBS-Student	$\dfrac{\Gamma\left\{\frac{\nu+1}{2}\right\}}{2\alpha\beta\,\Gamma(\nu+2)\sqrt{\nu\pi}}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\left(1+\frac{1}{\nu\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{\frac{-(\nu+1)}{2}}$
L(0, 1)	GBS-Laplace	$\dfrac{1}{4\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\exp\left\{-\frac{1}{\alpha}\left\lvert\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right\rvert\right\}$
Log(0, 1)	GBS-Logistic	$\dfrac{1}{2\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\dfrac{\exp\left\{\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right]\right\}}{\left\{1+\exp\left\{\frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right]\right\}\right\}^2}$
C(0, 1)	GBS-Cauchy	$\dfrac{1}{2\pi\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\left(1+\frac{1}{\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-1}$
KT(q, r, s)	GBS-KT	$\dfrac{sr^{\frac{2q-1}{2s}}\alpha^{2q-3}}{2\beta\Gamma\left(\frac{2q-1}{2s}\right)}\dfrac{\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}}{\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)^{1-q}}\exp\left\{-\frac{r}{\alpha^{2s}}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]^s\right\}$
PVII(q, r)	GBS-PVII	$\dfrac{\Gamma(q)}{2\alpha\beta\sqrt{r\pi}\,\Gamma(q-1/2)}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\left\{1+\frac{1}{\nu\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right\}^{-\frac{\nu+1}{2}}$
PE(r, s)	GBS-PE	$\dfrac{sr^{\frac{1}{2s}}}{2\alpha\beta\Gamma(\frac{1}{2s})}\left\{\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right\}\exp\left\{-\frac{r}{\alpha^{2s}}\left[\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right]^{2s}\right\}$

Table 2: The p.d.f of for some indicated distributions.

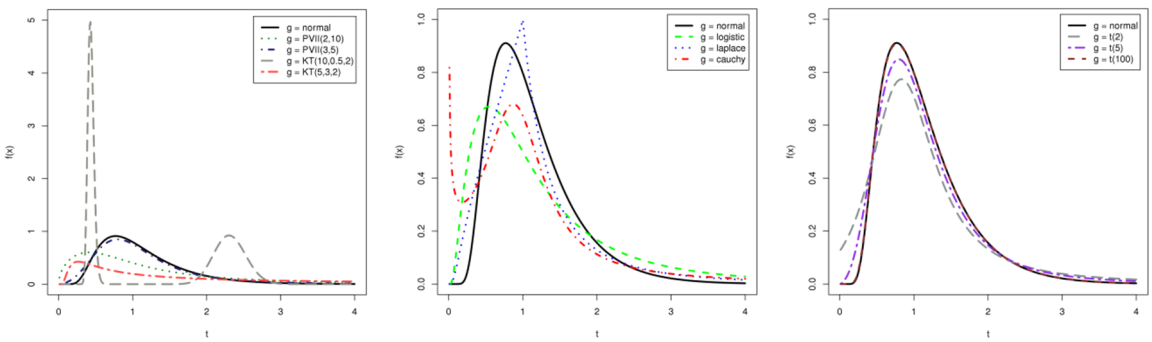

Figure 1: Plots of densities for given kernel .

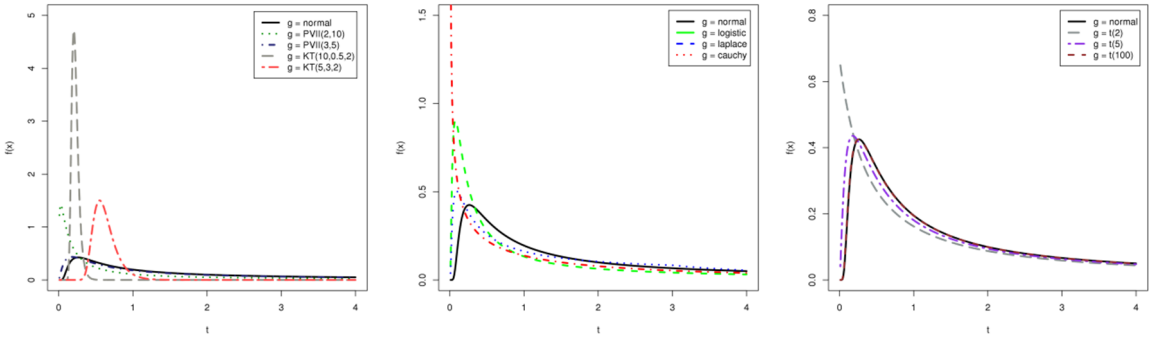

Figure 2: Plots of densities for given kernel .

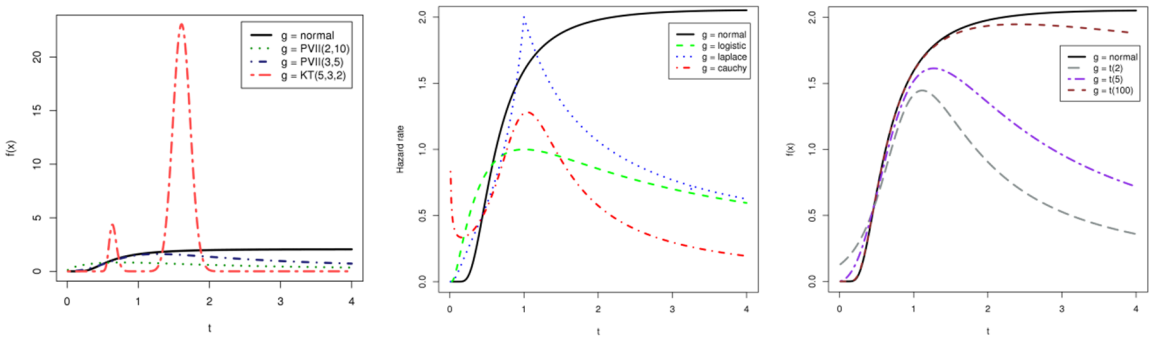

Figure 3: Plots of failures rates for given kernel .

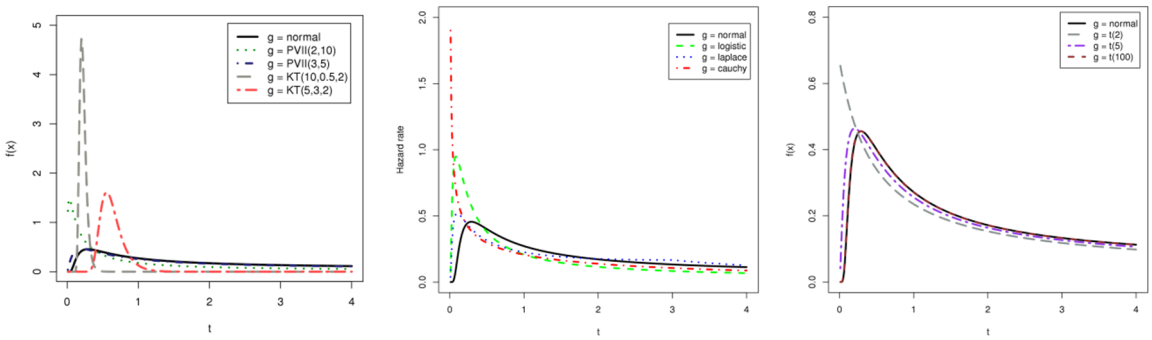

Figure 4: Plots of failures rates for given kernel .

3 Chi-squared type tests for right censored data

Following Bagdonavičius and Nikulin ([3], [4]), we describe a chi-squared test for testing composite parametric hypothesis when data are right censored.

Suppose that are failures time non-negative and independent and the probability density function of the random variable belong to a parametric family . The censoring variables are also non-negative and assumed to be random sample. Let us and are independent. We observed

(6)

where,

$$X_i = T_i \wedge C_i, \qquad \delta_i = 1_{\{T_i \le C_i\}}, i = 1, 2, \cdots, n.$$

Defined that

$$S(t, \boldsymbol{\theta}) = \boldsymbol{P}_{\boldsymbol{\theta}}(T > t); \;\; \lambda(t, \boldsymbol{\theta}) = \frac{f(t, \boldsymbol{\theta})}{S(t, \boldsymbol{\theta})}, \qquad \Lambda(t, \boldsymbol{\theta}) = -\ln\{S(t, \boldsymbol{\theta})\}, \boldsymbol{\theta} \in \Theta \subseteq R^m,$$

be the survival, hazard rate and cumulative hazard functions, respectively. Denote by G_i and g_i are the survival and the density function of the censoring time C_i, respectively. Supposing that the right censoring is non-informative which means that the function G_i does not depend on $\boldsymbol{\theta}$. So in this case, we obtain the following expressions for the likelihood function $L(\boldsymbol{\theta})$

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f^{\delta_i}(X_i, \boldsymbol{\theta}) S^{1-\delta_i}(X_i, \boldsymbol{\theta}) g^{1-\delta_i}(C_i) G^{\delta_i}(C_i).$$

So the members with G_i and g_i do not contain $\boldsymbol{\theta}$, so they can be rejected. The likelihood function is obtained

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f^{\delta_i}(X_i, \boldsymbol{\theta}) S^{1-\delta_i}(X_i, \boldsymbol{\theta}) = \prod_{i=1}^{n} \lambda^{\delta_i}(X_i, \boldsymbol{\theta}) S(X_i, \boldsymbol{\theta}). \tag{7}$$

The estimator $\widehat{\boldsymbol{\theta}}_n$ maximizing the likelihood function $L(\boldsymbol{\theta})$. The log-likelihood function is

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \{\delta_i \ln \lambda(X_i, \boldsymbol{\theta}) + \ln S(X_i, \boldsymbol{\theta})\} = \sum_{i=1}^{n} \{\delta_i \ln \lambda(X_i, \boldsymbol{\theta}) - \Lambda(X_i, \boldsymbol{\theta})\}. \tag{8}$$

The maximum likelihood estimator $\widehat{\boldsymbol{\theta}}_n$ satisfies the system equations

$$\dot{\ell}(\widehat{\boldsymbol{\theta}}_n) = \boldsymbol{0}_m,$$

where $\dot{\ell}(\boldsymbol{\theta})$ are the score vectors

$$\dot{\ell}(\boldsymbol{\theta}) = \frac{\partial}{\partial \theta} \ell(\boldsymbol{\theta}) = \left(\frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_1}, \frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_2}, \cdots, \frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_m}\right)^T.$$

The Fisher information matrix is defined as

$$\boldsymbol{I}(\boldsymbol{\theta}) = -E_\theta \ddot{\ell}(\boldsymbol{\theta}),$$

where

$$\ddot{\ell}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \delta_i \frac{\partial^2}{\partial \boldsymbol{\theta}^2} \ln \lambda(X_i, \boldsymbol{\theta}) - \sum_{i=1}^{n} \frac{\partial^2}{\partial \boldsymbol{\theta}^2} \Lambda(X_i, \boldsymbol{\theta}).$$

Supposing that $\boldsymbol{\theta}_0$ is the true value of $\boldsymbol{\theta}$, under some regularity conditions, we have

$$\widehat{\boldsymbol{\theta}}_n \xrightarrow{P} \boldsymbol{\theta}_0; \;\; \sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) = i^{-1}(\boldsymbol{\theta}_0) \frac{1}{\sqrt{n}} \dot{\ell}(\boldsymbol{\theta}_0) + O_P(1), \frac{-1}{\sqrt{n}} \ddot{\ell}(\widehat{\boldsymbol{\theta}}_n) \xrightarrow{P} i(\boldsymbol{\theta}_0),$$

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} N_m(0, i^{-1}(\boldsymbol{\theta}_0)); \;\; \frac{1}{\sqrt{n}} \dot{\ell}(\boldsymbol{\theta}_0) \xrightarrow{d} N_m(0, i(\boldsymbol{\theta}_0)),$$

where, $\widehat{\boldsymbol{\theta}}_n$ are the maximum likelihood estimation of $\boldsymbol{\theta}$ and the matrix

$$i(\boldsymbol{\theta}_0) = \lim_{n \to \infty} \frac{\boldsymbol{I}(\boldsymbol{\theta}_0)}{n}.$$

For any $t \ge 0$, set

$$N_i(t) = 1_{\{t \ge X_i, \delta_i = 1\}} = \begin{cases} 1, & \text{if } t \ge X_i \text{ and } \delta_i = 1, \\ 0, & \text{if } 0 \le t \le X_i. \end{cases}; \;\; Y_i(t) = 1_{\{t \le X_i\}} = \begin{cases} 1, & \text{if } t \le X_i, \\ 0, & \text{if } t > X_i. \end{cases}$$

$$N(t) = \sum_{i=1}^{n} N_i(t), \qquad Y(t) = \sum_{i=1}^{n} Y_i(t).$$

The process $N(t)$ shows the number of observed failures in the interval $[0, t]$ and the process $Y(t)$ shows the number of objects which are "at risk" just prior to time t. The sample (6) is equivalent to the sample

$$(N_1(t), Y_1(t), t \geq 0), (N_2(t), Y_2(t), t \geq 0), \cdots, (N_n(t), Y_n(t), t \geq 0). \quad (9)$$

For the sample (9), the parametric log-likelihood function can be written by expression follows

$$\ell(\boldsymbol{\theta}) = \int_0^{+\infty} \{\ln \lambda(u, \boldsymbol{\theta})\, dN(u) - Y(u)\lambda(u, \boldsymbol{\theta})\} du.$$

The score function is

$$\dot{\ell}(\boldsymbol{\theta}) = \int_0^{\infty} \frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda(u, \boldsymbol{\theta}) \{dN(u) - Y(u)\lambda(u, \boldsymbol{\theta}) du\} = \int_0^{\infty} \frac{\partial}{\partial \theta} \ln \lambda(u, \boldsymbol{\theta})\, dM(u, \boldsymbol{\theta}),$$

and

$$\ddot{\ell}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \int_0^{\infty} \frac{\partial^2}{\partial \boldsymbol{\theta}^2} \ln \lambda(u, \boldsymbol{\theta}) dM_i(u, \boldsymbol{\theta}) - \sum_{i=1}^{n} \int_0^{\infty} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda(u, \boldsymbol{\theta})\right) \left(\frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda(u, \boldsymbol{\theta})\right)^T \lambda(u, \boldsymbol{\theta}) Y_i(u) du,$$

where, $M_i(t, \boldsymbol{\theta}) = N_i(t) - \int_0^t Y_i(u)\lambda(u, \boldsymbol{\theta}) du$, $(\boldsymbol{\theta} \in \Theta)$ is the zero mean martingale with respect to the filtration generated by the data.

Suppose that the processes N_i and Y_i are observed for finite time $\tau > 0$, which means that at time τ, observation on all surviving objects are censored, and so instead of using censoring time C_i. In this case, the matrix Fisher information can be written as

$$\boldsymbol{I}(\boldsymbol{\theta}) = -E_{\boldsymbol{\theta}} \ddot{\ell}(\boldsymbol{\theta}) = E_{\theta} \sum_{i=1}^{n} \int_0^{\infty} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda(u, \boldsymbol{\theta})\right) \left(\frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda(u, \boldsymbol{\theta})\right)^T \lambda(u, \boldsymbol{\theta}) Y_i(u) du.$$

Let be consider next the hypothesis

$$H_0 : F(x) \in \mathcal{F}_0 = \{F_0(x, \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta \subseteq R^m\},$$

here, $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_m)^T$ are an unknown m-dimensional parameters and F_0 is a known distribution function.

Subdividing the interval $[0, \tau]$ into $k > m$ smaller intervals $I_j = (a_{j-1}, a_j]$, with $a_0 = 0$, $a_k = \tau$, and denote by

$$U_j = N(a_j) - N(a_{j-1}),$$

the number of observed failures in the jth interval I_j, $(j = 1, 2, \cdots, k)$. Let

$$e_j = \int_{a_{j-1}}^{a_j} \lambda(u, \widehat{\boldsymbol{\theta}}_n) Y(u) du.$$

A chi-squared test which was proposed by Bagdonavičius and Nikulin [21], based on the vector

$$\mathcal{Z} = (Z_1, Z_1, \cdots, Z_1)^T, \text{ with } Z_j = \frac{1}{\sqrt{n}}(U_j - e_j), j = 1, 2, \cdots, k. \quad (10)$$

Under the conditions

1) There exists a neighborhood Θ_0 of $\boldsymbol{\theta}_0$ such that for all n and $\boldsymbol{\theta} \in \Theta_0$, and almost all $t \in [0, \tau]$, the partial derivatives of $\lambda(t, \boldsymbol{\theta})$ of the first, second and third order with respect to $\boldsymbol{\theta}$ exist and are continuous in $\boldsymbol{\theta}$ for $\boldsymbol{\theta} \in \Theta_0$. Moreover, they are bound in $[0, \tau] \times \Theta_0$ and the log-likelihood function may be differentiated three times with respect to $\boldsymbol{\theta} \in \Theta_0$, by interchanging the order of integration and differentiation.

2) $\lambda(t, \boldsymbol{\theta})$ is bound away from zero in $[0, \tau] \times \Theta_0$.

3) A positive deterministic function $y(t)$ exists such that $\sup_{t \in [0,\tau]} \left|\frac{Y(t)}{n} - y(t)\right| \xrightarrow{P} 0$.

4) Under condition 1) - 3), the matrix $i(\boldsymbol{\theta}_0) = \lim_{n \to \infty} \frac{I(\boldsymbol{\theta}_0)}{n}$ is positive definite.

The statistic of Bagdonavičius and Nikulin given as

$$\boldsymbol{Y}_n^2(\widehat{\boldsymbol{\theta}}_n) = \mathcal{Z}^T\hat{\Sigma}^-\mathcal{Z}, \tag{11}$$

where, $\hat{\Sigma}^-$ is the general inverse matrix of the covariance matrix $\hat{\Sigma}$,

$$\hat{\Sigma} = \hat{\mathcal{A}} - \hat{\mathcal{C}}^T\hat{\boldsymbol{I}}^{-1}\hat{\mathcal{C}},$$
$$\hat{\Sigma}^- = \hat{\mathcal{A}}^{-1} + \hat{\mathcal{A}}^{-1}\hat{\mathcal{C}}^T\hat{\mathcal{G}}^{-1}\hat{\mathcal{C}}\hat{\mathcal{A}}^{-1}, \qquad \hat{\mathcal{G}} = \hat{\boldsymbol{I}} - \hat{\mathcal{C}}\hat{\mathcal{A}}^{-1}\hat{\mathcal{C}}^T, \tag{12}$$

$\hat{\mathcal{A}}$ is the diagonal $k \times k$ matrix with the elements $A_j = \frac{U_j}{n}$ on the diagonal, $\hat{\mathcal{A}}^{-1}$ is inverse matrix of $\hat{\mathcal{A}}$, and

$$\hat{\mathcal{C}} = [\hat{C}_{lj}]_{m\times k}, \text{with } \hat{C}_{lj} = \frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\frac{\partial \ln\lambda(X_i,\theta)}{\partial\theta_l}, l = 1,2,\cdots, m, \; j = 1, 2,\cdots, k, \tag{13}$$

$$\hat{\boldsymbol{I}} = [\hat{\imath}_{ll'}]_{m\times m}, \text{with } \hat{\imath}_{ll'} = \frac{1}{n}\sum_{i=1}^{n}\delta_i\frac{\partial \ln\lambda(X_i,\ \theta)}{\partial\theta_l}\frac{\partial \ln\lambda(X_i,\ \theta)}{\partial\theta_{l'}}, l, l' = 1,2,\cdots, \quad m. \tag{14}$$

From the definition of $\mathcal{Z}$ in (10), the test statistic $Y_n^2(\hat{\theta}_n)$ should be written as

$$\boldsymbol{Y}_n^2(\widehat{\boldsymbol{\theta}}_n) = \boldsymbol{X}_n^2 + \boldsymbol{Q}, \tag{15}$$

where,

$$\boldsymbol{X}_n^2 = \sum_{j=1}^{k}\frac{(U_j - e_j)^2}{U_j}, \qquad \boldsymbol{Q} = \widehat{\mathcal{W}}^T\hat{\mathcal{G}}^-\widehat{\mathcal{W}}, \qquad \widehat{\mathcal{W}} = \hat{\mathcal{C}}\hat{\mathcal{A}}^{-1}\mathcal{Z}, \qquad \hat{\mathcal{G}} = \hat{\boldsymbol{I}} - \hat{\mathcal{C}}\hat{\mathcal{A}}^{-1}\hat{\mathcal{C}}^T.$$

Under the hypothesis H_0, the limiting distribution of the statistics $\boldsymbol{Y}_n^2(\widehat{\boldsymbol{\theta}}_n)$ is chi-squared with $r = rank(\Sigma^-)$ degrees of freedom that is,

$$\lim_{n\to\infty} P\{\boldsymbol{Y}_n^2(\widehat{\boldsymbol{\theta}}_n) > x \mid H_0\} = P\{\chi_r^2 > x\}, \text{for any } x > 0.$$

Statistical inference for the hypothesis $\boldsymbol{H_0}$: The null hypothesis H_0 is rejected with approximate significance level α if $\boldsymbol{Y}_n^2(\widehat{\boldsymbol{\theta}}_n) > \chi_\alpha^2(r)$ or $\boldsymbol{Y}_n^2(\widehat{\boldsymbol{\theta}}_n) < \chi_{1-\alpha}^2(r)$ depending on an alternative, where $\chi_\alpha^2(r)$ and $\chi_{1-\alpha}^2(r)$ corresponding are the upper and lower α percentage points of the χ_r^2 distribution, respectively.

Using the method of interval selection which is proposed by Bagdonavičius, and Nikulin [20], we used a_j as the random data function. Define

$$E_k = \sum_{i=1}^{n}\Lambda(X_i, \widehat{\boldsymbol{\theta}}_n), \; E_j = \frac{j}{k}E_k, \; j = 1,2,\cdots,k.$$

Denote by $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ the ordered sample from $X_1, X_2, \cdots, X_n$. Set

$$b_i = (n-i)\Lambda(X_{(i)}, \widehat{\boldsymbol{\theta}}_n) + \sum_{l=1}^{i}\Lambda(X_{(l)}, \widehat{\boldsymbol{\theta}}_n), \qquad i = 1,2,\cdots,n,$$

if i is the smallest natural number verifying $b_{i-1} \le E_j \le b_i$ then $\hat{a}_j$ verifying the equality

$$(n-i+1)\Lambda(\hat{a}_j, \widehat{\boldsymbol{\theta}}_n) + \sum_{l=1}^{i-1}\Lambda(X_{(l)}, \widehat{\boldsymbol{\theta}}_n) = E_j$$

So

$$\hat{a}_j = \Lambda^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1}\Lambda(X_{(l)}, \widehat{\boldsymbol{\theta}}_n)}{n-i+1}, \widehat{\boldsymbol{\theta}}_n\right); \; \hat{a}_k = \max\{X_{(n)}, \tau\}, (j = 1,2,\cdots,k-1). \tag{16}$$

where Λ^{-1} is the inverse of the function Λ. We have: $0 < \hat{a}_1 < \hat{a}_2 < \cdots < \hat{a}_k$, with this choice of intervals, then $e_j = \frac{E_k}{k}$, for all j.

Application for GBS distributions: In particular, we shall give chi-squared tests NRR for the hypothesis H_0 that the data X_i are coming from the GBS distributions with the probability

density, cumulative distribution, hazard rate, survival and cumulative hazard functions give in formulas (3), (4) and (5), respectively.

The GBS log-likelihood functions $\ell(\boldsymbol{\theta}), (\boldsymbol{\theta} = (\alpha, \beta)^T)$ is

$$\ell(\boldsymbol{\theta}) = -\delta \ln \alpha - \delta \ln \beta + \sum_{i=1}^{n} \delta_i \ln\left\{\left(\frac{\beta}{X_i}\right)^{\frac{1}{2}} + \left(\frac{\beta}{X_i}\right)^{\frac{3}{2}}\right\} + \sum_{i=1}^{n} \delta_i \ln\{g(K_i(\alpha,\beta))\}$$
$$+ \sum_{i=1}^{n} \delta_i \ln\{1 - F_Z(a_i(\alpha,\beta))\}$$

Let $\widehat{\boldsymbol{\theta}}_n = (\hat{\alpha}, \hat{\beta})^T$ be maximum likelihood estimations which are solutions of the non-linear system equations

$$\left(\dot{\ell}_\alpha(\boldsymbol{\theta}), \dot{\ell}_\beta(\boldsymbol{\theta})\right) = \mathbf{0}_2.$$

Using the formula (13) – (14), the elements $\hat{\imath}_{ll'}, (l, l' = 1, 2)$ of the Fisher information matrix $\widehat{\boldsymbol{I}} = [\hat{\imath}_{ll'}]_{2\times 2}$ are

$$\hat{\imath}_{11} = \frac{1}{n\hat{\alpha}^2} \sum_{i=1}^{n} \delta_i \left[-1 + \widehat{K}_i(\hat{\alpha},\hat{\beta}) v\left(\widehat{K}_i(\hat{\alpha},\hat{\beta})\right) - \frac{\hat{A}_i(\hat{\alpha},\hat{\beta}) f_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}{1 - F_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}\right]^2,$$

$$\hat{\imath}_{22} = \frac{1}{n\hat{\beta}^2} \sum_{i=1}^{n} \delta_i \left[-1 + \frac{1}{2}\frac{1 + 3\frac{\hat{\beta}}{X_i}}{1 + \frac{\hat{\beta}}{X_i}} + \frac{1}{2}\hat{A}_i(\hat{\alpha},\hat{\beta})\hat{B}_i(\hat{\alpha},\hat{\beta}) v\left(\widehat{K}_i(\hat{\alpha},\hat{\beta})\right) - \frac{1}{2}\frac{\hat{B}_i(\hat{\alpha},\hat{\beta}) f_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}{1 - F_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}\right]^2,$$

$$\hat{\imath}_{12} = \frac{1}{n\hat{\alpha}\hat{\beta}} \sum_{i=1}^{n} \delta_i \left[-1 + \widehat{K}_i(\hat{\alpha},\hat{\beta}) v\left(\widehat{K}_i(\hat{\alpha},\hat{\beta})\right) - \frac{\hat{A}_i(\hat{\alpha},\hat{\beta}) f_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}{1 - F_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}\right] \times$$
$$\times \left[-1 + \frac{1}{2}\frac{1 + 3\frac{\hat{\beta}}{X_i}}{1 + \frac{\hat{\beta}}{X_i}} + \frac{1}{2}\hat{A}_i(\hat{\alpha},\hat{\beta})\hat{B}_i(\hat{\alpha},\hat{\beta}) v\left(\widehat{K}_i(\hat{\alpha},\hat{\beta})\right) - \frac{1}{2}\frac{\hat{B}_i(\hat{\alpha},\hat{\beta}) f_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}{1 - F_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}\right],$$

and the matrix $\hat{C} = [\hat{C}_{lj}]_{2\times k}$ given by

$$\hat{C}_{1j} = \frac{1}{n\hat{\alpha}} \sum_{i: X_i \in I_j} \delta_i \left[-1 + \widehat{K}_i(\hat{\alpha},\hat{\beta}) v\left(\widehat{K}_i(\hat{\alpha},\hat{\beta})\right) - \frac{\hat{A}_i(\hat{\alpha},\hat{\beta}) f_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}{1 - F_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}\right],$$

$$\hat{C}_{2j} = \frac{1}{n\hat{\beta}} \sum_{i: X_i \in I_j} \delta_i \left[-1 + \frac{1}{2}\frac{1 + 3\frac{\hat{\beta}}{X_i}}{1 + \frac{\hat{\beta}}{X_i}} + \frac{1}{2}\hat{A}_i(\hat{\alpha},\hat{\beta})\hat{B}_i(\hat{\alpha},\hat{\beta}) v\left(\widehat{K}_i(\hat{\alpha},\hat{\beta})\right) - \frac{1}{2}\frac{\hat{B}_i(\hat{\alpha},\hat{\beta}) f_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}{1 - F_Z(\hat{A}_i(\hat{\alpha},\hat{\beta}))}\right].$$

where,

$$\hat{A}_i(\hat{\alpha},\hat{\beta}) = \frac{1}{\hat{\alpha}}\left\{\left(\frac{X_i}{\hat{\beta}}\right)^{\frac{1}{2}} - \left(\frac{\hat{\beta}}{X_i}\right)^{\frac{1}{2}}\right\}, \qquad \hat{B}_i(\hat{\alpha},\hat{\beta}) = \frac{1}{\hat{\alpha}}\left\{\left(\frac{X_i}{\hat{\beta}}\right)^{\frac{1}{2}} + \left(\frac{\hat{\beta}}{X_i}\right)^{\frac{1}{2}}\right\},$$

$$\widehat{K}_i(\hat{\alpha},\hat{\beta}) = \frac{1}{\hat{\alpha}^2}\left\{\frac{X_i}{\hat{\beta}} + \frac{\hat{\beta}}{X_i} - 2\right\}, i = 1, 2, \cdots, n.$$

and $f_Z(u) = c\, g(u^2)$, $F_Z(\cdot)$ are the probability density function and cumulative function of the random variable $Z \approx EC(0, 1;\ g)$ which follows a standard symmetrical distribution in R with the kernel $g(\cdot)$, respectively, and

$$v(u) = -2w(u);\ w(u) = \frac{g'(u)}{g(u)},\ \ u > 0,$$

are the transformations functions of kernel function $g(u)$, and $w'(u)$ is the derivative of $w(u)$ ([15], [16]). Table 3 below shown some transformations functions $w(u)$ and its derivative $w'(u), (u > 0)$ corresponding with kernel $g(u)$ of indicated Elliptic distributions $EC(0, 1;\ g)$.

	$N(0,1)$	$t(\nu)$	$L(0,1)$	$Log(0,1)$	$PE(\nu)$
$w(u)$	$\frac{-1}{2}$	$-\frac{\nu+1}{2(\nu+u)}$	$-\frac{1}{2\sqrt{u}}$	$-\frac{1}{2\sqrt{u}}\tanh\left(\frac{\sqrt{u}}{2}\right)$	$-\frac{\nu}{2}u^{\nu-1}$
$w'(u)$	0	$\frac{\nu+1}{2(\nu+u)^2}$	$\frac{1}{4u\sqrt{u}}$	$\frac{\sinh\sqrt{u}-\sqrt{u}}{4u\sqrt{u}[1+\cosh\sqrt{u}]}$	$-\frac{\nu[\nu-1]}{2}u^{\nu-2}$

Table 3: Transformations functions *w(u)* and its derivative *w'(u), (u > 0)* for kernel *g(u)* of the indicated Elliptic distributions $EC(0,1;\ g)$.

Chi-squared test for GBS distributions: Under the hypothesis H_0, the matrix $\hat{\mathcal{G}}$ is non-degenerate. So, the hypothesis H_0 is rejected with approximate significance level α if

$$\boldsymbol{Y}_n^2(\widehat{\boldsymbol{\theta}}_n) > \chi_k^2(\alpha).$$

It is necessary to note that: $PE(\nu) \cong KT(1,0.5,\nu), N(0,\ 1) \cong KT(1,0.5,1), L(0,1) \cong PE(0.5), t(\nu) \cong PVII([\nu+1]/2,\nu), Log(0,1) \cong LIII(1)$.

Thus, the next section, we consider goodness of fit test for following five distributions: GBS-$N(0,\ 1)$ which known as BS distribution, GBS-Laplace, GBS-Logistic, GBS-$t(\nu)$ and GBS-Cauchy distributions.

4 Real study

All distributions presented in the next two examples by using R statistics software, we analyze the goodness of fit test for the parametric generalized BS distributions in two studies to a data of breast cancer which set from research of Boag (1949) and the data from a laboratory investigation in which the vaginas of rats were painted with the carcinogen DMBA of Pike (1966).

4.1 Analysis of breast cancer data

Boag [10] was presented the survival times for 121 patients treated for cancer of the breast in one particular hospital during the years 1929-1938 which given in table below. The times are in months, and asterisks denote censoring times. This data included 66 observations and 55 censoring times.

0.3	7.4*	13.5	16.8	21.0	29.1	37*	41	45*	52	60*	78	105*	
129*	0.3*	7.5	14.4	17.2	21.1	30	38	41	46*	54	61*	80	
109*	129*	4.0*	8.4	14.4	17.3	23.0	31	38*	41*	46*	55*	62*	
83*	109*	139*	4.0*	8.4	14.4	17.3	23.0	31	38*	41*	46*	55*	62*
	83*	109*	139*	5.0	8.4	14.8	17.5	23.4*	31	38*	42	47*	56
	65*	88*	111*	154*	5.6	10.3	15.5*	17.9	23.6	32	39*	43*	48
	57*	65*	89	115*	6.2	11.0	15.7	19.8	24.0	35	39*	43*	49*
	58*	67*	90	117*	6.3	11.8	16.2	20.4	24.0	35	40	43*	51
	59*	67*	93*	125*	6.6	12.2	16.3	20.9	27.9	37*	40*	44	51
	60	68*	96*	126	6.8	12.3	16.5	21.0	28.2	37*	40*	45*	51*
	60*	69*	103*	127*									

Firstly, we consider the hypothesis H_0 that the survival times for 121 breast cancer patients belongs the Birnbaum-Saunders distribution. In this case, MLE's of the parameters $\boldsymbol{\theta} = (\alpha,\ \beta)^T$ of the BS distribution are $\widehat{\boldsymbol{\theta}}_n = (2.04798, 47.11415)^T$.

Choosing the sub-intervals $k = 6$. The values of a_j, the frequency vector $\mathcal{Z}$ and the elements of the matrix $\hat{\mathcal{C}}$ give in table follows.

j	1	2	3	4	5	6
a_j	5.312384	9.476277	15.352179	24.889209	41.374625	154.000010
U_j	2	8	9	19	15	13
e_j	12.40306	12.40306	12.40306	12.40306	12.40306	12.40306
Z_j	-0.945732	-0.400278	-0.309368	0.599721	0.236085	0.054267
$\hat{C}_{1j}$	0.149316	0.015361	-0.008443	-0.042270	-0.052283	-0.058149
$\hat{C}_{2j}$	-0.003347	-0.000867	-0.000594	-0.001079	-0.000913	-0.001021

The matrix information of Fisher $\hat{\boldsymbol{I}} = [\hat{\imath}_{ll'}]_{2\times 2}$ and the matrix $\hat{\mathcal{G}}$ are

$$\hat{\boldsymbol{I}} = \begin{bmatrix} 2.62227 & -0.055142 \\ -0.055142 & 0.001280 \end{bmatrix}, \hat{\mathcal{G}} = \begin{bmatrix} 1.203914 & -0.025996 \\ -0.025996 & 0.000562 \end{bmatrix}.$$

We continued this data for GBS distributions for another kernel g: Logistic, Laplace, Cauchy and $t(\nu)$ distribution with the same sub-intervals. The results give in table 4 follow.

Distribution	$\hat{\boldsymbol{\theta}}_n = (\hat{\alpha}, \hat{\beta})^T$	$\boldsymbol{X}_n^2$	$\boldsymbol{Q}$	$\boldsymbol{Y}_n^2$	$pv_{0.05}$-value
GBS-Cauchy	$(0.95391, 47.25487)^T$	148.2783	3.593348	151.8717	0
BGS-$L(0,1)$	$(1.36965, 51.99999)^T$	9.916332	0.772807	10.68914	0.0984723
GBS-$Log(0,1)$	$(0.95750, 54.63219)^T$	6.586821	1.861331	8.448152	0.2070737
GBS-$t(100)$	$(1.89550, 50.25370)^T$	20.70194	8.548100	29.25004	$5.4553.10^{-5}$
GBS-$t(5)$	$(1.33215, 55.86467)^T$	5.652789	0.897494	6.550284	0.3644431

Table 4: MLE's of $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta})^{\boldsymbol{T}}$, values of $\boldsymbol{Y}_n^2$ and p-values with indicated kernel distributions, data of Boag (1949).

In this example, we suggest that GBS with kernel g: Normal, *t(100)* and Cauchy are strongly rejected and GBS with Logistic, Laplace and $t(5)$ kernels are very well in concordance with the survival times for breast cancer patients treated of Boag. Figure 5 below illustrates the curve of Kaplan-Meier estimate of survival function, with the curve of GBS survival functions corresponding with the kernel indicatives.

4.2 Analysis of the times until a carcinoma appeared

Pike [34] gave some data from a laboratory investigation in which the vaginas of rats were painted with the carcinogen DMBA, and the number of days T until a carcinoma appeared was recorded. The data below are for a group of 19 rats (Group 1 in Pike's paper). The two observations with asterisks are censoring times.

143, 164, 188, 188, 190, 192, 206, 209, 213, 216, 220, 227, 230, 234, 246, 265, 304, 216^*, 244^*.

This data analyzed by Lawless [25], pp.188 where he suggested that probability plots for two parameters Weibull distribution. By using the NRR statistics for two parameters Weibull distribution, we obtain the value of NRR statistics $\boldsymbol{Y}_n^2 = 6.658669$ with p-value at level significance $\alpha = 0.05$ is $pv_{0.05} = 0.08361068$.

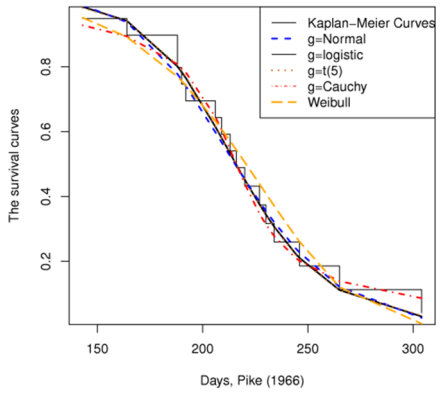

Figure 5: GBS with indicated kernel, Weibull and Kaplan-Meier estimate of for data of Boag (1949).

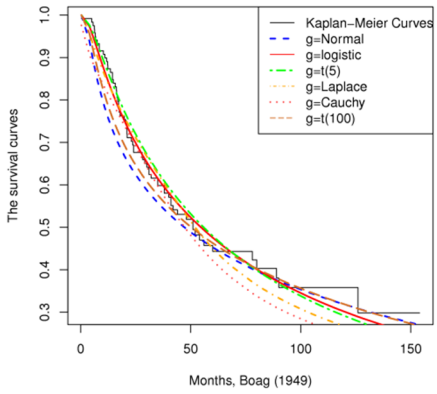

Figure 6: GBS with indicated kernel and Kaplan-Meier estimate of for the data of Pike (1966).

We consider next the hypotheses that the data above follows the GBS distributions in the cases of kernel : Standard Normal distribution , Standard Logistic distribution , Standard Cauchy distribution . Choosing grouping intervals , the results given in table 5 below.

Distribution					-value
GBS-		3.125849	16.35656	19.48241	0.0006316
GBS-		1.000369	2.885825	3.886194	0.4216269
GBS-		0.749583	0.325662	1.075246	0.8981795
GBS-		0.745397	0.3637746	1.109172	0.8928143

Table 5: MLE's of , values of and *p*-values with indicated kernel distributions, data of Pike (1966).

We plot the estimated GBS survivor functions correspond indicated kernel and the Kaplan-Meier estimate for the data of Pike (1966) in Figure 6.

In this example, it is clear that the data are the best in concord with GBS-Logistic, GBS-Cauchy and GBS- distributions, it also acceptes for two parameters Weibull distribution. However, these data contradict the BS distribution much very strongly.

5 Summary and conclusion

In this paper, we have presented a modifier Chi-squared goodness-of-fit test for generalized Birnbaum-Saunders distributions. The results obtained in our examples show that the considered families are in accordance with lifetimes data. In addition, its hazard rate functions can be uni-modal or bimodal by adjusting the values of its parameters and its kernel . So, it is necessary to use it as a baseline hazard rate functions in the parametric survival model. We would like to thank our colleagues, PhD. R. Tahir and N. Saaidia for valuable comments, which helped us improve the presentation.

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