CONNECTIVITY PROBABILITY OF RANDOM GRAPH GENERATED BY POINT POISSON FLOW

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ABSTRACT

In different applications (for an example in the mining engineering) a problem of a definition of a set in two or three dimension spaces by a finite set of points origins. This problem consists of a determination in the finite set of some subset of points sufficiently close to each other. A solution of this problem consists of two parts. Primarily initial finite set of points is approximated by point Poisson flow in some area which is widely used in the stochastic geometry [1, sections 5, 6]. But a concept of a proximity is analyzed using methods of the random graph theory like a concept of maximal connectivity component [2] - [4]. This concept origins in a junction of the combinatory probability theory and of the graph theory. An analysis of these concepts and mathematical constructions leads to a generalization of the random graph theory theorems onto graphs generated by point Poisson flow in some area.

1. MAIN RESULTS AND THEIR PROOFS

Following [1] consider the Poisson flow $\Lambda(n)$ of points $x(1),...,x(\tau(n))$ with the intensity $\lambda_n = n$ in three dimension cube A with unit length edges. Contrast each pair of points $x(i), x(j)$ the Euclidean distance $\rho(x(i), x(j))$ and introduce Boolean variable $z(x(i), x(j)) = 1$, if $\rho(x(i), x(j)) < r(n)$ and $z(x(i), x(j)) = 0$ in opposite case. Here $r(n)$ is some positive number dependent on n. Construct the random graph $\Gamma(n)$ with $\tau(n)$ vertices and the adjacency matrix $\|z(x(i), x(j))\|_{i,j=1}^{\tau(n)}$. Denote $P(n)$ the connectivity probability of this graph, $\overline{P}(n) = 1 - P(n)$.

Theorem 1. Suppose that $r(n) = (n\varphi(n))^{-1/3}$, $\varphi(n) \to \infty$, $n \to \infty$. Then the disconnection probability $\overline{P}(n) \to 1$.

Proof. Denote the center of the cube A by O and describe around O the ball $U_1(n)$ with the radius $r_1(n) = (n/\varphi_1(n))^{-1/3}$, $\varphi_1(n) \to \infty$, $\varphi_1(n)/n \to 0$, $\varphi_1^2(n)/\varphi(n) \to 0$, $n \to \infty$.

Then the probability that a positive number of the flow $\Lambda(n)$ points turns out in the ball $U_1(n)$ is

$$P_1(n) = 1 - \exp\left(-\frac{4\pi n \cdot \left((n/\varphi_1(n))^{-1/3}\right)^3}{3}\right) = 1 - \exp\left(-\frac{4\pi\varphi_1(n)}{3}\right) \to 1,\ n \to \infty .$$

Construct further the ball $U_2(n)$with the center O and with the radius

$$r_2(n) = r_1(n) + r(n) = (n/\varphi_1(n))^{-1/3} + (n\varphi(n))^{-1/3} .$$

Consider the ball layer $U_2(n) \setminus U_1(n)$ and using the condition $\varphi_1^2(n)/\varphi(n) \to 0$, $n \to \infty$, calculate its volume

$$V_2(n)=\frac{4\pi\left(r_2^3(n)-r_1^3(n)\right)}{3}=4\pi n^{-1}\left(\varphi_1^2(n)/\varphi(n)\right)^{1/3}\left(1+\mathrm{O}\left(\left(\varphi_1^2(n)/\varphi(n)\right)^{1/3}\right)\right),\ n\to\infty$$

Then for $n\to\infty$ the probability $P_2(n)$ of an absence in the set $U_2(n)\setminus U_1(n)$ points of the flow $\Lambda(n)$ satisfies the formula:

$$P_2(n)=\exp(-nV_2(n))=\exp\left(-4\pi n n^{-1}\left(\varphi_1^2(n)/\varphi(n)\right)^{1/3}\left(1+\mathrm{O}\left(\left(\varphi_1^2(n)/\varphi(n)\right)^{1/3}\right)\right)\right)\to 1.$$

Build now the set $A\setminus U_2(n)$ and calculate the probability $P_3(n)$ that a positive number of the flow $\Lambda(n)$ points belong to this set using the condition $\varphi_1^2(n)/\varphi(n)\to 0$, $n\to\infty$,

$$P_3(n)=1-\exp(-n(1-V_2(n)))=1-\exp\left(-n\left(1-4\pi n^{-1}\left(\varphi_1^2(n)/\varphi(n)\right)^{1/3}\left(1+\mathrm{O}\left(\left(\varphi_1^2(n)/\varphi(n)\right)^{1/3}\right)\right)\right)\right)\to 1,$$
$$n\to\infty.$$

So the probability of an intersection of the following events:

1) in the ball $U_1(n)$ there is positive number of the flow $\Lambda(n)$ points;
2) in the ball layer $U_2(n)\setminus U_1$ there are not points of the flow $\Lambda(n)$;
3) in the set $A\setminus U_2(n)$ there is a positive number of the flow $\Lambda(n)$ points

tends to the unit for $n\to\infty$. As a distance from any point of the ball $U_1(n)$ to each point of the set $A\setminus U_2(n)$ is not smaller than $r(n)$, so these events intersection contains in the event: the random graph $\Gamma(n)$ is not connected. Consequently the following limit relation is true: $\overline{P}(n)\to 1$, $n\to\infty$.

Theorem 2. Suppose that $r(n)=(n/\psi(n))^{-1/3}$ where $\psi(n)/n\to 0$, $\psi(n)-\ln n\to\infty$, $n\to\infty$. Then the connectivity probability $P(n)\to 1$, $n\to\infty$.

Proof. From the condition $\psi(n)-\ln n\to\infty$, $n\to\infty$, we obtain that $r(n)\to 0$, $n\to\infty$. Assume that the cube A and the cube A_n with the edge length $r(n)/\sqrt{6}$ have common centre and are homothetic. By a parallel hyphenation of the cube A_n on distances fold to $r(n)/\sqrt{6}$ by each coordinate construct a family of all cubes intersected with the cube A. An bundling of these cubes contains the cube A. A complete number $N(n)$ of all cubes from this family for some finite and positive number C is smaller than $Cn/\psi(n)$. The probability $P'(n)$ that a positive number of the flow $\Lambda(n)$ points occurs in some cube of this family equals

$$P'(n)=1-\exp\left(-n\cdot(n/\psi(n))^{-1}/6\sqrt{6}\right)=1-\exp\left(-\psi(n)/6\sqrt{6}\right).$$

Consequently the probability $P''(n)$ of these $N(n)$ events intersection $L(n)$ equals

$$P''(n)=(P'(n))^{N(n)}=\left(1-\exp\left(-\psi(n)/6\sqrt{6}\right)\right)^{N(n)}\geq\left(1-\exp\left(-\psi(n)/6\sqrt{6}\right)\right)^{Cn/\psi(n)},$$

so

$$0\geq(\ln P''(n))^{N(n)}=Cn\ln\left(1-\exp\left(-\psi(n)/6\sqrt{6}\right)\right)/\psi(n)\geq$$
$$\geq-\frac{Cn\exp(-\psi(n))}{6\sqrt{6}\psi(n)\left(1-\exp\left(-(n)/6\sqrt{6}\right)\right)}=R(n).$$

From the condition $\psi(n)-\ln n\to\infty, n\to\infty$, we have that $R(n)\to 0$, $n\to\infty$. As a result the probability of the event $L(n)$ satisfies the formula $P''(n)\to 1$, $n\to\infty$. A distance between each two points belonging to incident cubes (with common face) is not larger than $r(n)$. So the event that the graph $\Gamma(n)$ is connected contains the event $L(n)$ and the inequality $P(n)\geq P''(n)$ is true. As a result we have that

$$1\geq\liminf_{n\to\infty}P(n)\geq\lim_{n\to\infty}P''(n)\geq 1$$

The theorem 2 is proved completely.

2. CONCLUSION

Remark 1. In the conditions of Theorem 2 for any γ, $1/2 < \gamma < 1$, with a probability tending to the unit for $n \to \infty$, the graph $\Gamma(n)$ has single connectivity component with a number of vertices larger than $\gamma\tau(n)$. Indeed the event that there is single connectivity component with the number of vertices larger than $\gamma\tau(n)$, includes the event that the graph $\Gamma(n)$ is connected. From Theorem 2 the probability of the last event tends to the unit for $n \to \infty$. Analogous concept of a giant connectivity component but for a complete graph with independently working edges was introduced and was analysed in [2].

Remark 2. Theorems 1, 2 may be reformulated in terms of the threshold function of the connectivity as a graph theory property [2], [3, Section 10]. If $r(n) << n^{-1/3}$ then with the probability $\overline{P}(n) \to 1$, $n \to \infty$, the graph $\Gamma = \Gamma(n)$ in unconnected. If $r(n) << n^{-1/3}$ then with the probability $P(n) \to 1$, $n \to \infty$, the graph $\Gamma(n)$ is connected. In the last statement it is necessary to make a single refinement of the condition $r(n) << n^{-1/3}$ in the form $r(n) = (n/\psi(n))^{-1/3}$, $\psi(n) - \ln n \to \infty$, $n \to \infty$.

Remark 3. More modern treatment of introduced concepts but for complete graphs with independent and randomly working edges was made in [4].

Remark 4. In Theorems 1, 2 the cube A may be replaced by another geometrical bodies for an example by a ball.

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