SURVIVAL MODELS OF SOME POLITICAL PROCESSES

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ABSTRACT

We extend the Probabilistic ideas from stochastic processes (queuing theory and reliability) on creation of some realistic models for studying several governing political formations, and find their survival characteristics. These models were presented at the Sixth and Seventh International Conferences on Mathematical Models in Reliability (Moscow 2009, and Beijing 2011). Our focus is on a "democracy" model, where the times of survival (existence at the political scene, duration of stay in leading coalition, governing survivability, life time distribution, longevity, etc.) can be derived from the model. Markovian models of spending time in certain sets of states are explored, and some discussion on statistical properties and evaluations are presented. We are confident that other political schemes also can be modeled using appropriate probabilistic tools.

1 INSTRUCTIONS

Modeling politics based on certain scientific concepts and its inclusion into quantitative models is a challenging task. The authors of this work did an extensive review of some successful attempts in political modeling by mathematical means such as: Cioffi-Revilla (2009) recent model of a political system; the recent textbook of Clark et al. (2008) on modeling of preferences in political science; the Taylor's book (1995) with ideas for discrete and deductive mathematical approaches in international conflict resolution; Doyle nonlinear methods (2000) to describe and solve existing political questions; Monroe's (1997) evaluations of the current state of empirical political theory and guidelines to future developments in political science; the Ordeshook (1986), Morrow (1994), Hafer (2007) approaches based on the use of games theory in modeling political processes, and some non mathematical ideas as these, presented in J. March (1994), Then we started working on an approach based on construction of specific probabilistic analysis of components that form political processes inn conditions of an open democracy. In our opinion, formal political theory seeks to develop formal, mathematical models of political, demographic, and economic processes. The above mentioned authors in their books and articles, make an attempt to integrate the modern developments of the applied mathematical theories (games, decision making, multiple, interactive decisions) with models of demand and supply of public goods, and social-choice theory, part of what the political structures are considered.

The articles of Esa and Dimitrov (2009, 2011) are an illustration of how probability tools can be used to model basic components in the big political games. Reliability and service system theories provide a good assembly of approaches in analysis of various sides of the products (risks, and costs inclusive) and characteristics in the political activities. Encouraged by the welcome reception of these presentations on behalf of the experts in applied statistical modeling at the MMR' (2009 and 2011) Forums, we continue working in the same direction. We found more sites our approach may reveal. In the present work we use the results of these simple models to illustrate various important additional characteristics of the political subjects. In our opinion, appropriate models can be made to study totalitarian schemes, monarchy, parliamentary kingdoms, even some non existing, but virtually possible political structures. And this is our reason to consider probability models in politics a challenging area of applied probability and statistics.

2 CHARACTERISTICS OF THE DEMOCRATIC POLITICAL PARTY MODEL

First we use the results from the model of a political party, considered as a formation within a finite population of active individuals N. The existence (formation) of a party requires certain minimum of members, say M+1. Each free individual may decide to join a party at any time, as well as a member can quit the party at any time. Simultaneously, there exist a numerous pool of other parties which operate in a similar way, and their members are not allowed to join (or switch to) another party.

After discussing this mechanism and dynamics, Esa and Dimitrov (2009) introduced the following mathematical model of the political life in a country:

There is a population of N individuals (citizens of a country). These are r parties in the society. Each party is considered as a service system of N available seats (servers) in each. The rate of inputs from each free individual towards party (service system) j is λ_j . j=1,…,r. At system j the individual spends some random time S_j, and goes free of politics (back to its source). A simple Markovian model in the case of exponentially distributed service times S_j of parameters μ_j shows the stationary probabilities P_j that an individual is free (then we use j=0), or is member of the party j=1,2,…,r are given by the expressions

$$P_0 = \left(1+\frac{\lambda_1}{\mu_1}+\cdots+\frac{\lambda_r}{\mu_r}\right)^{-1}; \quad P_j = \frac{\lambda_j}{\mu_j}\left(1+\frac{\lambda_1}{\mu_1}+\cdots+\frac{\lambda_r}{\mu_r}\right)^{-1}, \quad j=1,2,\dots,r. \tag{1}$$

A multinomial model describes the entire spectrum of the party's life in the country, with N independent active *free i*ndividuals. The coordinates of the random vector $\vec{X}=(X_0, X_1, \dots, X_r)$ represent the number of individuals members of each party $X_0 + X_1 + \ldots + X_r = N$. They are distributed according to the multinomial law

$$P(X_0=k_0, X_1=k_1, \dots, X_r=k_r) = \frac{N!}{k_0!k_1!\dots k_r!}P_0^{k_0}P_1^{k_1}\dots P_r^{k_r}, \quad k_0+k_1+\ldots+k_r = N. \tag{2}$$

Hence, the chance of the j^{th} party to exist (at a minimum M_j+1 *members* required for this purpose) is having Binomial probability $P(X_j \geq M_j+1)=1 - B(M_j; N, P_j)$ with the P_j, given by (1), and $B(k; N, p)$ is notation for c.d.f. of Binomial distribution witn parameters N and p, and k its argument. The probability generating function of this distribution (2) is given by the expression

$$P(z_0, z_1, \dots, z_r) = E(z_0^{X_0} z_1^{X_1} \dots z_r^{X_r}) = (P_0 z_0 + P_1 z_1 + \ldots + P_r z_r)^N, \quad |z_j| \leq 1. \tag{3}$$

It allows particular calculation of various average characteristics, correlations etc.

The average number of members of a party j is $E(X_j)=NP_j$, and its variation equals $V(X_j)=NP_j(1 - P_j)$.

The correlation coefficients between the counts X_i and X_j of the parties labeled as i and j are given by the expressions

$$\rho_{ij} = Corr(X_i, X_j) = -\sqrt{\frac{P_i}{1-P_i}}\sqrt{\frac{P_j}{1-P_j}}, \quad i \neq j, \quad i, j=0,1, \dots, r.$$

The correlation is always negative, since parties compete for one and the same pool of potential members. Interesting quantities are the correlation coefficients of the parties i=1,…,r with the "party" of the free individuals, for which subscript j is zero. Their absolute values may be used as a kind of rating indices for the parties in the country. The more the value, the higher the rating is. Interesting observation here is, that the correlation coefficients between parties (as well as the rating) do no depend on the total population size N.

Probability for party *j* to be dissolved must be considered under condition that party is active, i.e. given that $X_j \geq M_j+1$. This is the probability that someone of its members will decide to quit the platform when its members are at the critical number M_j+1. Thus this probability equals

$$d_j = P\{X_j(t+\Delta t) = M_j \mid X_j(t) \geq M_j + 1\} = \binom{N}{M_j+1} P_j^{M_j+1}(1-P_j)^{N-M_j+1} \frac{(M_j+1)\mu_j \Delta t}{1-B(M_j;N,P_j)}.$$

Probability for party (platform) *j* not to exist (to be in a "sleep" state) equals to the measure of the chance for the model to spend in one of the states X_j=0, 1, …, M_j, and therefore, is given by the expression

$$P(X_j \leq M_j) = B(M_j; N, P_j)$$

with the P_j, determined by (1).

Further dynamic analysis will allow to determine the **duration of existence of a party**. Let us note that when we look at the Markovian process of the changes in the platform (party's) states, we may imaginary consider another, absorbing Markov chain whose absorbing states are these when X_j=0, 1, …, M_j, and the states $X_j=M_j+1$, M_j+2, …, N are transient. Then, the duration of the existence of party (platform) j on the political scene will be equal to the time the above described Markovian process spends in the sets of its transient states.

Denote by $B_{k_0,k_1,\cdots,k_r}$ the average time the process spends at the transition set of states if it starts at a state given by the left hand side of equation 2. Taking into account the infinitesimal intensities of the process interstate transitions, and the respective average spending times immediately before a transition and the respective sojourn times in the same state and in the transient sets after the transition, we arrive to the following system of equations for the sojourn times $B_{k_0,k_1,\cdots,k_r}$, valid for $k_j > M_j+1$, for all j = 1, …, r**:**

$$B_{k_0,k_1,\cdots,k_r} = \sum_{i=1}^{r} \sum_{k_i=M_i+1}^{N} k_i \mu_i \frac{B_{k_0,k_1,\cdots,k_i-1,\cdots,k_r}}{k_0 \sum_{l=1}^{r} \lambda_l + \sum_{l=1}^{r} k_l \mu_l} + \frac{1}{k_0 \sum_{l=1}^{r} \lambda_l + \sum_{l=1}^{r} k_l \mu_l} +$$

$$+ \sum_{i=1}^{r} \sum_{k_i=M_i+1}^{N} (k_0+1)\lambda_i \frac{B_{k_0+1,k_1,\cdots,k_i,\cdots,k_r}}{k_0 \sum_{l=1}^{r} \lambda_l + \sum_{l=1}^{r} k_l \mu_l}. \quad (4)$$

On the boundary layers where some, or several $k_i = M_i + 1$ the equations (4) are still valid with $B_{k_0,k_1,\cdots,k_i-1,\cdots,k_r} = 0$. Also is true that $B_{k_0,k_1,\cdots,k_r} = 0$ if $k_j \leq M_j$, for any j = 1, …, r. System (4) always has a solution since the chain is absorbing. This solution can be found by the method of inverse matrices.

The expected life time of party *j* is then given by the expression

$$B_j = \sum_{k_j=M_j+1}^{N} \binom{N}{k_j} P_j^{k_j}(1-P_j)^{N-k_j} \sum_{\substack{i=0 \\ i\neq j}}^{r} \sum_{\substack{k_j=M_j+1 \\ i\neq j}}^{N} \mathrm{P}(\mathrm{X}_0 = \mathrm{k}_0, \mathrm{X}_1 = \mathrm{k}_1, \ldots, \mathrm{X}_r = \mathrm{k}_r) B_{k_0,k_1,\cdots,k_r} \quad (5)$$

The use of the introduced parameters in respective statistical data may allow practical estimation of these parameters and give the answer to various interesting statistical questions.

3 CHARACTERISTICS OF THE ELECTION MODEL

The party model of the previous section was used in [5] to create respective election model. It is assumed the following configurations before the elections. There are C coalitions registered for the peoples vote. A coalition may consist of one ore several parties. The "Zero" party is made by those who are not members of any party. They vote with probability p, or not vote with probability q=1-p. A vote goes to coalition C_j with probability Q_j proportional to the intensities to join the parties with indices $i_1,\ldots,i_{C_j}$ from this coalition, i.e.

$$Q_j = \frac{\lambda_{i_1} + \cdots + \lambda_{i_{C_j}}}{\lambda_1 + \lambda_2 + \cdots + \lambda_r}, \quad j=1,\ \ldots,\ C. \tag{6}$$

Party members vote for the coalition where their party belongs. Under some additional assumptions it is found that the random vector $(T_0, T_1, \ldots T_C)$ of votes given to the coalitions in the elections has multinomial distribution with probability generating function of the voting results $(T_0, T_1, \ldots T_C)$ given by the expression

$$T(z_0, z_1, \ldots, z_C) = \left((1-p)P_0 z_0 + \sum_{j=1}^{C} (pP_0 Q_j + P_{i_1} + \cdots P_{i_{C_j}}) z_j\right)^N, \tag{7}$$

Here P_i, Q_j are given by expressions in (1) and (6), and T_0 is the number of those who do not vote. Hence, the random number of voters for coalition C_j has Binomial distribution with parameters N and $\alpha_j = pP_0 Q_j + P_{i_1} + \cdots + Pi_{C_j}$. Roughly speaking, the number of votes for a coalition equals to the sum of its party members and the votes of non-party people who may vote for this coalition.

Knowledge of the distribution of the random variables Tj allows calculation of various interesting explicit and/or average characteristics related to the specific electoral laws. For instance:

The average number of votes for coalition T_j equals $N\alpha_j$.

The probability that coalition T_j does not survive the requirements to pass the minimum γ% percentage barrier is given by the expression

$$P\{T_j < \gamma\ V/100) = B(\frac{\gamma\ V}{100};\ N,\ \alpha_j). \tag{8}$$

Here $V=pE(X_0)$ is the expected number of voters in the elections, X_0 is the number of members of "the zero party" with the marginal distribution as given by equation (2), and $B(.)$ is notation for the Binomial c.d.f. The complement to 1 of the probability in (8) is the probability for this coalition to survive the elections. Since here the work is mostly with binomially distributed r.v.'s, all the conventional approximations to the Binomial distribution (Poisson, normal) are legitimate tools to simplify the calculations. We omit these details.

The provability that coalition C_j is the winner in the elections will be determined from the requirement

$$T_j = \max\{T_1, \ldots, T_C). \tag{9}$$

When take into account (7) we easily find that the probability (9) to be fulfilled equals

$$\beta_j = \sum_{\substack{k_j > k_1, \cdots, k_C \\ k_0 + k_1 + \cdots + k_C = N}}^{N} \frac{N!}{k_0! k_1! \cdots k_C!} [(1-p)P_p]^{k_0} \alpha_1^{k_1} \ldots \alpha_C^{k_C}. \tag{10}$$

It is intuitively clear that the coalition with the highest value of the probability α_j is the expected winner. However, (10) allows to evaluate probabilities for any other coalition to win.

One last remark here is that the results in votes are negatively correlated random variables. The correlation matrix among the coordinates of the random vector $(T_0, T_1, \ldots T_C)$ is given by the entries as shown in the case for the counts X_i and X_j of the parties in the previous section, where instead the probabilities P_j the quantities α_j must be used.

In many cases, the votes from the losing coalitions are distributed proportionally between the winning coalitions, according to the numbers of actual votes approved for the winners. These potential extra votes may increase the elected members from a coalition. Respective conditional distributions are also available based on the described here model. This will be subject of another study.

4 CHARACTERISTICS OF THE GOVERNING MODEL

The process of formation of governing coalition between the winners in the election has a complex structure. Here we enter in the complexity of issues well described in chapters 3 and 4 by J. March (1994). It is a challenging task to make mathematical models based on these descriptions of decision making. Usually, the coalition with major sits in the National Assembly (NA) takes responsibility to form a governing coalition. It negotiates with groups with smaller sits in the NA, until gets more than K supporters among the sits in the NA. Then the government is ready to be formed. This government survives as long as it is supported by at least K members in the NA. Assuming that the government is supported by G coalitions, with $M_1,\ldots,M_G$ representatives in the NA, we modeled the government as a system with subsystems connected in series, with variable number of functioning components in each subsystem (the coalitions in the government). With the notation $F_j(t)$ for the distribution function of the time of random duration that a member of the NA from coalition j keeps his/her loyalty to the governmental formation (and decisions assumed independent between the members of the NA), we derive in [5] that the government survives time of duration t is given by the rule

$$P\{L > t\} = \prod_{j=1}^{G} \sum_{i=K_j}^{M_j} \binom{M_j}{i} \left(1 - F_j(t)\right)^i F^{M_j - i}(t). \tag{11}$$

Here K is a number showing the minimum number of members required for a coalition to exist as an entity in the NA, and L is the life time of this government.

If the next elections are scheduled after expiration of time of duration T, the average life time of this government will be presented by the quantity

$$\mu_T = E(L \mid T) = \int_0^T P\{L > t\}\ dt \tag{12}$$

However, this is politics, and the life is dynamically changed. There are times when the parties in opposition call for non-confidentiality vote against this particular government. The results from the vote are modeled by making use of the models from sections 2 and 3. Such votes are binary (Pro or Con where the votes "abstain" are actually in favor of no confidence). Each coalition may have its own probability p_j for a member to vote "Pro". The random number N_j of "pro"-votes in each coalition C_j is binomial of parameters (Y_j, p_j), where Y_j is the random number of surviving supporters in the j^{th} governing coalition. Therefore, the chance to survive a non-confidentiality vote at time t has probability

$$P(N_1 + \cdots N_C \geq K,\, L > t) = \sum_{j=1}^{C} \sum_{k_1 + \cdots k_C \geq K} \sum_{i=K_j}^{M_j} \binom{M_j}{i} \left(1 - F_j(t)\right)^i F^{M_j - i}(t) \binom{i}{k_j} p_j^{k_j} (1 - p_j)^{i - k_j} \tag{13}$$

If the calls from opposition for non-confidentiality vote for the government form a point process of certain kind, the survival probability of the government will decrease proportionally to the product of probabilities to survive each non- confidentiality vote. If just one no confidence vote is supposed to be induced with a uniform distribution within the assumed interval between elections, the probability, say *a,* for surviving it equals to the integral of the expression on the right hand side of (3). Then the expected life of this government will be evaluated by the expression

$$\mu(a) = a\mu_T + \frac{1}{2}(1-a)\mu_T = \frac{\mu_T}{2}(1+a).$$

Complications under other assumptions are evident, but not worthless to be discussed.

5 CONCLUSIONS

Political processes offer interesting area of applications of various mathematical modeling approaches and theories. We discuss probability models by keeping close to the processes of formation of political units and activities to their natural components. The uncertain elements are naturally included into specific probabilistic relationships. The obtained analytical results produce promising particular characteristics, and offer a lot of field for discussions, statistical considerations and interesting applications.

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