SEQUENTIAL ALGORITHMS OF GRAPH NODES FACTORIZATION

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In this paper algorithms of a factorization of graph nodes which are used in a processing of data connected with different extreme situations are constructed. We assume that information about a new node and incident edges arrives at each step sequentially. In non oriented graph we suppose that two nodes are equivalent if there is a way between them. In oriented graph we assume that two nodes are equivalent if there is a cycle which includes these nodes. In this paper algorithms of such factorization are constructed. These algorithms need $O(n^2)$ arithmetic operations where n is a number of graph nodes.

1. Factorization of nodes in non oriented graph

Two nodes of a non oriented graph belong to a same connectivity component (are equivalent [1,§3]), if in the graph there is a way which connects these nodes.

1. On the step 1 we put that there is the single node 1 and the number of connectivity components equals $q=1$ and the connectivity component $K_1=\{1\}$.
2. Assume that on the step n there are nodes $1,...,n;$ and $q \le n$ connectivity components $K_1,...,K_q$

$$K_i \cap K_j = \varnothing,\ i \ne j,\ \bigcup_{i=1}^{q} K_i = \{1,...,n\}.$$

3. On the step $n+1$ we receive an information about $a_j, 1 \le j \le n$: $a_j = 1$, if the nodes $n+1, j$ are connected by (non oriented) edge, in opposite case $a_j = 0$. Calculate $c_j = \vee_{j \in K_i} a_j$, $1 \le i \le q$, using n arithmetic operations.
4. Define the indexes set $I = \{i : c_i = 1\} = \{i_0, i_1, ..., i_r\}$, $1 < i_0 < i_1 < ... < i_r \le q$, using no more than n arithmetic operations.
5. From the sequence $\{i_0+1,...,i_1-1,i_1+1,...,i_2-1,i_2+1,...,i_r-1,i_r+1,...,q\}$ with $q-i_0$ integers remove the numbers $i_1,...,i_r$ and transform it into the sequence with $q-i_0-r$ integers $\{i_0+1,...,i_1-1,i_1+1,...,i_2-1,i_2+1,...,i_r-1,i_r+1,...,q\} = \{l(i_0+1),...,l(q-r)\}$ using no more than n arithmetic operations.
6. Assume that $I \notin \varnothing$ then put $K_i := K_i$, $1 \le i \le i_0$, $K_0 = \{n+1\} \cup \left[\bigcup_{i \in I} K_i\right]$, $K_i := K_{l(i)}$, $i_0 + 1 \le i \le q - r$, $q := q - r$, using no more than n arithmetic operations.

If $I \in \varnothing$ then put $K_{q+1} := \{n+1\}$, $q := q+1$ using 2 arithmetic operations. Consequently the transformation of the connectivity components set on the step $n+1$ needs no more then $4n$ aarithmetic operations for the non oriented graph. So the calculation complexity of suggested algorithm on n steps is no more then $2n^2$ aarithmetic operations.

2. Factorization of nodes in oriented graph

Say that two nodes of an oriented graph are equivalent [1,§3] if there is a cycle which contains them. It is obvious that this binary relation is reflexive, symmetric and transitive and so is a relation of equivalence on the set of nodes of the oriented graph. On the set of equivalence classes introduce the following binary relation: $p \succeq q$ if in the initial graph there is a way from any node of the class p to any node of the class q. It is clear that this binary relation is reflexive, transitive and anti symmetric and so is a relation of a partial order. [1,§4]. Then we contrast to this relation a zero-one matrix in which in the cell (p,q) there is 1 if $p \succeq q$ and 0 in other cases.

We suggest the following sequential algorithm of the oriented graph nodes factorization. On the first step we have a single node which is in the single cluster. The matrix a of the partial order here consists of the single unit. Assume that on the n-the step there is the set I of clusters and the one-zero matrix a which characterizes the relation of the partial order $\succeq$ between them.

On the step $n+1$ new node $n+1$ and two edges appear. One of these edges runs into the cluster p and another runs from the cluster q into the node $n+1$. Then new clusters and the partial order matrix a are constructed as follows. Denote

$$K_p = \{k \in I_n : p \succeq k\}, \quad R_q = \{k \in I_n : k \succeq q\}, \tag{1}$$

$$A = K_p \bigcap R_q, A_1 = K_p \setminus A, \; A_2 = R_q \setminus A, \; B = I \setminus \left(A \bigcup A_1 \bigcup A_2 \right). \tag{2}$$

The new node $n+1$ and the clusters from the set A form new cluster $(n+1)$. The matrix a is divided into 16 rectangular boxes which are created by the cluster $n+1$ and the sets of clusters A_1, A_2, B. To describe these boxes introduce auxiliary designations $\mathcal{A}$ of a sub matrix which coincides with analogous sub matrix in previous version of a, $\mathcal{E}$ is a sub matrix consisting of units and $\mathcal{O}$ is a sub matrix consisting of zeros. In a cell "$\ldots \to \ldots$" of a left set "..." is a vector-column and right set "..." is
a vector-string consisting of clusters. The matrix a contains the following cells:

1. $(n+1) \to (n+1)$ which has the form $\mathcal{E}$ because $(n+1)$ is the cluster;
2. $(n+1) \to A_1$ has the form $\mathcal{E}$ by a definition of the clusters set A_1;
3. $A_2 \to (n+1)$ has the form $\mathcal{E}$ by a definition of the clusters set A_2;
4. $A_2 \to A_1$ has the form $\mathcal{E}$ because in the graph on the step n+1 there is a way from a cluster $j \in A_2$ to a cluster $i \in A_1$ which passes through the node $n+1$;
5. $A_1 \to (n+1)$ has the form $\mathcal{O}$ because in opposite case a part of clusters from A_1 joins the set A;
6. $(n+1) \to A_2$ has the form $\mathcal{O}$ because in opposite case a part of clusters from the set A_2 joins the set A;
7. $A_1 \to A_2$ has the form $\mathcal{O}$ because in opposite case a part of clusters from the sets A_1, A_2 joins the set A_0;
8. $(n+1) \to B$ has the form $\mathcal{O}$ because in opposite case a part of clusters from the set B joins the set A_1;
9. $B \to (n+1)$ has the form $\mathcal{O}$ because in opposite case a part of clusters from the set B joins the set A_2;
10. $A_1 \to B$ has the form $\mathcal{O}$ because in opposite case a part of clusters from the set B joins the set A_1;
11. $B \to A_2$ has the form $\mathcal{O}$ because in opposite case a part of clusters from the set B joins the set A_2;
12. $B \to B$ has the form $\mathcal{A}$ because in opposite case a way $i \to k_1 \to k_2 \to j \to (n+1) \to i$, $i \in A_1$, $j \in A_2$, $k_1 \in B, k_2 \in B$ appears and so the clusters $k_1, k_2 \in B$;

13. $A_1 \to A_1$ has the form $\mathcal{A}$ because in opposite case a way $i \to k \to (n+1), i \in A_1, k \in B, j \in A_2$ appears and so $k \in A_1 \cap A_2$;
14. $A_2 \to A_2$ has the form $\mathcal{A}$ because in opposite case a way $i \to k \to (n+1)$, $i \in A_1$, $k \in B$, $j \in A_2$ appears and so $k \in A_1 \cap A_2$;
15. $B \to A_1$ has the form $\mathcal{A}$ because in opposite case a way $k \to j \to (n+1) \to i$, $i \in A_1$, $k \in B$, $j \in A_2$ appears and so $k \in A_2$;
16. $A_2 \to B$ has the form $\mathcal{A}$ because in opposite case a way $k \to i \to (n+1) \to j$, $i \in A_1$, $k \in B$, $j \in A_2$ appears and so $k \in A_1$.

The transformation of the matrix a on the transition from the step n to the step $n+1$ needs the sets A, A_1, A_2, B definition by the formulas (1), (2) and demands $O(n)$ arithmetic operations and $O(n^2)$ operations of an assignment. Consequently a calculation of the connectivity components and of the partial order matrix up to the step n demands $O(n^2)$ arithmetic operations and $O(n^3)$ assignment operations.

Remark 1. This solution remains correct if the single edge from the node $n+1$ is replaced by a few edges and the single edge to the node $n+1$ is replaced by a few edges also. Numbers of these additional edges is no more than some finite m which does not depend on n. Denote by P the set of clusters in which new edges come into and by Q the set of clusters from which new edges come into the node $n+1$. Then the sets A, A_1, A_2, B are defined as follows. Assume that

$$K_p = \{k \in I_n : p \succeq k\},\ p \in P,\ R_q = \{k \in I_n : k \succeq q\},\ q \in Q,$$
$$K' = \bigcup_{p \in P} K_p,\ R = \bigcup_{q \in Q} R_q,\ A = K' \cap R'.$$

New node $n+1$ and clusters from the set A create new cluster which we denote by $(n+1)$ and put $A_1 := K' \setminus A,\ A_2 := R' \setminus A,\ B := I \setminus (A \cup A_1 \cup A_2), I := I \cup n+1.$

Describe now an algorithm of a construction of the clusters set and the matrix of partial order on this set in general case. On the step 1 there is the single node 1, the set of clusters $K = \{1\}$ and the matrix of the partial order characterized by the formula $a(1,1) = 1$. Assume that on the step n there is the set of clusters K. These clusters create a partitioning of the set $I = \{1,...,n\}$ into non intersected subsets,

Each cluster $k \in K$ is indexed by a maximal number of its nodes.

On the set K we have unit-zero matrix $a = \|a(p,q)\|_{p,q \in K}$ which characterizes the relation of partial order „$\succeq$": $a(p,q) = 1$ if $p \succeq k$ and $a(p,q) = 0$ in opposite case.
From the new node $n+1$ no more than m edges exit into the set $\mathcal{P} \subseteq I$ of nodes and no more than come into the node $n+1$ from the set $\mathcal{Q} \subseteq I$ of n nodes. Denote by P, Q the sets of clusters generated by nodes of the sets $\mathcal{P}, \mathcal{Q}$ and define

$$K_p = \{k \in K : a(p,k) = 1\},\ p \in P;\ R_q = \{k \in K : a(k,q) = 1\},\ q \in Q;$$

$$K' = \bigcup_{p \in P} K_p ,\ R = \bigcup_{q \in Q} R_q ,\ A = K' \bigcap R' .$$

The new node $n+1$ and the clusters from the set A create new cluster $(n+1)$, put
$A_1 := K' \setminus A,\ A_2 := R' \setminus A,\ B := K \setminus \left(A \bigcup A_1 \bigcup A_2 \right),\quad I := I \bigcup (n+1)$.
Then a calculation of the matrix a on the step $n+1$ satisfies the formulas
$a(n+1, n+1) = 1,\ a(n+1, A_1) := \mathcal{E},\ a(A_2, n+1) := \mathcal{E},\ a(A_2, A_1) := \mathcal{E},$
$a(A_1, n+1) := O,\ a(n+1, A_2) = O,\ a(A_1, A_2) = O,\ a(B, B) = O,$
$a(B, n+1) = O,\ \ a(A_1, B) = O,\ a(B, A_2) = O.$

References

1. Kurosh A.G. Lectures on general algebra. Moscow: Phizmatlit. 1962.