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REGRESSION EQUATIONS FOR MARSHALL-OLKIN TRIVARIATE EXPONENTIAL DISTRIBUTION AND RELIABILITY MEASURES OF RELATED THREE-UNIT SYSTEMS

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ABSTRACT

Marshall and Olkin(1967) proposed a multivariate exponential distribution and derived some properties including the moment generating function. Proschan and Sullo(1976) provide the probability density function which involves some tedious notations. In this paper, we provide an explicit expression for the probability density function in the trivariate case and derive the conditional distributions, regression equations and the moment generating function. By considering four three unit systems with trivariate exponential failure time distribution, we derive the reliability measures of the systems.

Key words: Conditional distribution, moment generating function, multiple regression, performance measure, trivariate exponential.

1 Introduction

Marshall-Olkin(1967) proposed a multivariate exponential distribution as a model arising out of Poisson shocks. The distribution is not absolutely continuous and so the distribution received considerable attention among the researchers. Bemis et al(1972) provide a probability distribution which is not absolutely continuous with respect to the Lebesgue measure in R^2. Inference for bivariate exponential distribution was discussed by Arnold(1968), Bemis et al(1972), Bhattacharya and Johnson(1973) among others. For the trivariate case with equal marginals, Samanta(1983) discussed the problem of testing independence. Proschan and Sullo(1976) discussed parameter estimation for multivariate case and proposed the probability density function(pdf) involving tedious notations.

The aim of this paper is to provide an explicit expression for the pdf in the trivariate case and derive the conditional distributions, regression equations and the moment generating function. Further, by considering four three unit systems with trivariate exponential failure time distribution, we derive the reliability measures of the systems. Section 2 proposes the pdf whose bivariate marginal is the one given by Bemis et al(1972). Section 3 derives the conditional distributions, both univariate and bivariate. Further the multiple regression equations are obtained and shown that they are not linear. The moment generating function is obtained in Section 4. Finally in Section 5, three component standby, parallel, series and relay systems with trivariate exponential failure times are discussed and the performance measures are obtained.

2 The probability density function

The survival function of the Marshall-Olkin trivariate exponential distribution (MOTVE) is of the form

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$$\overline{F}(x_1, x_2, x_3) = \exp\left\{-\sum_{i=1}^{3}\lambda_i x_i - \sum_{i<}\sum_{j}\lambda_{ij}\left(x_i \vee x_j\right) - \lambda_{123}\left(x_1 \vee x_2 \vee x_3\right)\right\},\ x_1, x_2, x_3 \geq 0.$$

Here $\lambda_1, \lambda_2, \lambda_3 > 0$, $\lambda_{12}, \lambda_{13}, \lambda_{23}, \lambda_{123} \geq 0$ and $x_1 \vee x_2 \vee .x_3 = \max(x_1, x_2, x_3)$.

Let $\lambda_1^* = \lambda_1 + \lambda_{12} + \lambda_{13} + \lambda_{123}$, $\lambda_2^* = \lambda_2 + \lambda_{12} + \lambda_{23} + \lambda_{123}$, $\lambda_3^* = \lambda_3 + \lambda_{13} + \lambda_{23} + \lambda_{123}$, $\lambda_{ji} = \lambda_{ij}$, $i < j$ and $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_{12} + \lambda_{13} + \lambda_{23} + \lambda_{123}$.

We propose the following pdf for the MOTVE distribution

$$f(x_1, x_2, x_3) = \begin{cases} \lambda_i(\lambda_j + \lambda_{ij})\lambda_k^* \exp\left\{-\lambda_i x_i - \left(\lambda_j + \lambda_{ij}\right)x_j - \lambda_k^* x_k\right\}, & x_i < x_j < x_k \\ \lambda_i(\lambda_{jk} + \lambda_{123})\exp\left\{-\lambda_i x_i - \left(\lambda - \lambda_i\right)x_j\right\}, & x_i < x_j = x_k \\ \lambda_{jk}\lambda_i^* \exp\left\{-(\lambda_j + \lambda_k + \lambda_{jk})x_j - \lambda_i^* x_i\right\}, & x_j = x_k < x_i \\ \lambda_{123}\exp\left(-\lambda x_1\right), & x_1 = x_2 = x_3 \end{cases} \quad \ldots(2.1)$$

Remark 2.1 It can be verified that the total integral is one. While integrating the pdf, there are 6 triple integrals, 6 double integrals and one single integral. It may be noted that we have the same number of cases as in Samanta(1983) and (2.1) reduces to the one given in the paper for equal marginals situation.

2.1 Bivariate marginals

It is known that the bivariate marginals are bivariate exponential. We use the pdf approach to derive the bivariate marginals.

Theorem 2.1 The pdf of (X_1, X_2) at (x_1, x_2) is given by

$$f_{12}(x_1, x_2) = \begin{cases} \left(\lambda_1 + \lambda_{13}\right)\lambda_2^* e\mathrm{xp}\left\{-\left(\lambda_1 + \lambda_{13}\right)x_1 - \lambda_2^* x_2\right\}, & x_1 < x_2 \\ \left(\lambda_2 + \lambda_{23}\right)\lambda_1^* \exp\left\{-\left(\lambda_2 + \lambda_{23}\right)x_2 - \lambda_1^* x_1\right\}, & x_1 > x_2 \\ \left(\lambda_{12} + \lambda_{123}\right)\exp\left\{-\left(\lambda - \lambda_3\right)x_1\right\}, & x_1 = x_2. \end{cases}$$

Proof The pdf of (X_1, X_2) at (x_1, x_2) is $f_{12}(x_1, x_2) = \int_0^\infty f(x_1, x_2, x_3)\,dx_3$.

Three cases arise according as $x_1 < x_2$, $x_1 > x_2$ and $x_1 = x_2$.

Case 1: $x_1 < x_2$

In this case

$$f(x_1, x_2, x_3) = \begin{cases} \lambda_3\left(\lambda_1 + \lambda_{13}\right)\lambda_2^* \exp\left\{-\lambda_3 x_3 - \left(\lambda_1 + \lambda_{13}\right)x_1 - \lambda_2^* x_2\right\}, & x_3 < x_1 < x_2 \\ \lambda_1\left(\lambda_3 + \lambda_{13}\right)\lambda_2^* \exp\left\{-\lambda_1 x_1 - \left(\lambda_3 + \lambda_{13}\right)x_3 - \lambda_2^* x_2\right\}, & x_1 < x_3 < x_2 \\ \lambda_1\left(\lambda_2 + \lambda_{12}\right)\lambda_3^* \exp\left\{-\lambda_1 x_1 - \left(\lambda_2 + \lambda_{12}\right)x_2 - \lambda_3^* x_3\right\}, & x_1 < x_2 < x_3 \\ \lambda_1\left(\lambda_{23} + \lambda_{123}\right)\exp\left\{-\lambda_1 x_1 - \left(\lambda + \lambda_1\right)x_2\right\}, & x_1 < x_2 = x_3 \\ \lambda_{13}\lambda_2^* \exp\left\{-\left(\lambda_1 + \lambda_3 + \lambda_{13}\right)x_1 - \lambda_2^* x_2\right\}, & x_2 > x_1 = x_3 \end{cases}$$

Thus

$$f_{12}(x_1,x_2)=\int_0^{x_1} f(x_1,x_2,x_3)dx_3+\int_{x_1}^{x_2} f(x_1,x_2,x_3)dx_3+\int_{x_2}^{\infty} f(x_1,x_2,x_3)dx_3+f(x_1,x_2,x_1)+f(x_1,x_2,x_2)$$

$$=\int_0^{x_1}\lambda_3(\lambda_1+\lambda_{13})\lambda_2^*\exp\{-(\lambda_1+\lambda_{13})x_1-\lambda_2^*x_2-\lambda_3x_3\}dx_3+$$

$$\int_{x_1}^{x_2}\lambda_1(\lambda_3+\lambda_{13})\lambda_2^*\exp\{-\lambda_1x_1-\lambda_2^*x_2-(\lambda_3+\lambda_{13})x_3\}dx_3+$$

$$\int_{x_2}^{\infty}\lambda_1(\lambda_2+\lambda_{12})\lambda_3^*\exp\{-\lambda_1x_1-(\lambda_2+\lambda_{12})x_2-\lambda_3^*x_3\}dx_3+$$

$$\lambda_{13}\lambda_2^*\exp\{-(\lambda_1+\lambda_3+\lambda_{13})x_1-\lambda_2^*x_2\}+\lambda_1(\lambda_{23}+\lambda_{123})\exp\{-\lambda_1x_1-(\lambda-\lambda_1)x_2\}$$

$$=(\lambda_1+\lambda_{13})\lambda_2^*\exp\{-(\lambda_1+\lambda_{13})x_1-\lambda_2^*x_2\}+(1-\exp\{-\lambda_3x_1\})$$

$$\lambda_1\lambda_2^*\exp\{-\lambda_1x_1-\lambda_2^*x_2\}\left(\exp\{-(\lambda_3+\lambda_{13})x_1-(\lambda_1+\lambda_{13})x_2\}\right)$$

$$+\lambda_1(\lambda_2+\lambda_{12})\exp\{-\lambda_1x_1-(\lambda_2+\lambda_{12})x_2\}.\exp\{-\lambda_3^*x_2\}+$$

$$\lambda_{13}\lambda_2^*\exp\{-(\lambda_1+\lambda_3+\lambda_{13})x_1-\lambda_2^*x_2\}+\lambda_1(\lambda_{23}+\lambda_{123})\exp\{-\lambda_1x_1-(\lambda-\lambda_1)x_2\}$$

$$=(\lambda_1+\lambda_{13})\lambda_2^*\exp\{-(\lambda_1+\lambda_{13})x_1-\lambda_2^*x_2\}.$$

Case 2: $x_1>x_2$

As in Case 1, we can show that

$$f_{12}(x_1,x_2)=(\lambda_2+\lambda_{23})\lambda_1^*\exp\{-(\lambda_2+\lambda_{23})x_2-\lambda_1^*x_1\}.$$

Case 3: $x_1=x_2$

In this case

$$f(x_1,x_2,x_3)=\begin{cases}\lambda_3(\lambda_{12}+\lambda_{123})\exp\{-\lambda_3x_3-(\lambda-\lambda_3)x_1\}, & x_3<x_1=x_2\\ \lambda_{12}\lambda_3^*\exp\{-(\lambda_1+\lambda_2+\lambda_{12})x_1-\lambda_3^*x_3\}, & x_3>x_1=x_2\\ \lambda_{123}\,\mathrm{e}xp(-\lambda x_1), & x_1=x_2=x_3.\end{cases}$$

$$\textit{Thus}\quad f_{12}(x_1,x_2)=\int_0^{x_1}\lambda_3(\lambda_{12}+\lambda_{123})\exp\{-\lambda_3x_3-(\lambda-\lambda_3)x_1\}dx_3$$

$$+\int_{x_1}^{\infty}\lambda_{12}\lambda_3^*\exp\{-(\lambda_1+\lambda_2+\lambda_{12})x_1-\lambda_3^*x_3\}+\lambda_{123}\exp(-\lambda x_1)$$

$$=(\lambda_{12}+\lambda_{123})\exp\{-(\lambda-\lambda_3)x_1\}$$

Hence the theorem.

Remark 2.2 Thus $(X_1,X_2)\sim BVE(\lambda_1+\lambda_{13},\lambda_2+\lambda_{23},\lambda_{12}+\lambda_{123})$, in view of Bemis et al(1972). Similarly, it can be shown that the other two bivariate distributions are also bivariate exponential.

3 Conditional distributions and regressions

In this section, we determine the univariate and bivariate conditional distributions.

3.1 Univariate conditional distributions

The conditional pdf of X_1 given $(X_2, X_3) = (x_2, x_3)$ is

$$f_{1.23}(x_1|x_2,x_3) = \frac{f(x_1,x_2,x_3)}{f_{23}(x_2,x_3)},\ 0 < x_1 < \infty .$$

Note that $f(x_1,x_2,x_3)$ is given in (2.1) and

$$f_{23}(x_2,x_3) = \begin{cases} (\lambda_2+\lambda_{12})\lambda_3^* \exp\{-(\lambda_2+\lambda_{12})x_2 - \lambda_3^* x_3\}, & x_2 < x_3 \\ (\lambda_3+\lambda_{13})\lambda_2^* \exp\{-\lambda_2^* x_2 - (\lambda_3+\lambda_{13})x_3\}, & x_2 > x_3 \\ (\lambda_{23}+\lambda_{123})\exp\{-(\lambda-\lambda_1)x_2\}, & x_2 = x_3. \end{cases}$$

Three cases arise according as $x_2 < x_3, x_2 > x_3, x_2 = x_3$.

Case 1: $x_2 < x_3$

$$\begin{aligned} f_{1.23}(x_1|x_2,x_3) = {} & \lambda_1 \exp(-\lambda_1 x_1), && x_1 < x_2 \\ & \frac{\lambda_2(\lambda_1+\lambda_{12})}{(\lambda_2+\lambda_{12})}\exp\{-(\lambda_1+\lambda_{12})x_1 + \lambda_{12}x_2\}, && x_2 < x_1 < x_3 \\ & \frac{\lambda_2(\lambda_3+\lambda_{23})\lambda_1^*}{(\lambda_2+\lambda_{12})\lambda_3^*}\exp\{-\lambda_1^* x_1 + \lambda_{12}x_2 + (\lambda_{13}+\lambda_{123})x_3\}, && x_3 < x_1 \\ & \frac{\lambda_{12}}{(\lambda_2+\lambda_{12})}\exp\{-\lambda_1 x_2\}, && x_1 = x_2 \\ & \frac{\lambda_2(\lambda_{13}+\lambda_{123})}{(\lambda_2+\lambda_{12})\lambda_3^*}\exp\{-(\lambda_1+\lambda_{12})x_3 + \lambda_{12}x_2\}, && x_1 = x_3. \end{aligned}$$

Case 2: $x_2 > x_3$

$$\begin{aligned} f_{1.23}(x_1|x_2,x_3) = {} & \lambda_1 \exp\{-\lambda_1 x_1\}, && x_1 < x_3 \\ & \frac{\lambda_3(\lambda_1+\lambda_{13})}{(\lambda_3+\lambda_{13})}\exp\{-(\lambda_1+\lambda_{13})x_1 + \lambda_{13}x_3\}, && x_3 < x_1 < x_2 \\ & \frac{\lambda_3(\lambda_2+\lambda_{23})\lambda_1^*}{(\lambda_3+\lambda_{13})\lambda_2^*}\exp\{-\lambda_1^* x_1 + \lambda_{13}x_3 + (\lambda_{12}+\lambda_{123})x_2\}, && x_2 < x_1 \\ & \frac{\lambda_{13}}{(\lambda_3+\lambda_{13})}\exp(-\lambda_1 x_3), && x_1 = x_3 \\ & \frac{\lambda_3(\lambda_{12}+\lambda_{123})}{(\lambda_3+\lambda_{13})\lambda_2^*}\exp\{-(\lambda_1+\lambda_{13})x_2 + \lambda_{13}x_3\}, && x_1 = x_2 . \end{aligned}$$

Case 3: $x_2 = x_3$

$$\begin{aligned} f_{1.23}(x_1|x_2,x_3) = {} & \lambda_1 \exp(-\lambda_1 x_1), && x_1 < x_2 \\ & \frac{\lambda_{23}\lambda_1^*}{(\lambda_{23}+\lambda_{123})}\exp\{-\lambda_1^* x_1 + (\lambda_{12}+\lambda_{13}+\lambda_{123})x_2\}, && x_2 < x_1 \\ & \frac{\lambda_{123}}{(\lambda_{23}+\lambda_{123})}\exp(-\lambda_1 x_2), && x_1 = x_2 . \end{aligned}$$

Remark 3.1 Similarly, one can find the other two univariate conditional distributions.

Theorem 3.1 The regression equation of X_1 on (X_2, X_3) is

$$X_1 = \begin{cases} \dfrac{1}{\lambda_1} + \left\{ \dfrac{\lambda_3}{(\lambda_1+\lambda_{13})(\lambda_3+\lambda_{13})} - \dfrac{1}{\lambda_1} \right\} \exp(-\lambda_1 X_3) \\ + \left\{ \dfrac{\lambda_3(\lambda_2+\lambda_{23})}{(\lambda_3+\lambda_{13})\lambda_1^*\lambda_2^*} - \dfrac{\lambda_3}{(\lambda_1+\lambda_{13})(\lambda_3+\lambda_{13})} \right\} \exp\{\lambda_{13}X_3 - (\lambda_1+\lambda_{13})X_2\}, & X_2 > X_3 \\ \dfrac{1}{\lambda_1} + \left\{ \dfrac{\lambda_2}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})} - \dfrac{1}{\lambda_1} \right\} \exp(-\lambda_1 X_2) \\ + \left\{ \dfrac{\lambda_2(\lambda_3+\lambda_{23})}{(\lambda_2+\lambda_{12})\lambda_1^*\lambda_3^*} - \dfrac{\lambda_2}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})} \right\} \exp\{\lambda_{12}X_2 - (\lambda_1+\lambda_{12})X_3\}, & X_2 < X_3 \\ \dfrac{1}{\lambda_1} + \left\{ \dfrac{\lambda_{23}}{(\lambda_{23}+\lambda_{123})\lambda_1^*} - \dfrac{1}{\lambda_1} \right\} \exp(-\lambda_1 X_2), & X_2 = X_3. \end{cases}$$

Proof For $x_2 > x_3$,

$$E[X_1|(X_2,X_3)=(x_2,x_3)] = \int_0^{x_3} x_1\lambda_1 \exp\{-\lambda_1 x_1\} dx_1 + \int_{x_3}^{x_2} x_1 \frac{\lambda_3(\lambda_1+\lambda_{13})}{(\lambda_3+\lambda_{13})} \exp\{\lambda_{13}x_3 - (\lambda_1+\lambda_{13})x_1\} dx_1 +$$

$$\int_{x_2}^{\infty} x_1 \frac{\lambda_3(\lambda_2+\lambda_{23})\lambda_1^*}{(\lambda_3+\lambda_{13})\lambda_2^*} \exp\{-\lambda_1^* x_1 + (\lambda_{12}+\lambda_{123})x_2 + \lambda_{13}x_3\} dx_1$$

$$+ x_3 \frac{\lambda_{13}}{(\lambda_3+\lambda_{13})} \exp(-\lambda_1 x_3) + x_2 \frac{\lambda_3(\lambda_{12}+\lambda_{123})}{(\lambda_3+\lambda_{13})\lambda_2^*} \exp\{-(\lambda_1+\lambda_{13})x_2\}$$

$$= \frac{1}{\lambda_1} + \left\{ \frac{\lambda_3}{(\lambda_1+\lambda_{13})(\lambda_3+\lambda_{13})} - \frac{1}{\lambda_1} \right\} \exp(-\lambda_1 x_3) +$$

$$\left\{ \frac{\lambda_3(\lambda_2+\lambda_{23})}{(\lambda_3+\lambda_{13})\lambda_2^*\lambda_1^*} - \frac{\lambda_3}{(\lambda_1+\lambda_{13})(\lambda_3+\lambda_{13})} \right\} \exp\{-(\lambda_1+\lambda_{13})x_2 + \lambda_{13}x_3\}$$

Similarly one can derive the regression equation for $x_2 < x_3$.

For $x_2 = x_3$,

$$E[X_1|(X_2,X_3)=(x_2,x_3)] = \int_0^{x_3} x_1\lambda_1 \exp\{-\lambda_1 x_1\} dx_1 +$$

$$x_2 \frac{\lambda_{123}}{(\lambda_{23}+\lambda_{123})} \exp\{-\lambda_1 x_2\} \int_{x_2}^{\infty} x_1 \frac{\lambda_{23}\lambda_1^*}{(\lambda_{23}+\lambda_{123})} \exp\{-\lambda_1^* x_1 + (\lambda_{12}+\lambda_{13}+\lambda_{123})x_2\} dx_1$$

$$= \frac{1}{\lambda_1} + \left\{ \frac{\lambda_{23}}{(\lambda_{23}+\lambda_{123})\lambda_1^*} - \frac{1}{\lambda_1} \right\} \exp(-\lambda_1 x_2)$$

Hence the theorem.

Remark 3.2 The other two regression equations can be derived in a similar manner. Thus the multiple regression equations are non-linear.

3.2 Bivariate Conditional Distributions

Let us find the conditional distribution of (X_1, X_2) given $X_3 = x_3$.

Note that $f_3(x_3) = \lambda_3^* \exp(-\lambda_3^* x_3), x_3 > 0$

The conditional pdf of (X_1, X_2) given $X_3 = x_3$ is

$$f_{12.3}(x_1, x_2 | x_3) = \frac{f(x_1, x_2, x_3)}{f_3(x_3)},\ x_1, x_2 > 0.$$

Here three cases arise.

For $x_1 < x_2$

$$f_{12.3}(x_1, x_2 | x_3) = \begin{cases} \dfrac{\lambda_3(\lambda_1 + \lambda_{13})\lambda_2^*}{\lambda_3^*} \exp\{-(\lambda_1 + \lambda_{13})x_1 - \lambda_2^* x_2 + (\lambda_{12} + \lambda_{23} + \lambda_{123})x_3\}, & x_3 < x_1 \\ \dfrac{\lambda_1(\lambda_3 + \lambda_{13})\lambda_2^*}{\lambda_3^*} \exp\{-\lambda_1 x_1 - \lambda_2^* x_2 + (\lambda_{23} + \lambda_{123})x_3\}, & x_1 < x_3 < x_2 \\ \dfrac{\lambda_{13}\lambda_2^*}{\lambda_3^*} \exp\{-\lambda_2^* x_2 - (\lambda_1 - \lambda_{23} - \lambda_{123})x_3\}, & x_1 = x_3 \\ \dfrac{\lambda_1(\lambda_{23} + \lambda_{123})}{\lambda_3^*} \exp\{-\lambda_1 x_1 - (\lambda_2 + \lambda_{12})x_3\}, & x_2 = x_3. \end{cases}$$

For $x_1 > x_2$

$$f_{12.3}(x_1, x_2 | x_3) = \begin{cases} \dfrac{\lambda_3(\lambda_2 + \lambda_{23})\lambda_1^*}{\lambda_3^*} \exp\{-(\lambda_2 + \lambda_{23})x_2 - \lambda_1^* x_1 + (\lambda_{12} + \lambda_{13} + \lambda_{123})x_3\}, & x_3 < x_2 \\ \dfrac{\lambda_2(\lambda_3 + \lambda_{23})\lambda_1^*}{\lambda_3^*} \exp\{-\lambda_1^* x_1 - \lambda_2 x_2 + (\lambda_{13} + \lambda_{123})x_3\}, & x_2 < x_3 < x_1 \\ \lambda_2(\lambda_1 + \lambda_{12})\exp\{-(\lambda_1 + \lambda_{12})x_1 - \lambda_2 x_2\}, & x_1 < x_3 \\ \dfrac{\lambda_{23}\lambda_1^*}{\lambda_3^*} \exp\{-\lambda_1^* x_1 - (\lambda_2 - \lambda_{13} - \lambda_{123})x_3\}, & x_2 = x_3 \\ \dfrac{\lambda_2(\lambda_{13} + \lambda_{123})}{\lambda_3^*} \exp\{-\lambda_2 x_2 - (\lambda_1 + \lambda_{12})x_3\}, & x_1 = x_3 \end{cases}$$

For $x_1 = x_2$

$$f_{12.3}(x_1, x_2 | x_3) = \begin{cases} \dfrac{\lambda_3(\lambda_{12} + \lambda_{123})}{\lambda_3^*} \exp\{-(\lambda - \lambda_3)x_1 + (\lambda_{12} + \lambda_{23} + \lambda_{123})x_3\}, & x_3 < x_1 \\ \lambda_{12} \exp\{-(\lambda_1 + \lambda_2 + \lambda_{12})x_1\}, & x_1 < x_3 \\ \dfrac{\lambda_{122}}{\lambda_3^*} \exp\{-(\lambda_1 + \lambda_2 + \lambda_{12})x_3\}, & x_1 = x_3 \end{cases}$$

Remark 3.3 It can be verified with routine but tedious integration, that $f_{12.3}(x_1, x_2 | x_3)$ is a pdf. Similarly one can derive the other two bivariate conditional distributions.

4 Moment generating function

Theorem 4.1 The mgf of (X_1, X_2, X_3) at (t_1, t_2, t_3) is given by

$$M(t_1, t_2, t_3) = = \frac{\lambda_1 \lambda_2^* \lambda_3^*}{(\lambda_2^* - t_2)(\lambda_3^* - t_3)(\lambda - t_1 - t_2 - t_3)} + \frac{\lambda_1(\lambda_{23} + \lambda_{123})t_2 t_3}{(\lambda_2^* - t_2)(\lambda_3^* - t_3)(\lambda - t_1 - t_2 - t_3)(\lambda - \lambda_1 - t_2 - t_3)}$$
$$+ \frac{\lambda_2 \lambda_1^* \lambda_3^*}{(\lambda_1^* - t_1)(\lambda_3^* - t_3)(\lambda - t_1 - t_2 - t_3)} + \frac{\lambda_2(\lambda_{13} + \lambda_{123})t_1 t_3}{(\lambda_1^* - t_1)(\lambda_3^* - t_3)(\lambda - t_1 - t_2 - t_3)(\lambda - \lambda_2 - t_1 - t_3)}$$

$$+\frac{\lambda_3\lambda_1^*\lambda_2^*}{(\lambda_1^*-t_1)(\lambda_2^*-t_2)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_3(\lambda_{12}+\lambda_{123})t_1t_2}{(\lambda_1^*-t_1)(\lambda_2^*-t_2)(\lambda-t_1-t_2-t_3)(\lambda-\lambda_3-t_1-t_2)}$$

$$+\frac{\lambda_{12}t_3}{(\lambda-t_1-t_2-t_3)(\lambda_3^*-t_3)}+\frac{\lambda_{13}t_2}{(\lambda-t_1-t_2-t_3)(\lambda_2^*-t_2)}+\frac{\lambda_{23}t_1}{(\lambda-t_1-t_2-t_3)(\lambda_1^*-t_1)}$$

$$+\frac{\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}}{(\lambda-t_1-t_2-t_3)}$$

Proof $M(t_1,t_2,t_3)=E\{\exp(t_1X_1+t_2X_2+t_3X_3)\}$

$$=\frac{\lambda_1(\lambda_2+\lambda_{12})\lambda_3^*}{(\lambda_3^*-t_3)(\lambda-\lambda_1-t_2-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_1(\lambda_3+\lambda_{13})\lambda_2^*}{(\lambda_2^*-t_2)(\lambda-\lambda_1-t_2-t_3)(\lambda-t_1-t_2-t_3)}$$

$$+\frac{\lambda_2(\lambda_1+\lambda_{12})\lambda_3^*}{(\lambda_3^*-t_3)(\lambda-\lambda_2-t_2-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_2(\lambda_3+\lambda_{23})\lambda_1^*}{(\lambda_1^*-t_1)(\lambda-\lambda_2-t_2-t_3)(\lambda-t_1-t_2-t_3)}$$

$$+\frac{\lambda_3(\lambda_1+\lambda_{13})\lambda_2^*}{(\lambda_2^*-t_2)(\lambda-\lambda_3-t_1-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_3(\lambda_2+\lambda_{23})\lambda_1^*}{(\lambda_1^*-t_1)(\lambda-\lambda_3-t_2-t_3)(\lambda-t_1-t_2-t_3)}$$

$$+\frac{\lambda_1(\lambda_{23}+\lambda_{123})}{(\lambda-\lambda_1-t_2-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_2(\lambda_{13}+\lambda_{123})}{(\lambda-\lambda_2-t_1-t_3)(\lambda-t_1-t_2-t_3)}$$

$$+\frac{\lambda_3(\lambda_{12}+\lambda_{123})}{(\lambda-\lambda_3-t_1-t_2)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_{12}\lambda_3^*}{(\lambda-t_1-t_2-t_3)(\lambda_3^*-t_3)}+$$

$$+\frac{\lambda_{13}\lambda_2^*}{(\lambda-t_1-t_2-t_3)(\lambda_2^*-t_2)}+\frac{\lambda_{23}\lambda_1^*}{(\lambda-t_1-t_2-t_3)(\lambda_1^*-t_1)}+\frac{\lambda_{123}}{(\lambda-t_1-t_2-t_3)}$$

$$=\left\{\frac{\lambda_1(\lambda_2+\lambda_{12})\lambda_3^*}{(\lambda_3^*-t_3)(\lambda-\lambda_1-t_2-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_1(\lambda_3+\lambda_{13})\lambda_2^*}{(\lambda_2^*-t_2)(\lambda-\lambda_1-t_2-t_3)(\lambda-t_1-t_2-t_3)}\right.$$

$$\left.+\frac{\lambda_1(\lambda_{23}+\lambda_{123})}{(\lambda-\lambda_1-t_2-t_3)(\lambda-t_1-t_2-t_3)}\right\}+\left\{\frac{\lambda_2(\lambda_1+\lambda_{12})\lambda_3^*}{(\lambda_3^*-t_3)(\lambda-\lambda_2-t_1-t_3)(\lambda-t_1-t_2-t_3)}\right.$$

$$\left.+\frac{\lambda_2(\lambda_3+\lambda_{23})\lambda_1^*}{(\lambda_1^*-t_1)(\lambda-\lambda_2-t_1-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_2(\lambda_{13}+\lambda_{123})}{(\lambda-\lambda_2-t_1-t_3)(\lambda-t_1-t_2-t_3)}\right\}$$

$$+\left\{\frac{\lambda_3(\lambda_2+\lambda_{23})\lambda_1^*}{(\lambda_1^*-t_1)(\lambda-\lambda_3-t_1-t_2)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_3(\lambda_1+\lambda_{13})\lambda_2^*}{(\lambda_2^*-t_2)(\lambda-\lambda_3-t_1-t_2)(\lambda-t_1-t_2-t_3)}\right.$$

$$\left.+\frac{\lambda_3(\lambda_{12}+\lambda_{123})}{(\lambda-\lambda_3-t_1-t_2)(\lambda-t_1-t_2-t_3)}\right\}+\frac{\lambda_{12}\lambda_3^*}{(\lambda_3^*-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_{13}\lambda_2^*}{(\lambda_2^*-t_2)(\lambda-t_1-t_2-t_3)}$$

$$+\frac{\lambda_{23}\lambda_1^*}{(\lambda_1^*-t_1)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_{123}}{(\lambda-t_1-t_2-t_3)}$$

$$=\frac{\lambda_1\lambda_2^*\lambda_3^*}{(\lambda_2^*-t_2)(\lambda_3^*-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_1(\lambda_{23}+\lambda_{123})t_2t_3}{(\lambda_2^*-t_2)(\lambda_3^*-t_3)(\lambda-t_1-t_2-t_3)(\lambda-\lambda_1-t_2-t_3)}$$

$$+\frac{\lambda_2\lambda_1^*\lambda_3^*}{(\lambda_1^*-t_1)(\lambda_3^*-t_3)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_2(\lambda_{13}+\lambda_{123})t_1t_3}{(\lambda_1^*-t_1)(\lambda_3^*-t_3)(\lambda-t_1-t_2-t_3)(\lambda-\lambda_2-t_1-t_3)}$$

$$+\frac{\lambda_3\lambda_1^*\lambda_2^*}{(\lambda_1^*-t_1)(\lambda_2^*-t_2)(\lambda-t_1-t_2-t_3)}+\frac{\lambda_3(\lambda_{12}+\lambda_{123})t_1t_2}{(\lambda_1^*-t_1)(\lambda_2^*-t_2)(\lambda-t_1-t_2-t_3)(\lambda-\lambda_3-t_1-t_2)}$$

$$+\frac{\lambda_{12}t_3}{(\lambda-t_1-t_2-t_3)(\lambda_3^*-t_3)}+\frac{\lambda_{13}t_2}{(\lambda-t_1-t_2-t_3)(\lambda_2^*-t_2)}+\frac{\lambda_{23}t_1}{(\lambda-t_1-t_2-t_3)(\lambda_1^*-t_1)}$$
$$+\frac{\lambda_{12}+\lambda_{13}+\lambda_{23}+\lambda_{123}}{(\lambda-t_1-t_2-t_3)}$$

Remark 4.1 The mgf of (X_1, X_2) at (t_1, t_2) is $M(t_1, t_2, 0)$ and reduces to the mgf of BVE $(\lambda_1+\lambda_{13}, \lambda_2+\lambda_{23}, \lambda_{12}+\lambda_{123})$ at (t_1, t_2) in view of Barlow and Proschan(1975).

5 Reliability measures of systems with MOTVE components

In this section we derive the performance measures associated with four three component systems, assuming that the component failure times have a joint MOTVE distribution.

5.1 Standby system

Consider a three unit standby system with component failure times X_1, X_2, X_3 respectively. Then the system failure time is $T=\sum_{i=1}^{3}X_i$. Assume that the component failure times are identically distributed. As in Samantha(1983), take $\lambda_1=\lambda_2=\lambda_3=\beta_1$, $\lambda_{12}=\lambda_{13}=\lambda_{23}=\beta_2$, $\lambda_{123}=\beta_3$. Let us first find the mgf of T.

Define $\alpha_1=\beta_1+2\beta_2+\beta_3$, $\alpha_2=2\beta_1+3\beta_2+\beta_3$ and $\alpha_3=3\beta_1+3\beta_2+\beta_3$. From Theorem 4.1,

$$M(t_1,t_2,t_3)=\beta_1(\beta_1+\beta_2)\alpha_1$$
$$\left\{\frac{1}{(\alpha_1-t_1)(\alpha_2-t_1-t_2)(\alpha_3-t_1-t_2-t_3)}+\frac{1}{(\alpha_1-t_1)(\alpha_2-t_1-t_3)(\alpha_3-t_1-t_2-t_3)}\right\}$$
$$+\left\{\frac{1}{(\alpha_1-t_2)(\alpha_2-t_1-t_2)(\alpha_3-t_1-t_2-t_3)}+\frac{1}{(\alpha_1-t_2)(\alpha_2-t_2-t_3)(\alpha_3-t_1-t_2-t_3)}\right\}$$
$$+\left\{\frac{1}{(\alpha_1-t_3)(\alpha_2-t_1-t_3)(\alpha_3-t_1-t_2-t_3)}+\frac{1}{(\alpha_1-t_3)(\alpha_2-t_2-t_3)(\alpha_3-t_1-t_2-t_3)}\right\}$$
$$+\beta_1(\beta_2+\beta_3)\left\{\frac{1}{(\alpha_2-t_1-t_2)(\alpha_3-t_1-t_2-t_3)}+\frac{1}{(\alpha_2-t_1-t_3)(\alpha_3-t_1-t_2-t_3)}+\right.$$
$$\left.\frac{1}{(\alpha_2-t_2-t_3)(\alpha_3-t_1-t_2-t_3)}\right\}+\frac{\beta_3}{(\alpha_3-t_1-t_2-t_3)}$$

Thus the mgf of $X_1+X_2+X_3$ at t is,

$$M^*(t)=\frac{6\beta_1(\beta_1+\beta_2)\alpha_1}{(\alpha_1-t)(\alpha_2-2t)(\alpha_3-3t)}+\frac{3\beta_1(\beta_2+\beta_3)}{(\alpha_2-2t)(\alpha_3-3t)}+\frac{3\beta_2\alpha_1}{(\alpha_1-t)(\alpha_3-3t)}+\frac{\beta_3}{(\alpha_3-3t)}$$

Note that the mgf exists for $t<\min\left\{\alpha_1,\frac{\alpha_2}{2},\frac{\alpha_3}{3}\right\}$

Resolving the first three terms into partial fractions and simplifying we get

$$M^*(t)=\frac{6\beta_1(\beta_1+\beta_2)\alpha_1}{(\beta_2+\beta_3)(3\beta_2+2\beta_3)(\alpha_1-t)}-\frac{24\beta_1(\beta_1+\beta_2)\alpha_1}{(\beta_2+\beta_3)(3\beta_2+\beta_3)(\alpha_2-2t)}$$
$$+\frac{54\beta_1(\beta_1+\beta_2)\alpha_1}{(3\beta_2+2\beta_3)(3\beta_2+\beta_3)(\alpha_3-3t)}+\frac{-6\beta_1(\beta_2+\beta_3)}{(3\beta_2+\beta_3)(\alpha_2-2t)}+\frac{9\beta_1(\beta_2+\beta_3)}{(3\beta_2+\beta_3)(\alpha_3-3t)}$$
$$+\frac{-3\beta_2\alpha_1}{(3\beta_2+2\beta_3)(\alpha_2-2t)}+\frac{9\beta_2\alpha_1}{(3\beta_2+2\beta_3)(\alpha_3-3t)}+\frac{\beta_3}{(\alpha_3-3t)}$$

Let us express the mgf as the weighted average of three exponential mgfs.

Define $M_1(t)=\left(1-\frac{t}{\alpha_1}\right)^{-1}, M_2(t)=\left(1-\frac{2t}{\alpha_2}\right)^{-1}$ and $M_3(t)=\left(1-\frac{3t}{\alpha_3}\right)^{-1}$

Then $M^*(t)=w_1M_1(t)+w_2M_2(t)+w_3M_3(t)$, where

$$w_1=\frac{6\beta_1(\beta_1+\beta_2)}{(\beta_2+\beta_3)(3\beta_2+2\beta_3)}-\frac{3\beta_2}{(3\beta_2+2\beta_3)}$$
$$w_2=-\frac{24\beta_1(\beta_1+\beta_2)\alpha_1}{(\beta_2+\beta_3)(3\beta_2+\beta_3)\alpha_2}+\frac{-6\beta_1(\beta_2+\beta_3)}{(3\beta_2+\beta_3)\alpha_2} \text{ and}$$
$$w_3=\frac{54\beta_1(\beta_1+\beta_2)\alpha_1}{(3\beta_2+2\beta_3)(3\beta_2+\beta_3)\alpha_3}+\frac{9\beta_1(\beta_2+\beta_3)}{(3\beta_2+\beta_3)\alpha_3}+\frac{9\beta_2\alpha_1}{(3\beta_2+2\beta_3)\alpha_3}+\frac{\beta_3}{\alpha_3}$$

It can be verified that $w_1+w_2+w_3=1$.

Therefore, the reliability function $R(t)=\sum_{i=1}^{3}w_i\exp\{-(\alpha_i/i)t\}$, t > 0, and the MTBF $=\sum_{i=1}^{3}i\,w_i/\alpha_i$.

5.2 Parallel system

Consider a three unit parallel system with component failure times X_1, X_2, X_3 respectively. Then the system failure time is $T=\underset{1\le i\le 3}{Max}\ X_i$.

The distribution function of T at x is

$$G(x)=1-\overline{F}(x,0,0)-\overline{F}(0,x,0)-\overline{F}(0,0,x)+\overline{F}(x,x,0)+\overline{F}(x,0,x)+\overline{F}(0,x,x)-\overline{F}(x,x,x)$$

Therefore the reliability function of the system is,

$$R(t)=\sum_{i=1}^{3}\exp(-\lambda_i^*t)-\sum_{i=1}^{3}\exp\{-(\lambda-\lambda_i)t\}+\exp(-\lambda t),\ t>0.$$

The MTBF is

$$MTBF=\sum_{i=1}^{3}\frac{1}{\lambda_i^*}-\sum_{i=1}^{3}\frac{1}{(\lambda-\lambda_i)}+\frac{1}{\lambda}.$$

5.3 Series system

Consider a three unit series system with component failure times X_1, X_2, X_3 respectively. Then the system failure time is $T=\underset{1\le i\le 3}{Min}\ X_i$.

The reliability function is $R(t)=\overline{F}(t,t,t)$
$=\exp\{-\lambda t\}$, t > 0.

$$MTBF = \frac{1}{\lambda t}.$$

5.4 Relay system

A three component relay system operates if and only if component 1 and at least one of the remaining two components operate. The system failure time is T = $X_1 \wedge (X_2 \vee X_3)$. (Barlow and Proschan, 1975). It seems natural to assume that X_2 and X_3 are identically distributed. Thus we assume that

$$\overline{F}(x_1, x_2, x_3) = \exp\{-\lambda_1 x_1 - \lambda_2 (x_2 + x_3) - \lambda_{12}(x_1 \vee x_2 + x_1 \vee x_3) - \lambda_{23}(x_2 \vee x_3) - \lambda_{123}(x_1 \vee x_2 \vee x_3)\}$$

Note that (X_1, X_2) and (X_1, X_3) are identically distributed.

The reliability function is

$$R(t) = \overline{F}(t,t,0) + \overline{F}(t,0,t) - \overline{F}(t,t,t)$$

$$= 2\exp\{-(\lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})t\} - \exp\{-(\lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})t\},\ t > 0.$$

The MTBF is

$$MTBF = \frac{2}{(\lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})} - \frac{1}{(\lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})}$$

$$= \frac{(\lambda_1 + 3\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})}{(\lambda_1 + \lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})(\lambda_1 + 2\lambda_2 + 2\lambda_{12} + \lambda_{23} + \lambda_{123})}.$$

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