RELIABILITY OPTOMIZATION OF COMPLEX SYSTEMS

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ABSTRACT

The method based on the results of the joint model linking a semi-Markov modelling of the system operation process with a multistate approach to system reliability and the linear programming are proposed to the operation and reliability optimization of complex technical systems at the variable operation conditions. The method consists in determining the optimal values of limit transient probabilities at the system operation states that maximize the system lifetimes in the reliability state subsets. The proposed method is applied to the operation and reliability optimization of the exemplary technical multistate non-homogeneous system composed of a series-parallel and a series-"*m* out of *l*" subsystems linked in series that is changing its reliability structure and its components reliability parameters at its variable operation conditions.

1 INTRODUCTION

The complex technical systems reliability improvement and decreasing the risk of exceeding a critical reliability state are of great value in the industrial practice (Kołowrocki, Soszyńska-Budny, 2011; Kuo, Prasad, 2000; Kuo, Zuo 2003; Vercellis, 2009). In everyday practice, there are needed the tools that could be applied to improving the reliability characteristics of the multistate systems operating at variable conditions. There are needed the tools allowing for finding the distributions and the expected values of the optimal times until the exceeding by the system the reliability critical state, the optimal system risk function and the moment when the system risk function exceeds a permitted level and allowing for changing their operation processes after comparing the values of these characteristics with their values before their operation processes optimization in order to improve their reliability (Klabjan, Adelman, 2008; Kołowrocki, Soszyńska-Budny, 2009, 2010, 2011; Lisnianski, Levitin 2003, Tang, Yin, Xi, 2007).

2 COMPLEX SYSTEM RELIABILITY AND OPERATION PROCESS OPTIMIZATION

Considering the equation (25) (Kołowrocki, Soszyńska-Budny, 2013), it is natural to assume that the system operation process has a significant influence on the system reliability. This influence is also clearly expressed in the equation (26) (Kołowrocki, Soszyńska-Budny, 2013) for the mean values of the system unconditional lifetimes in the reliability state subsets.

From the linear equation (26) (Kołowrocki, Soszyńska-Budny, 2013), we can see that the mean value of the system unconditional lifetime $M(u)$, $u = 1,2,...,z,$ is determined by the limit values of transient probabilities p_b, $b = 1,2,...,\nu,$ of the system operation process at the operation states given by (8) (Kołowrocki, Soszyńska-Budny, 2013) and the mean values $M_b(u)$, $b = 1,2,...,\nu,$ $u = 1,2,...,z,$ of the system conditional lifetimes in the reliability state subsets $\{u, u+1,...,z\}$, $u = 1,2,...,z,$ given by (27) (Kołowrocki, Soszyńska-Budny, 2013). Therefore, the system lifetime optimization approach based on the linear programming (Klabjan, Adelman, 2008; Kołowrocki, Soszyńska-Budny, 2009, 2010, 2011).

can be proposed. Namely, we may look for the corresponding optimal values $\dot{p}_b$, $b = 1,2,...,\nu$, of the transient probabilities p_b, $b = 1,2,...,\nu$, of the system operation process at the operation states to maximize the mean value $M(u)$ of the unconditional system lifetimes in the reliability state subsets $\{u, u+1,..., z\}$, $u = 1,2,..., z$, under the assumption that the mean values $M_b(u)$, $b = 1,2,...,\nu$, $u = 1,2,..., z$, of the system conditional lifetimes in the reliability state subsets are fixed. As a special and practically important case of the above formulated system lifetime optimization problem, if r, $r = 1,2,..., z$, is a system critical reliability state, we may look for the optimal values $\dot{p}_b$, $b = 1,2,...,\nu$, of the transient probabilities p_b, $b = 1,2,...,\nu$, of the system operation process at the system operation states to maximize the mean value $M(r)$ of the unconditional system lifetime in the reliability state subset $\{r, r+1, ,..., z\}$, $r = 1,2,..., z$, under the assumption that the mean values $M_b(r)$, $b = 1,2,...,\nu$, $r = 1,2,..., z$, of the system conditional lifetimes in this reliability state subset are fixed. More exactly, we may formulate the optimization problem as a linear programming model with the objective function of the following form

$$M(r) = \sum_{b=1}^{\nu} p_b M_b(r) \tag{1}$$

for a fixed $r \in \{1,2,..., z\}$ and with the following bound constraints

$$\breve{p}_b \le p_b \le \hat{p}_b, \; b = 1,2,...,\nu, \tag{2}$$

$$\sum_{b=1}^{\nu} p_b = 1, \tag{3}$$

where

$$M_b(r), \; M_b(r) \ge 0, \; b = 1,2,...,\nu, \tag{4}$$

are fixed mean values of the system conditional lifetimes in the reliability state subset $\{r, r+1,..., z\}$ and

$$\breve{p}_b, \; 0 \le \breve{p}_b \le 1 \text{ and } \hat{p}_b, \; 0 \le \hat{p}_b \le 1, \; \breve{p}_b \le \hat{p}_b, \; b = 1,2,...,\nu, \tag{5}$$

are lower and upper bounds of the unknown transient probabilities p_b, $b = 1,2,...,\nu$, respectively.
Now, we can obtain the optimal solution of the formulated by (1)-(5) the linear programming problem, i.e. we can find the optimal values $\dot{p}_b$ of the transient probabilities p_b, $b = 1,2,...,\nu$, that maximize the objective function given by (1).
First, we arrange the system conditional lifetime mean values $M_b(r)$, $b = 1,2,...,\nu$, in non-increasing order

$$M_{b_1}(r) \ge M_{b_2}(r) \ge \ldots \ge M_{b_\nu}(r), \text{ where } b_i \in \{1,2,...,\nu\} \text{ for } i = 1,2,...,\nu.$$

Next, we substitute

$$x_i = p_{b_i}, \; \breve{x}_i = \breve{p}_{b_i}, \; \hat{x}_i = \hat{p}_{b_i} \text{ for } i = 1,2,...,\nu \tag{6}$$

and we maximize with respect to x_i, $i = 1,2,...,\nu$, the linear form (1) that after this transformation takes the form

$$M(r) = \sum_{i=1}^{\nu} x_i M_{b_i}(r)$$
(7)

for a fixed $r \in \{1,2,..., z\}$ with the following bound constraints

$$\breve{x}_i \le x_i \le \hat{x}_i, \; i = 1,2,...,\nu, \quad (8)$$

$$\sum_{i=1}^{\nu} x_i = 1,$$
(9)

where

$$M_{b_i}(r), \; M_{b_i}(r) \ge 0, \; i = 1,2,...,\nu,$$

are fixed mean values of the system conditional lifetimes in the reliability state subset $\{r, r+1,..., z\}$ arranged in non-increasing order and

$$\breve{x}_i, \; 0 \le \breve{x}_i \le 1 \text{ and } \hat{x}_i, \; 0 \le \hat{x}_i \le 1, \; \breve{x}_i \le \hat{x}_i, \; i = 1,2,...,\nu, \quad (10)$$

are lower and upper bounds of the unknown probabilities x_i, $i = 1,2,...,\nu$, respectively.
To find the optimal values of x_i, $i = 1,2,...,\nu$, we define

$$\breve{x} = \sum_{i=1}^{\nu} \breve{x}_i, \; ₣ = 1 - \breve{x} \quad (11)$$

and

$$\breve{x}^0 = 0, \; \hat{x}^0 = 0 \text{ and } \breve{x}^I = \sum_{i=1}^{I} \breve{x}_i, \; \hat{x}^I = \sum_{i=1}^{I} \hat{x}_i \text{ for } I = 1,2,...,\nu. \quad (12)$$

Next, we find the largest value $I \in \{0,1,...,\nu\}$ such that

$$\hat{x}^I - \breve{x}^I < ₣$$
(13)

and we fix the optimal solution that maximize (7) in the following way:
i) if $I = 0$, the optimal solution is

$$\dot{x}_1 = ₣ + \breve{x}_1 \text{ and } \dot{x}_i = \breve{x}_i \text{ for } i = 2,3,...,\nu; \quad (14)$$

ii) if $0 < I < \nu$, the optimal solution is

$$\dot{x}_i = \hat{x}_i \text{ for } i = 1,2,..., I, \; \dot{x}_{I+1} = ₣ - \hat{x}^I + \breve{x}^I + \breve{x}_{I+1} \text{ and } \dot{x}_i = \breve{x}_i$$

for $i = I+2, I+3, ..., \nu;$ (15)

iii) if $I = \nu$, the optimal solution is

$$\dot{x}_i = \hat{x}_i \text{ for } i = 1,2,...,\nu. \quad (16)$$

Finally, after making the inverse to (6) substitution, we get the optimal limit transient probabilities

$$\dot{p}_{b_i} = \dot{x}_i \text{ for } i = 1,2,...,\nu, \quad (17)$$

that maximize the system mean lifetime in the reliability state subset $\{r, r+1, ..., z\}$, defined by the linear form (1), giving its maximum value in the following form

$$\dot{M}(r) = \sum_{b=1}^{\nu} \dot{p}_b M_b(r) \quad (18)$$

for a fixed $r \in \{1,2,...,z\}$.

From the expression (18) for the maximum mean value $\dot{M}(r)$ of the system unconditional lifetime in the reliability state subset $\{r, r+1, ..., z\}$, replacing in it the critical reliability state r by the reliability state u, $u = 1,2,...,z$, we obtain the corresponding optimal solutions for the mean values of the system unconditional lifetimes in the reliability state subsets $\{u, u+1, ..., z\}$ of the form

$$\dot{M}(u) = \sum_{b=1}^{\nu} \dot{p}_b M_b(u) \text{ for } u = 1,2,...,z. \quad (19)$$

Further, according to (24)-(25) (Kołowrocki, Soszyńska-Budny, 2013), the corresponding optimal unconditional multistate reliability function of the system is the vector

$$\dot{\boldsymbol{R}}(t,\cdot) = [1, \dot{\boldsymbol{R}}(t,1), ..., \dot{\boldsymbol{R}}(t,z)], \quad (20)$$

with the coordinates given by

$$\dot{\boldsymbol{R}}(t,u) \cong \sum_{b=1}^{\nu} \dot{p}_b [\boldsymbol{R}(t,u)]^{(b)} \text{ for } t \ge 0,\ u = 1,2,...,z. \quad (21)$$

And, by (29) (Kołowrocki, Soszyńska-Budny, 2013), the optimal solutions for the mean values of the system unconditional lifetimes in the particular reliability states are

$$\dot{\bar{M}}(u) = \dot{M}(u) - \dot{M}(u+1),\ u =,1,...,z-1,\ \dot{\bar{M}}(z) = \dot{M}(z).$$
(22)

Moreover, considering (30) and (31) (Kołowrocki, Soszyńska-Budny, 2013), the corresponding optimal system risk function and the optimal moment when the risk exceeds a permitted level δ, respectively are given by

$$\dot{\boldsymbol{r}}(t) = 1 - \dot{\boldsymbol{R}}(t,r),\ t \ge 0,$$
(23)

and

$$\dot{\tau} = \dot{\boldsymbol{r}}^{-1}(\delta), \tag{24}$$

where $\dot{\boldsymbol{R}}(t,r)$ is given by (21) for $u = r$ and $\dot{\boldsymbol{r}}^{-1}(t)$, if it exists, is the inverse function of the optimal risk function $\dot{\boldsymbol{r}}(t)$.

Replacing in (8) (Kołowrocki, Soszyńska-Budny, 2013) the limit transient probabilities p_b of the system operation process at the operation states by their optimal values $\dot{p}_b$, maximizing the mean value $M(r)$ of the system lifetime in the reliability states subset $\{r, r+1, ..., z\}$ defined by (1) and the mean values m_b of the unconditional sojourn times at the operation states by their corresponding unknown optimal values $\dot{m}_b$, we get the system of equations

$$\dot{p}_b = \frac{\pi_b \dot{m}_b}{\sum_{l=1}^{v} \pi_l \dot{m}_l}, \quad b = 1,2,...,v. \tag{25}$$

After simple transformations the above system takes the form

$$(\dot{p}_1 - 1)\pi_1 \dot{m}_1 + \dot{p}_1 \pi_2 \dot{m}_2 + ... + \dot{p}_1 \pi_v \dot{m}_v = 0$$

$$\dot{p}_2 \pi_1 \dot{m}_1 + (\dot{p}_2 - 1)\pi_2 \dot{m}_2 + ... + \dot{p}_2 \pi_v \dot{m}_v = 0$$

$$...$$

(26)

$$\dot{p}_v \pi_1 \dot{m}_1 + \dot{p}_v \pi_2 \dot{m}_2 + ... + (\dot{p}_v - 1)\pi_v \dot{m}_v = 0,$$

where $\dot{m}_b$ are unknown variables we want to find, $\dot{p}_b$ are optimal transient probabilities determined by (17) and π_b are steady probabilities determined by (9) (Kołowrocki, Soszyńska-Budny, 2013).

Since the system of equations (26) is homogeneous and it can be proved that the determinant of its main matrix is equal to zero, then it has nonzero solutions and moreover, these solutions are ambiguous. Thus, if we fix some of the optimal values $\dot{m}_b$ of the mean values m_b of the unconditional sojourn times at the operation states, for instance by arbitrary fixing one or a few of them, we may find the values of the remaining once and this way to get the solution of this equation.

Having this solution, it is also possible to look for the optimal values $\dot{m}_{bl}$ of the mean values m_{bl} of the conditional sojourn times at the operation states using the following system of equations

$$\sum_{l=1}^{v} p_{bl} \dot{m}_{bl} = \dot{m}_b, \quad b = 1,2,...,v, \tag{27}$$

obtained from (7) (Kołowrocki, Soszyńska-Budny, 2013) by replacing m_b by $\dot{m}_b$ and m_{bl} by $\dot{m}_{bl}$, were p_{bl} are known probabilities of the system operation process transitions between the operation states z_b i z_l, $b,l = 1,2,...,v$, $b \neq l$, defined by (2) (Kołowrocki, Soszyńska-Budny, 2013).

Another very useful and much easier to be applied in practice tool that can help in planning the operation processes of the complex technical systems are the system operation process optimal mean values of the total system operation process sojourn times $\hat{\theta}_b$ at the particular operation states z_b, $b = 1,2,...,v,$ during the fixed system operation time θ, that can be obtain by the replacing in the formula (10) (Kołowrocki, Soszyńska-Budny, 2013) the transient probabilities p_b at the operation states z_b by their optimal values $\dot{p}_b$ and resulting in the following expession

$$\dot{\hat{M}}_b = \dot{E}[\hat{\theta}_b] = \dot{p}_b \theta, \ \ b = 1,2,...,v. \tag{28}$$

The knowledge of the optimal values $\dot{m}_b$ of the mean values of the unconditional sojourn times and the optimal values $\dot{m}_{bl}$ of the mean values of the conditional sojourn times at the operation states and the optimal mean values $\dot{\hat{M}}_b$ of the total sojourn times at the particular operation states during the fixed system operation time may by the basis for changing the complex technical systems operation processes in order to ensure these systems operation more reliable.

3 APPLICATION

We consider a series system S composed of the subsystems S_1 and S_2, with the scheme showed in Figures 1-3 (Kołowrocki, Soszyńska-Budny, 2013). This system reliability structure and its components reliability parameters depend on its changing in time operation states with arbitrarily fixed the number of the system operation process states $v = 4$ and their influence on the system reliability indicated in Sections 2-3 (Kołowrocki, Soszyńska-Budny, 2013) where its main reliability characteristics are predicted.

To find the optimal values of those system reliability characteristics, we conclude that the objective function defined by (1), in this case, as the exemplary system critical state is $r = 2$, according to (89) (Kołowrocki, Soszyńska-Budny, 2013), takes the form

$$M(2) = p_1 \cdot 25.00 + p_2 \cdot 14.88 + p_3 \cdot 13.04 + p_4 \cdot 7.04. \tag{29}$$

Arbitrarily assumed, the lower $\breve{p}_b$ and upper $\hat{p}_b$ bounds of the unknown optimal values of transient probabilities p_b, $b = 1,2,3,4,$ respectively are:

$$\breve{p}_1 = 0.201, \ \breve{p}_2 = 0.03, \ \breve{p}_3 = 0.245. \ \breve{p}_4 = 0.309;$$

$$\hat{p}_1 = 0.351, \ \hat{p}_2 = 0.105, \ \hat{p}_3 = 0.395, \ \hat{p}_4 = 0.459.$$

Therefore, according to (2)-(3), we assume the following bound constraints

$$0.201 \le p_1 \le 0.351, \ 0.030 \le p_2 \le 0.105,$$

$$0.245 \le p_3 \le 0.395, \ 0.309 \le p_4 \le 0.459. \tag{30}$$

$$\sum_{b=1}^{4} p_b = 1, \tag{31}$$

Now, before we find optimal values $\dot{p}_b$ of the transient probabilities p_b, $b = 1,2,3,4,$ that maximize the objective function (29), w arrange the system conditional lifetime mean values $M_b(2)$, $b = 1,2,3,4,$ in non-increasing order

$$M_1(2) \geq M_2(2) \geq M_3(2) \geq M_4(2).$$

Further, according to (6), we substitute

$$x_1 = p_1,\ x_2 = p_2,\ x_3 = p_3,\ x_4 = p_4, \tag{32}$$

and

$$\breve{x}_1 = \breve{p}_1 = 0.201,\ \breve{x}_2 = \breve{p}_2 = 0.030,\ \breve{x}_3 = \breve{p}_3 = 0.245,\ \breve{x}_4 = \breve{p}_4 = 0.309; \tag{33}$$

$$\widehat{x}_1 = \widehat{p}_1 = 0.351,\ \widehat{x}_2 = \widehat{p}_2 = 0.105,\ \widehat{x}_3 = \widehat{p}_3 = 0.395,\ \widehat{x}_4 = \widehat{p}_4 = 0.459, \tag{34}$$

and we maximize with respect to x_i, $i = 1,2,3,4$, the linear form (29) that according to (7)-(9) takes the form

$$M(2) = x_1 \cdot 25.00 + x_2 \cdot 14.88 + x_3 \cdot 13.04 + x_4 \cdot 7.04, \tag{35}$$

with the following bound constraints

$$0.201 \leq x_1 \leq 0.351,\ 0.030 \leq x_2 \leq 0.105,$$

$$0.245 \leq x_3 \leq 0.395,\ 0.309 \leq x_4 \leq 0.459. \tag{36}$$

$$\sum_{i=1}^{4} x_i = 1. \tag{37}$$

According to (11), we calculate

$$\breve{x} = \sum_{i=1}^{4} \breve{x}_i = 0.785,\ \breve{y} = 1 - \breve{x} = 1 - 0.785 = 0.215 \tag{38}$$

and according to (12), we determine

$$\breve{x}^0 = 0,\ \widehat{x}^0 = 0,\ \widehat{x}^0 - \breve{x}^0 = 0,$$

$$\breve{x}^1 = 0.201,\ \widehat{x}^1 = 0.351,\ \widehat{x}^1 - \breve{x}^1 = 0.150,$$

$$\breve{x}^2 = 0.231,\ \widehat{x}^2 = 0.456,\ \widehat{x}^2 - \breve{x}^2 = 0.225,$$

$$\breve{x}^3 = 0.476.\ \widehat{x}^3 = 0.851,\ \widehat{x}^3 - \breve{x}^3 = 0.375,$$

$$\breve{x}^4 = 0.785\ \ \widehat{x}^4 = 1.31\ \ \widehat{x}^4 - \breve{x}^4 = 0.525. \tag{39}$$

From the above, as according to (38), the inequality (13) takes the form

$$\widehat{x}^I - \breve{x}^I < 0.215, \tag{40}$$

it follows that the largest value $I \in \{0,1,2,3,4\}$ such that this inequality holds is $I = 1$.
Therefore, we fix the optimal solution that maximize linear function (35) according to the rule (15). Namely, we get

$$\dot{x}_1 = \hat{x}_1 = 0.351,$$

$$\dot{x}_2 = \hat{y} - \hat{x}^1 + \breve{x}^1 + \breve{x}_2 = 0.215 - 0.351 + 0.201 + 0.030 = 0.095,$$

$$\dot{x}_3 = \breve{x}_3 = 0.245,\ \ \dot{x}_4 = \breve{x}_4 = 0.309. \quad (41)$$

Finally, after making the inverse to (32) substitution, we get the optimal transient probabilities

$$\dot{p}_1 = \dot{x}_1 = 0.351,\ \ \dot{p}_2 = \dot{x}_2 = 0.095,\ \ \dot{p}_3 = \dot{x}_3 = 0.245,\ \ \dot{p}_4 = \dot{x}_4 = 0.309,$$
(42)

that maximize the exemplary system mean lifetime $M(2)$ in the reliability state subset $\{2,3\}$ expressed by the linear form (29) giving, according to (18) and (42), its optimal value

$$\dot{M}(2) = \dot{p}_1 \cdot 25.00 + \dot{p}_2 \cdot 14.88 + \dot{p}_3 \cdot 13.04 + \dot{p}_4 \cdot 7.04$$

$$= 0.351 \cdot 25.00 + 0.095 \cdot 14.88 + 0.245 \cdot 13.04 + 0.309 \cdot 7.07 \cong 15.56. \quad (43)$$

Substituting the optimal solution (42) into the formula (19), we obtain the optimal solution for the mean values of the exemplary system unconditional lifetimes in the reliability state subsets $\{1,2,3\}$ and $\{3\}$, that are as follows

$$\dot{M}(1) = \dot{p}_1 \cdot 27.78 + \dot{p}_2 \cdot 16.27 + \dot{p}_3 \cdot 14.82 + \dot{p}_4 \cdot 7.72$$

$$= 0.351 \cdot 27.78 + 0.095 \cdot 16.27 + 0.245 \cdot 14.82 + 0.309 \cdot 7.72 \cong 17.31, \quad (44)$$

$$\dot{M}(3) = \dot{p}_1 \cdot 22.73 + \dot{p}_2 \cdot 13.71 + \dot{p}_3 \cdot 11.48 + \dot{p}_4 \cdot 6.47$$

$$= 0.351 \cdot 22.73 + 0.095 \cdot 13.71 + 0.245 \cdot 11.48 + 0.309 \cdot 6.47 \cong 14.09 \quad (45)$$

and according to (22), the optimal values of the mean values of the system unconditional lifetimes in the particular reliability states 1, 2 and 3, respectively are

$$\dot{\overline{M}}(1) = \dot{M}(1) - \dot{M}(2) = 1.75,\ \dot{\overline{M}}(2) = \dot{M}(2) - \dot{M}(3) = 1.47,$$

$$\dot{\overline{M}}(3) = \dot{M}(3) = 14.09. \quad (46)$$

Moreover, according to (20)-(21), the corresponding optimal unconditional multistate reliability function of the system is of the form

$$\dot{\boldsymbol{R}}(t,\cdot) = [1,\ \dot{\boldsymbol{R}}(t,1),\ \dot{\boldsymbol{R}}(t,2),\ \dot{\boldsymbol{R}}(t,3)\,],\ t \geq 0, \quad (47)$$

with the coordinates given by

$$\dot{\boldsymbol{R}}(t,1) = 0.351\cdot[\boldsymbol{R}(t,1)]^{(1)} + 0.095\cdot[\boldsymbol{R}(t,1)]^{(2)} + 0.245\cdot[\boldsymbol{R}(t,1)]^{(3)}$$

$$+ 0.309\cdot[\boldsymbol{R}(t,1)]^{(4)} \text{ for } t \geq 0, \quad (48)$$

$$\dot{\boldsymbol{R}}(t,2) = 0.351\cdot[\boldsymbol{R}(t,2)]^{(1)} + 0.095\cdot[\boldsymbol{R}(t,2)]^{(2)} + 0.245\cdot[\boldsymbol{R}(t,2)]^{(3)}$$

$$+ 0.309\cdot[\boldsymbol{R}(t,2)]^{(4)} \text{ for } t \geq 0, \quad (49)$$

$$\dot{\boldsymbol{R}}(t,3) = 0.351\cdot[\boldsymbol{R}(t,3)]^{(1)} + 0.0095\cdot[\boldsymbol{R}(t,3)]^{(2)} + 0.245\cdot[\boldsymbol{R}(t,3)]^{(3)}$$

$$+ 0.309\cdot[\boldsymbol{R}(t,3)]^{(4)} \text{ for } t \geq 0, \quad (50)$$

where $[\boldsymbol{R}(t,1)]^{(b)}$, $[\boldsymbol{R}(t,2)]^{(b)}$, $[\boldsymbol{R}(t,3)]^{(b)}$, $b = 1,2,3,4,$ are fixed in Section 3 (Kołowrocki, Soszyńska-Budny, 2013).
The graph of the exemplary system optimal reliability function $\dot{\boldsymbol{R}}(t,\cdot)$ given by (47)-(50) is presented in Figure 1.

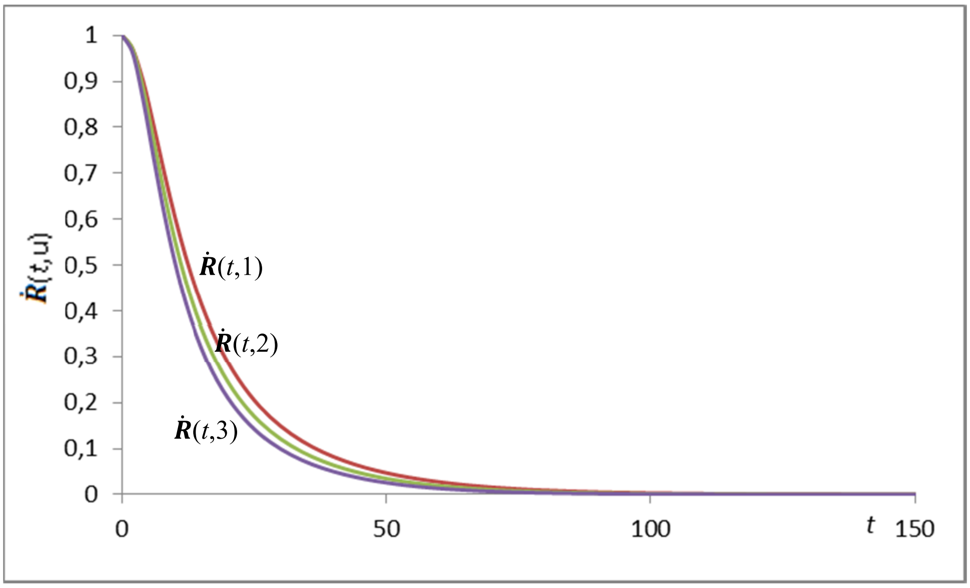

Fig. 1. The graph of the exemplary system optimal reliability function $\dot{R}(t,\cdot)$ coordinates

As the critical reliability state is r =2, then the exemplary system optimal system risk function, according to (23), is given by

$$\dot{\boldsymbol{r}}(t) = 1 - \dot{\boldsymbol{R}}(t,2) \text{ for } t \geq 0, \quad (51)$$

where $\dot{\boldsymbol{R}}(t,2)$ is given by (49).
Hence and considering (24), the moment when the optimal system risk function exceeds a permitted level, for instance δ = 0.025, is

$$\dot{\tau} = \dot{\boldsymbol{r}}^{-1}(\delta) \cong 2.55. \quad (52)$$

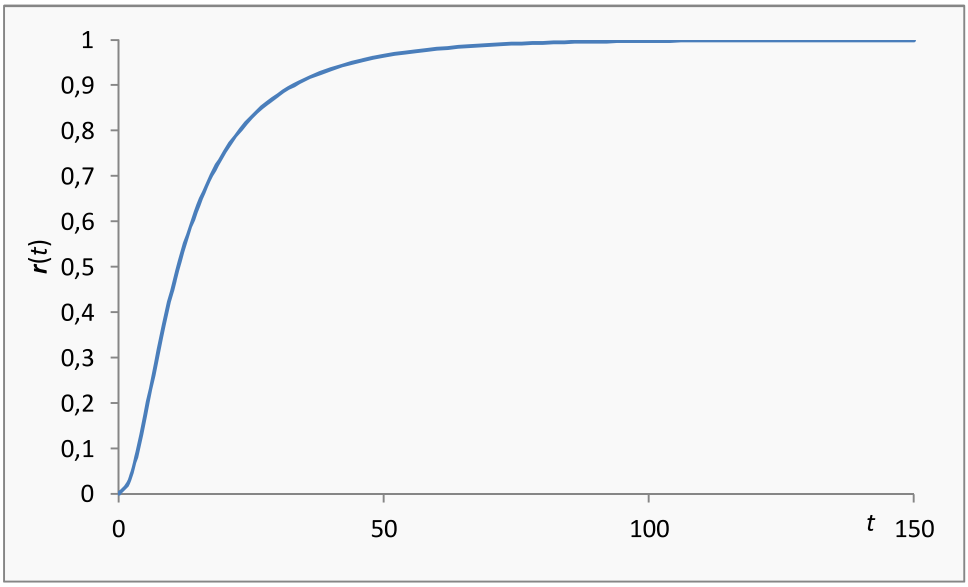

Fig. 2. The graph of the exemplary system optimal risk function $\dot{r}(t)$

Substituting the exemplary operation process optimal transient probabilities at operation states

$$\dot{p}_1 = 0.351,\ \dot{p}_2 = 0.095,\ \dot{p}_3 = 0.245,\ \dot{p}_4 = 0.309,$$

determined by (42) and the steady probabilities

$$\pi_1 \cong 0.236,\ \pi_2 \cong 0.169,\ \pi_3 \cong 0.234,\ \pi_4 \cong 0.361,$$

determined by (17) in Section 2 (Kołowrocki, Soszyńska-Budny, 2013) into (26), we get the following system of equations with the unknown optimal mean values $\dot{m}_b$ of the exemplary system operation process unconditional sojourn times at the operation states we are looking for

$$-0.153164\dot{m}_1 + 0.059319\dot{m}_2 + 0.082134\dot{m}_3 + 0.126711\dot{m}_4 = 0$$

$$0.02242\dot{m}_1 - 0.152945\dot{m}_2 + 0.02223\dot{m}_3 + 0.034295\dot{m}_4 = 0$$

$$0.05782\dot{m}_1 + 0.041405\dot{m}_2 - 0.17667\dot{m}_3 + 0.088445\dot{m}_4 = 0$$

$$0.072924\dot{m}_1 + 0.052221\dot{m}_2 + 0.072306\dot{m}_3 - 0.249451\dot{m}_4 = 0. \quad (53)$$

The determinant of the main matrix of the above homogeneous system of equations is equal to zero and therefore there are non-zero solutions of this system of equations that are ambiguous and dependent on one or more parameters. Thus, we may fix some of them and determine the remaining ones. To show the way of solving this system of equations, we may suppose that we are arbitrarily interested in fixing the value of $\dot{m}_4$ and we put

$$\dot{m}_4 = 400.$$

Substituting the above value into the system of equations (53), we get

$$-0.153164\dot{m}_1 + 0.059319\dot{m}_2 + 0.082134\dot{m}_3 = -50.6844$$

$$0.02242\dot{m}_1 - 0.152945\dot{m}_2 + 0.02223\dot{m}_3 = -13.7180$$

$$0.05782\dot{m}_1 + 0.041405\dot{m}_2 - 0.17667\dot{m}_3 = -35.3780$$

$$0.072924\dot{m}_1 + 0.052221\dot{m}_2 + 0.072306\,\dot{m}_3 = 99.7804$$

and we solve it with respect to $\dot{m}_1$, $\dot{m}_2$ and $\dot{m}_3$, after omitting its last equation. This way obtained solutions that satisfy (53), are

$$\dot{m}_1 \cong 689,\ \dot{m}_2 \cong 261,\ \dot{m}_3 \cong 487,\ \dot{m}_4 = 400. \tag{54}$$

It can be seen that these solution differ much from the values m_1, m_2, m_3 and m_4 estimated in Section 2 (Kołowrocki, Soszyńska-Budny, 2013) and given by (13)-(16) (Kołowrocki, Soszyńska-Budny, 2013).

Having these solutions, it is also possible to look for the optimal values $\dot{m}_{bl}$ of the mean values m_{bl} of the exemplary system operation process conditional sojourn times at operation states. Namely, substituting the values $\dot{m}_b$ instead of m_b, the probabilities

$$[p_{bl}] = \begin{bmatrix} 0 & 0.22 & 0.32 & 0.46 \\ 0.20 & 0 & 0.30 & 0.50 \\ 0.12 & 0.16 & 0 & 0.72 \\ 0.48 & 0.22 & 0.30 & 0 \end{bmatrix}$$

of the exemplary system operation process transitions between the operation states given by (11) in Section 2 (Kołowrocki, Soszyńska-Budny, 2013) and replacing m_{bl} by $\dot{m}_{bl}$ in (27), we get the following system of equations

$$0.22\dot{m}_{12} + 0.32\dot{m}_{13} + 0.46\dot{m}_{14} = 689$$

$$0.20\dot{m}_{21} + 0.30\dot{m}_{23} + 0.50\dot{m}_{24} = 261$$

$$0.12\dot{m}_{31} + 0.16\dot{m}_{32} + 0.72\dot{m}_{34} = 487$$

$$0.48\dot{m}_{41} + 0.2\dot{2}_{42} + 0.30\dot{m}_{14} = 400 \tag{55}$$

with the unknown optimal values $\dot{m}_{bl}$ we want to find.

As the solutions of the above system of equations are ambiguous, then we fix some of them, say that because of practically important reasons, and we find the remaining ones. For instance:

- we fix in the first equation $\dot{m}_{12} = 200,\ \dot{m}_{13} = 500$ and we find $\dot{m}_{14} \cong 1054$;
- we fix in the second equation $\dot{m}_{21} = 100,\ \dot{m}_{23} = 100$ and we find $\dot{m}_{24} \cong 422$;

- we fix in the third equation $\dot{m}_{31} = 900,\ \dot{m}_{32} = 500$ and we find $\dot{m}_{34} \cong 415$;
- we fix in the fourth equation $\dot{m}_{41} = 300,\ \dot{m}_{42} = 500$ and we find $\dot{m}_{43} \cong 487$. (56)

It can be seen that these solutions differ much from the mean values of the exemplary system conditional sojourn times at the particular operation states before its operation process optimization given by (12) (Kołowrocki, Soszyńska-Budny, 2013).
Another very useful and much easier to be applied in practice tool that can help in planning the operation process of the exemplary system are the system operation process optimal mean values of the total sojourn times at the particular operation states during the system operation time that by the same assumpion as in Section 2 (Kołowrocki, Soszyńska-Budny, 2013) is equel to $\theta = 1$year = 365 days. Under this assumption, after aplying (28), we get the optimal values of the exemplary system operation process total sojourn times at the particular operation states during 1 year

$$\dot{\hat{M}}_1 = \dot{E}[\hat{\theta}_1] = \dot{p}_1\theta = 0.341 \cdot 365 \cong 124.5,$$

$$\dot{\hat{M}}_2 = \dot{E}[\hat{\theta}_2] = \dot{p}_2\theta = 0.105 \cdot 365 \cong 38.3,$$

$$\dot{\hat{M}}_3 = \dot{E}[\hat{\theta}_3] = \dot{p}_3\theta = 0.245 \cdot 365 \cong 89.4,$$

$$\dot{\hat{M}}_4 = \dot{E}[\hat{\theta}_4] = \dot{p}_4\theta = 0.309 \cdot 365 \cong 112.8, \qquad (57)$$

that differ much from the values of $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$, $\hat{M}_4$, determined by (19) in Section 2 (Kołowrocki, Soszyńska-Budny, 2013).
In practice, the knowledge of the optimal values of $\dot{m}_b$ $\dot{m}_{bl}$ and $\dot{\hat{M}}_b$ given respectively by (54), (56), (57), can be very important and helpful for the system operation process planning and improving in order to make the system operation more reliable.

4 CONCLUSION

Presented in this paper tool is useful in reliability and operation optimization of a very wide class of real technical systems operating at the varying conditions that have an influence on changing their reliability structures and their components reliability parameters. The results can be interesting for reliability practitioners from various industrial sectors.

5 REFERENCES

Klabjan, D., Adelman, D. 2006. Existence of optimal policies for semi-Markov decision processes using duality for infinite linear programming. Siam J on Contr and Optim 44(6): 2104-2122.

Kołowrocki, K., Soszyńska, J. 2010. Reliability, availability and safety of complex technical systems: modeling – identification – prediction – optimization. *Journal of Polish Safety and Reliability Association, Summer Safety & Reliability Seminars* 1(1), 133-158.

Kołowrocki, K., Soszyńska-Budny, J. 2011. *Reliability and Safety of Complex Technical Systems and Processes: Modeling-Identification-Prediction-Optimization*. Springer.

Kołowrocki, K., Soszyńska-Budny, J. 2013. Modelling reliability of complex systems. *Reliability: Theory & Applications* 8(4): ….

Kuo, W., Prasad, V.R. 2000. An annotated overview of system-reliability optimization. IEEE Trans on Reliab 49(2): 176-187.

Kuo, W., Zuo, M. J. 2003. Optimal Reliability Modeling: Principles and Applications. Hoboken: John Wiley & Sons, Inc.

Lisnianski, A., Levitin, G. 2003. Multi-State System Reliability. Assessment, Optimisation and Applications. World Scientific Publishing Co. Pte. Ltd.

Tang, H., Yin, B.Q., Xi, H.S. 2007. Error bounds of optimization algorithms for semi-Markov decision processes. Int J of Sys Sci 38(9): 725-736.

Vercellis, S. 2009. *Data mining and optimization for decision making*. John Wiley & Sons.