MODELLING RELIABILITY OF COMPLEX SYSTEMS

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ABSTRACT

Modelling and prediction of the operation and reliability of technical systems related to their operation processes are presented. The emphasis is on multistate systems composed of ageing components and changing their reliability structures and their components reliability parameters during their operation processes that are called the complex systems. The integrated general model of complex systems' reliability, linking their reliability models and their operation processes models and considering variable at different operation states their reliability structures and their components reliability parameters is constructed. This theoretical tool is applied to modelling and prediction of the operation processes and reliability characteristics of the multistate non-homogeneous system composed of a series-parallel and a series-"m out of l" subsystems linked in series, changing its reliability structure and its components reliability parameters at variable operation conditions.

1 INTRODUCTION

Most real technical systems are very complex and it is difficult to analyze their reliability. Large numbers of components and subsystems and their operating complexity cause that the identification, evaluation and prediction of their reliability are complicated. The complexity of the systems' operation processes and their influence on changing in time the systems' structures and their components' reliability parameters are very often met in real practice. Thus, the practical importance of an approach linking the system reliability models and the system operation processes models into an integrated general model in reliability assessment of real technical systems is evident.

The convenient tools for analyzing these problems are semi-Markov modelling the systems' operation processes (Ferreir, Pacheco, 2007; Glynn, Hass, 2006; Habibullah et al. 2009; Kołowrocki, Soszyńska, 2009; Mercier, 2008; Soszyńska et al. 2010; Grabski, 2002; Kołowrocki, Soszyńska-Budny, 2011; Limnios, Oprisan, 2001; Kołowrocki 2008) multistate approach to the systems' reliability evaluation (Kołowrocki, Soszyńska, 2009; Xue, 1985; Xue, Yang 1995b; Kołowrocki, 2008). The common usage of the multistate systems' reliability models and the semi-Markov model for the systems' operation processes in order to construct the joint general system reliability model related to its operation process (Kołowrocki, 2006; Kołowrocki, 2007a; Kołowrocki 2007b; Kołowrocki, Soszyńska, 2006; Kołowrocki, Soszyńska, 2010, Soszyńska, 2007a; Soszyńska, 2007b; Kołowrocki, Soszyńska-Budny, 2011; Soszyńska 2007c; Kołowrocki et all 2008) and to apply it to the reliability analysis of complex technical systems is this paper main idea.

2 COMPLEX SYSTEM OPERATION PROCESS MODELLING

We assume that the system during its operation process is taking $v, v \in N$, different operation states $z_1, z_2, \ldots, z_v$. Further, we define the system operation process $Z(t)$, $t \in <0,+\infty)$, with discrete operation states from the set $\{z_1, z_2, \ldots, z_v\}$. Moreover, we assume that the system operation

process $Z(t)$ is a semi-Markov process (Kołowrocki, Soszyńska, 2009; Kołowrocki, Soszyńska, 2010; Grabski, 2002; Soszyńska, 2007b) with the conditional sojourn times θ_{bl} at the operation states z_b when its next operation state is z_l, $b, l = 1,2,...,v,$ $b \neq l$. Under these assumptions, the system operation process may be described by:
- the vector of the initial probabilities $p_b(0) = P(Z(0) = z_b)$, $b = 1,2,...,v,$ of the system operation process $Z(t)$ staying at particular operation states at the moment $t = 0$

$$[p_b(0)]_{1 \times v} = [p_1(0), p_2(0),..., p_v(0)]; \tag{1}$$

- the matrix of probabilities p_{bl}, $b, l = 1,2,...,v,$ $b \neq l,$ of the system operation process $Z(t)$ transitions between the operation states z_b and z_l

$$[p_{bl}]_{v \times v} = \begin{bmatrix} p_{11} \ p_{12} \ \cdots \ p_{1v} \\ p_{21} \ p_{22} \ \cdots \ p_{2v} \\ \cdots \\ p_{v1} \ p_{v2} \ \cdots \ p_{vv} \end{bmatrix}, \tag{2}$$

where by a formal agreement

$$p_{bb} = 0 \ \text{for} \ b = 1,2,...,v;$$

- the matrix of conditional distribution functions $H_{bl}(t) = P(\theta_{bl} < t)$, $b, l = 1,2,...,v,$ $b \neq l,$ of the system operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states

$$[H_{bl}(t)]_{v \times v} = \begin{bmatrix} H_{11}(t) \ H_{12}(t) \ldots H_{1v}(t) \\ H_{21}(t) \ H_{22}(t) \ldots H_{2v}(t) \\ \cdots \\ H_{v1}(t) \ H_{v2}(t) \ldots H_{vv}(t) \end{bmatrix}, \tag{3}$$

where by formal agreement

$$H_{bb}(t) = 0 \ \text{for} \ b = 1,2,...,v.$$

We introduce the matrix of the conditional density functions of the system operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states corresponding to the conditional distribution functions $H_{bl}(t)$

$$[h_{bl}(t)]_{v \times v} = \begin{bmatrix} h_{11}(t) \ h_{12}(t) \ldots h_{1v}(t) \\ h_{21}(t) \ h_{22}(t) \ldots h_{2v}(t) \\ \cdots \\ h_{v1}(t) \ h_{v2}(t) \ldots h_{vv}(t) \end{bmatrix}, \tag{4}$$

where

$$h_{bl}(t) = \frac{d}{dt}[H_{bl}(t)] \text{ for } b,l = 1,2,...,v,\ b \neq l,$$

and by formal agreement

$$h_{bb}(t) = 0 \text{ for } b = 1,2,...,v.$$

As the mean values $E[\theta_{bl}]$ of the conditional sojourn times θ_{bl} are given by

$$m_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t) = \int_0^\infty t h_{bl}(t)dt,\ \ b,l = 1,2,...,v,\ \ b \neq l, \quad (5)$$

then from the formula for total probability, it follows that the unconditional distribution functions of the sojourn times θ_b, $b = 1,2,...,v$, of the system operation process $Z(t)$ at the operation states z_b, $b = 1,2,...,v$, are given by (Grabski, 2002; Kołowrocki, Soszyńska-Budny, 2011; Soszyńska, 2007b; Limnios, Oprisan, 2001)

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t),\ \ b = 1,2,...,v. \quad (6)$$

Hence, the mean values $E[\theta_b]$ of the system operation process $Z(t)$ unconditional sojourn times θ_b, $b = 1,2,...,v$, at the operation states are given by

$$m_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} m_{bl},\ b = 1,2,...,v, \quad (7)$$

where m_{bl} are defined by the formula (5).
The limit values of the system operation process $Z(t)$ transient probabilities at the particular operation states

$$p_b(t) = P(Z(t) = z_b),\ t \in \langle 0,+\infty),\ \ b = 1,2,...,v,$$

are given by (Grabski, 2002; Kołowrocki, Soszyńska-Budny, 2011; Soszyńska, 2007b; Limnios, Oprisan, 2001)

$$p_b = \lim_{t\to\infty} p_b(t) = \frac{\pi_b m_b}{\sum_{l=1}^{v} \pi_l m_l},\ \ b = 1,2,...,v, \quad (8)$$

where m_b, $b = 1,2,...,v$, are given by (7), while the steady probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases} \quad (9)$$

In the case of a periodic system operation process, the limit transient probabilities p_b, $b = 1,2,...,v$, at the operation states defined by (8), are the long term proportions of the system operation process $Z(t)$ sojourn times at the particular operation states z_b, $b = 1,2,...,v$.
Other interesting characteristics of the system operation process $Z(t)$ possible to obtain are its total sojourn times $\hat{\theta}_b$ at the particular operation states z_b, $b = 1,2,...,v$, during the fixed system opetation time. It is well known (Grabski, 2002; Kołowrocki, Soszyńska-Budny, 2011; Soszyńska, 2007b; Limnios, Oprisan, 2001) that the system operation process total sojourn times $\hat{\theta}_b$ at the particular operation states z_b, for sufficiently large operation time θ, have approximately normal distributions with the expected value given by

$$\hat{M}_b = E[\hat{\theta}_b] = p_b\theta, \quad b = 1,2,...,v, \tag{10}$$

where p_b are given by (8).

Example
We consider a series system S composed of the subsystems S_1 and S_2, with the scheme showed in Figure 1.

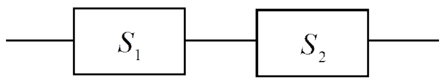

Fig. 1. The scheme of the exemplary system S reliability structure

We assume that the subsystem S_1 is a series-parallel system with the scheme given in Figure 2 and the subsystem S_2 illustrated in Figure 3 is either a series-parallel system or a series-“2 out of 4” system.

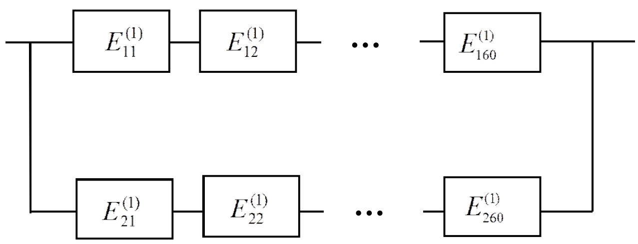

Fig. 2. The scheme of the subsystem S_1 reliability structure

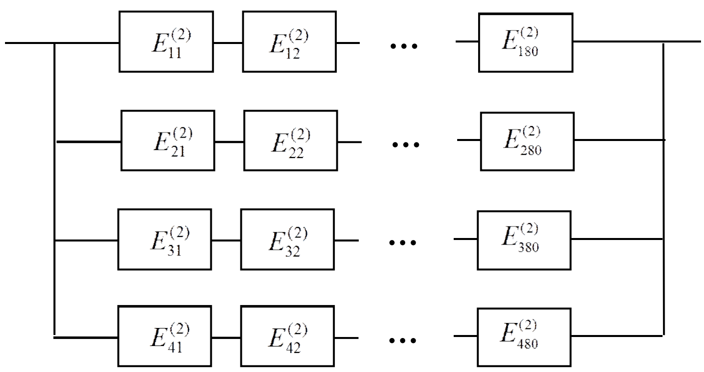

Fig. 3. The scheme of the subsystem S_2 reliability structure

The subsystems S_1 and S_2 are forming a general series reliability structure of the system presented in Figure 1. However, this system reliability structure and its subsystems and components reliability depend on its changing in time operation states (Kołowrocki, Soszyńska, 2009; Soszyńska, 2007b).
Under the assumption that the system operation conditions are changing in time, we arbitrarily fix the number of the system operation process states $\nu = 4$ and we distinguish the following as its operation states:

- an operation state z_1 − the system is composed of the subsystem S_1 with the scheme showed in Figure 2 that is a series-parallel system,
- an operation state z_2 − the system is composed of the subsystem S_2 with the scheme showed in Figure 3 that is a series-parallel system,
- an operation state z_3 − the system is a series system with the scheme showed in Figure 1 composed of the subsystems S_1 and S_2 that are series-parallel systems with the schemes respectively given in Figure 2 and Figure 3,
- an operation state z_4 − the system is a series system with the scheme showed in Figure 1 composed of the subsystem S_1 and S_2, while the subsystem S_1 is a series-parallel system with the scheme given in Figure 2 and the subsystem S_2 is a series-"2 out of 4" system with the scheme given in Figure 3.

The influence of the above system operation states changing on the changes of the exemplary system reliability structure is indicated in these operation states above definitions and illustrated in Figures 1-3. Its influence on the system components reliability will be defined in this example continuation in Section 3.
We arbitrarily assume that the probabilities p_{bl} of the exemplary system operation process transitions from operation state z_b into the operation state z_l are given in the matrix below

$$[p_{bl}] = \begin{bmatrix} 0 & 0.25 & 0.30 & 0.45 \\ 0.20 & 0 & 0.25 & 0.55 \\ 0.15 & 0.20 & 0 & 0.65 \\ 0.40 & 0.25 & 0.35 & 0 \end{bmatrix}. \tag{11}$$

We also arbitrarily fix the conditional mean values $m_{bl} = E[\theta_{bl}],\ b,l = 1,2,3,4,$ of the exemplary system sojourn times at the particular operation states as follows:

$$\begin{aligned} &m_{12} = 190,\ m_{13} = 480,\ m_{14} = 200, \\ &m_{21} = 100,\ m_{23} = 80,\ m_{24} = 60, \\ &m_{31} = 870,\ m_{32} = 480,\ m_{34} = 300, \\ &m_{41} = 320,\ m_{42} = 510,\ m_{43} = 440. \end{aligned} \tag{12}$$

This way, the exemplary system operation process is defined and we may find its main characteristics. Namely, applying (7), (11) and (12), the unconditional mean sojourn times at the particular operation states are given by:

$$m_1 = E[\theta_1] = p_{12}m_{12} + p_{13}m_{13} + p_{14}m_{14} = 0.25\cdot 190 + 0.30\cdot 480 + 0.45\cdot 200 = 281.5, \tag{13}$$

$$m_2 = E[\theta_2] = p_{21}m_{21} + p_{23}m_{23} + p_{24}m_{24} = 0.20\cdot 100 + 0.25\cdot 80 + 0.55\cdot 60 = 73.00, \tag{14}$$

$$m_3 = E[\theta_3] = p_{31}m_{31} + p_{32}m_{32} + p_{34}m_{34} = 0.15\cdot 870 + 0.20\cdot 480 + 0.65\cdot 300 = 421.5, \tag{15}$$

$$m_4 = E[\theta_4] = p_{41}m_{41} + p_{42}m_{42} + p_{43}m_{43} = 0.40\cdot 320 + 0.25\cdot 510 + 0.35\cdot 440 = 409.5. \tag{16}$$

Further, according to (9), the system of equations

$$\begin{cases} [\pi_1, \pi_2, \pi_3, \pi_4] = [\pi_1, \pi_2, \pi_3, \pi_4][p_{bl}]_{4x4} \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1, \end{cases}$$

after considering (11), takes the form

$$\begin{cases} \pi_1 = 0.20\pi_2 + 0.15\pi_3 + 0.40\pi_4 \\ \pi_2 = 0.25\pi_1 + 0.20\pi_3 + 0.25\pi_4 \\ \pi_3 = 0.30\pi_1 + 0.25\pi_2 + 0.35\pi_4 \\ \pi_4 = 0.45\pi_1 + 0.55\pi_2 + 0.65\pi_3 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1. \end{cases}$$

The approximate solutions of the above system of equations are:

$$\pi_1 \cong 0.216,\ \ \pi_2 \cong 0.191,\ \ \pi_3 \cong 0.237,\ \pi_4 \cong 0.356. \tag{17}$$

After considering the result (17) and (13)-(16), we have

$$\sum_{l=1}^{4} \pi_l m_l \cong 0.216\cdot 281.5 + 0.191\cdot 73.0 + 0.237\cdot 421.5 + 0.356\cdot 409.5 = 320.4245,$$

and according to (8), the limit values of the exemplary system operation process transient probabilities $p_b(t)$ at the operation states z_b are given by

$$p_1 = \frac{0.216\cdot 281.5}{320.4245} \cong 0.190,\ \ p_2 = \frac{0.191\cdot 73.0}{320.4245} \cong 0.043,$$
$$p_3 = \frac{0.237\cdot 421.5}{320.4245} \cong 0.312,\ \ p_4 = \frac{0.356\cdot 409.5}{320.4245} \cong 0.455. \tag{18}$$

Hence, the expected values of the total sojourn times $\hat{\theta}_b$, $b = 1,2,3,4$, of the exemplary system operation process at the particular operation states z_b, $b = 1,2,3,4$, during the fixed operation time $\theta = 1$ year = 365 days, after applying (9.10), amount:

$$\hat{M}_1 = E[\hat{\theta}_1] = 0.190\cdot 1 = 0.190 \text{ year} = 69.3 \text{ days},$$

$$\hat{M}_2 = E[\hat{\theta}_2] = 0.043\cdot 1 = 0.043 \text{ year} = 15.7 \text{ days},$$

$$\hat{M}_3 = E[\hat{\theta}_3] = 0.312\cdot 1 = 0.312 \text{ year} = 113.9 \text{ days},$$

$$\hat{M}_4 = E[\hat{\theta}_4] = 0.455 = 0.455 \text{ year} = 166.1 \text{ days}. \tag{19}$$

3 COMPLEX SYSTEM RELIABILITY MODELLING

We assume that the changes of the operation states of the system operation process $Z(t)$ have an influence on the system multistate components E_i, $i = 1,2,...,n,$ reliability and the system reliability structure as well. Consequently, we denote the system multistate component E_i, $i = 1,2,...,n,$ conditional lifetime in the reliability state subset $\{u, u+1,..., z\}$ while the system is at the operation state z_b, $b = 1,2,...,v,$ by $T_i^{(b)}(u)$ and its conditional reliability function by the vector

$$[R_i(t, \cdot)]^{(b)} = [1, [R_i(t,1)]^{(b)}, ..., [R_i(t,z)]^{(b)}], \tag{20}$$

with the coordinates defined by

$$[R_i(t,u)]^{(b)} = P(T_i^{(b)}(u) > t | Z(t) = z_b) \tag{21}$$

for $t \in <0, \infty)$, $u = 1,2,..., z$, $b = 1,2,..., v$.

The reliability function $[R_i(t,u)]^{(b)}$ is the conditional probability that the component E_i lifetime $T_i^{(b)}(u)$ in the reliability state subset $\{u, u+1,..., z\}$ is greater than t, while the system operation process $Z(t)$ is at the operation state z_b.

Similarly, we denote the system conditional lifetime in the reliability state subset $\{u, u+1,..., z\}$ while the system is at the operation state z_b, $b = 1,2,...,v,$ by $T^{(b)}(u)$ and the conditional reliability function of the system by the vector

$$[\boldsymbol{R}(t,\cdot)]^{(b)} = [1, [\boldsymbol{R}(t,1)]^{(b)}, ..., [\boldsymbol{R}(t,z)]^{(b)}], \tag{22}$$

with the coordinates defined by

$$[\boldsymbol{R}(t,u)]^{(b)} = P(T^{(b)}(u) > t | Z(t) = z_b) \tag{23}$$

for $t \in <0, \infty)$, $u = 1,2,..., z$, $b = 1,2,..., v$.

The reliability function $[\boldsymbol{R}(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the reliability state subset $\{u, u+1,..., z\}$ is greater than t, while the system operation process $Z(t)$ is at the operation state z_b.

Further, we denote the system unconditional lifetime in the reliability state subset $\{u, u+1,..., z\}$ by $T(u)$ and the unconditional reliability function of the system by the vector

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1), ..., \boldsymbol{R}(t,z)], \tag{24}$$

with the coordinates defined by

$$\boldsymbol{R}(t,u) = P(T(u) > t) \text{ for } t \in <0, \infty), \ u = 1,2,..., z.$$

In the case when the system operation time θ is large enough, the coordinates of the unconditional reliability function of the system defined by (24) are given by

$$\boldsymbol{R}(t,u) \cong \sum_{b=1}^{\nu} p_b [\boldsymbol{R}(t,u)]^{(b)} \text{ for } t \ge 0,\ u = 1,2,...,z, \tag{25}$$

where $[\boldsymbol{R}(t,u)]^{(b)}$, $u = 1,2,...,z$, $b = 1,2,...,\nu$, are the coordinates of the system conditional reliability functions defined by (23) and p_b, $b = 1,2,...,\nu$, are the system operation process limit transient probabilities given by (9).

Thus, the mean value $\mu(u) = E[T(u)]$ of the system unconditional lifetime $T(u)$ in the reliability state subset $\{u, u+1,...,z\}$ is given by (Kołowrocki, Soszyńska-Budny, 2011; Soszyńska, 2007b),

$$M(u) \cong \sum_{b=1}^{\nu} p_b M_b(u),\ u = 1,2,...,z, \tag{26}$$

where $M_b(u) = E[T^{(b)}(u)]$ are the mean values of the system conditional lifetimes $T^{(b)}(u)$ in the reliability state subset $\{u, u+1,...,z\}$ at the operation state z_b, $b = 1,2,...,\nu$, given by

$$M_b(u) = \int_0^{\infty} [\boldsymbol{R}(t,u)]^{(b)} dt,\ u = 1,2,...,z, \tag{27}$$

$[\boldsymbol{R}(t,u)]^{(b)}$, $u = 1,2,...,z$, $b = 1,2,...,\nu$, are defined by (23) and p_b are given by (9). Since the relationships between the system unconditional lifetimes $\overline{T}(u)$ in the particular reliability states and the system unconditional lifetimes $T(u)$ in the reliability state subsets can be expressed by

$$\overline{T}(u) = T(u) - T(u+1),\ u = 0,1,...,z-1,\ \overline{T}(z) = T(z), \tag{28}$$

then we get the following formulae for the mean values of the unconditional lifetimes of the system in particular reliability states

$$\overline{M}(u) = M(u) - M(u+1),\ u = 0,1,...,z-1,\ \overline{M}(z) = M(z), \tag{29}$$

where $M(u)$, $u = 0,1,...,z$, are given by (27).

Moreover, if $s(t)$ is the system reliability state at he moment t, $t \in <0,\infty)$, and r, $r \in \{1,2,...,z\}$, is the system critical reliability state, then the system risk function

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \le t),\ t \in <0,\infty),$$

defined as the probability that the system is in the subset of states worse than the critical state r, $r \in \{1,...,z\}$ while it was in the state z at the moment $t = 0$ is given by (Kołowrocki, Soszyńska-Budny, 2011)

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}(t,r),\ t \in <0,\infty), \tag{30}$$

where $\boldsymbol{R}(t,r)$ is the coordinate of the system unconditional reliability function given by (25) for $u = r$ and if τ is the moment when the system risk function exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{31}$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$ given by (30).
Further, we assume that the system components E_i, $i = 1,2,...,n,$ at the system operation states z_b, $b = 1,2,...,v,$ have the exponential reliability functions, i.e. their coordinates are given by

$$[R_i\ (t,u)]^{(b)} = P(T_i^{(b)}(u) > t | Z(t) = z_b) = \exp[-[\lambda_i\ (u)]^{(b)}\ t] \tag{32}$$

for $t \in < 0,\infty)$, $u = 1,2,...,z$, $b = 1,2,...,v$.
The reason for this strong assumption on the system components is that the exponential distribution has "no memory" (Kołowrocki, Soszyńska-Budny, 2011). Both of them, the assumption about the exponential reliability functions of the system components and this property, justify the following form of the formula (25) (Kołowrocki, Soszyńska-Budny, 2011)

$$\begin{aligned}\boldsymbol{R}(t,u) &\cong \sum_{b=1}^{v} p_b [\boldsymbol{R}(t,u)]^{(b)} \\ &= \sum_{b=1}^{v} p_b [\boldsymbol{R}(\exp[-[\lambda_1(u)]^{(b)} t], \exp[-[\lambda_2(u)]^{(b)} t], ..., \exp[-[\lambda_n(u)]^{(b)} t])]^{(b)}\end{aligned} \tag{33}$$

for $t \geq 0$, $u = 1,2,...,z$.

The application of the above formula and the results given in Chapter 3 of (Kołowrocki, Soszyńska-Budny, 2011) yield the following results formulated in the form of the following proposition.

Proposition 1
If components of the multi-state system at the operation state z_b, $b = 1,2,...,v,$ have the exponential reliability functions given by

$$[R_{ij}(t,\cdot)]^{(b)} = [1, [R_{ij}(t,1)]^{(b)}, \ldots, [R_{ij}(t,z)]^{(b)}],\ t \in (-\infty,\infty),\ b = 1,2,...,v,$$

where

$$[R_{ij}(t,u)]^{(b)} = \exp[-[\lambda_{ij}(u)]^{(b)} t] \text{ for } t \geq 0,\ [\lambda_{ij}(u)]^{(b)} > 0,\ i = 1,2,...,k,\ j = 1,2,...,l_i,\ u = 1,2,\ldots,z,$$
$$b = 1,2,...,v,$$

then its multistate unconditional reliability function is given by the vector:
i) for a series-parallel system with the structure shape parameters $k^{(b)}$, $l_i^{(b)}$, $i = 1,2,..., k^{(b)}$, at the operation state z_b, $b = 1,2,...,v$,

$$\boldsymbol{R}(t,\cdot) = [1,\ \boldsymbol{R}(t,1),..., \boldsymbol{R}(t,z)], \tag{34}$$

where

$$\boldsymbol{R}(t,u) \cong \sum_{b=1}^{v} \boldsymbol{p_b} \boldsymbol{R}_{k^{(b)};l_1^{(b)},l_2^{(b)},...,l_{k^{(b)}}^{(b)}}(t,u),\ u = 1,2,...,z, \tag{35}$$

$$\boldsymbol{R}_{k^{(b)};l_1^{(b)},l_2^{(b)},...,l_{k^{(b)}}^{(b)}}(t,u) = 1 - \prod_{i=1}^{k^{(b)}} [1 - \prod_{j=1}^{l_i^{(b)}} [R_{ij}(t,u)]^{(b)}]$$

$$= 1 - \prod_{i=1}^{k^{(b)}} [1 - \exp[-\sum_{j=1}^{l_i^{(b)}} [\lambda_{ij}(u)]^{(b)} t]], \; t \ge 0, \; u = 1,2,...,z, \; b = 1,2,...,\nu; \qquad (36)$$

ii) for a series-"*m* out of *k*" system with the structure shape parameters $m^{(b)}$, $k^{(b)}$, $l_i^{(b)}$, $i = 1,2,..., k^{(b)}$, at the operation state z_b, $b = 1,2,...,\nu$,

$$\boldsymbol{R}(t,\cdot) = [1, \; \boldsymbol{R}(t,1),..., \boldsymbol{R}(t,z)], \qquad (37)$$

where

$$\boldsymbol{R}(t,u) \cong \sum_{\boldsymbol{b=1}}^{\nu} \boldsymbol{p_b} \boldsymbol{R}^{m^{(b)}}_{k^{(b)}; l_1^{(b)}, l_2^{(b)}, ..., l_{k^{(b)}}^{(b)}}(t,u), \; u = 1,2,...,z, \qquad (38)$$

$$\boldsymbol{R}^{m^{(b)}}_{k^{(b)}; l_1^{(b)}, l_2^{(b)}, ..., l_{k^{(b)}}^{(b)}}(t,u) = 1 - \sum_{\substack{r_1, r_2, ..., r_k = 0 \\ r_1 + r_2 + ... + r_k \le m^{(b)} - 1}}^{1} \prod_{i=1}^{k^{(b)}} [\prod_{j=1}^{l_i^{(b)}} [R_{ij}(t,u)]^{(b)} t]]^{r_i} \; [1 - \prod_{j=1}^{l_i^{(b)}} R_{ij}(t,u)]^{(b)}]]^{1-r_i}$$

$$= 1 - \sum_{\substack{r_1, r_2, ..., r_k = 0 \\ r_1 + r_2 + ... + r_k \le m^{(b)} - 1}}^{1} \prod_{i=1}^{k^{(b)}} [\prod_{j=1}^{l_i^{(b)}} \exp[-[\lambda_{ij}(u)]^{(b)} t]]^{r_i}$$

$$\cdot [1 - \prod_{j=1}^{l_i^{(b)}} \exp[-[\lambda_{ij}(u)]^{(b)} t]]^{1-r_i}, \; t \ge 0, u = 1,2,...,z, \; b = 1,2,...,\nu. \qquad (39)$$

- ***Example*** (continuation)
- In Section 2, it is fixed that the exemplary system reliability structure and its subsystems and components reliability depend on its changing in time operation states. Considering the assumptions and agreements of these sections, we assume that its subsystems S_υ, $\upsilon = 1,2,$ are composed of four-state, i.e. z = 3, components $E_{ij}^{(\upsilon)}$, $\upsilon = 1,2,$ having the conditional reliability functions given by the vector

$$[R_{ij}^{(\upsilon)}(t,\cdot)]^{(b)} = [1, \; [R_{ij}^{(\upsilon)}(t,1)]^{(b)}, [R_{ij}^{(\upsilon)}(t,2)]^{(b)}, [R_{ij}^{(\upsilon)}(t,3)]^{(b)}], \; b = 1,2,3,4,$$

with the exponential co-ordinates

$$[R_{ij}^{(\upsilon)}(t,1)]^{(b)} = \exp[-[\lambda_{ij}^{(\upsilon)}(1)]^{(b)}],$$
$$[R_{ij}^{(\upsilon)}(t,2)]^{(b)} = \exp[-[\lambda_{ij}^{(\upsilon)}(2)]^{(b)}],$$
$$[R_{ij}^{(\upsilon)}(t,3)]^{(b)} = \exp[-[\lambda_{ij}^{(\upsilon)}(3)]^{(b)}],$$

different at various operation states z_b, $b = 1,2,3,4,$ and with the intensities of departure from the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, respectively

$$[\lambda_{ij}^{(\upsilon)}(1)]^{(b)}, \; [\lambda_{ij}^{(\upsilon)}(2)]^{(b)}, \; [\lambda_{ij}^{(\upsilon)}(3)]^{(b)}, \; b = 1,2,3,4.$$

The influence of the system operation states changing on the changes of the system reliability structure and its components reliability functions is as follows.

At the system operation state z_1, the system is composed of the series-parallel subsystem S_1 with the structure showed in Figure 2, containing two identical series subsystems ($k^{(1)} = 2$), each composed of sixty components ($l_1^{(1)} = 60$, $l_2^{(1)} = 60$) with the exponential reliability functions. In both series subsystems of the subsystem S_1 there are respectively:

- the components $E_{ij}^{(1)}$, $i = 1,2$, $j = 1,2,...,40$, with the conditional reliability function coordinates

$$[R_{ij}^{(1)}(t,1)]^{(1)} = \exp[-0.0008t],\ [R_{ij}^{(1)}(t,2)]^{(1)} = \exp[-0.0009t],$$
$$[R_{ij}^{(1)}(t,3)]^{(1)} = \exp[-0.0010t],\ i = 1,2,\ j = 1,2,...,40;$$

- the components $E_{ij}^{(1)}$, $i = 1,2$, $j = 41,42,...,60$, with the conditional reliability function coordinates

$$[R_{ij}^{(1)}(t,1)]^{(1)} = \exp[-0.0011t],\ [R_{ij}^{(1)}(t,2)]^{(1)} = \exp[-0.0012t],$$
$$[R_{ij}^{(1)}(t,3)]^{(1)} = \exp[-0.0013t],\ i = 1,2,\ j = 41,42,...,60.$$

Thus, at the operational state z_1, the system is identical with the subsystem S_1 that is a four-state series-parallel system with its structure shape parameters , $l_1^{(1)} = 60$, $l_2^{(1)} = 60$, and according to the formulae appearing after Definition 3.11 in (Kołowrocki, Soszyńska-Budny, 2011) and (34)-(36), its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(1)} = [1, [\boldsymbol{R}(t,1)]^{(1)}, [\boldsymbol{R}(t,2)]^{(1)}, [\boldsymbol{R}(t,3)]^{(1)}],\ t \geq 0, \tag{40}$$

where

$$\begin{aligned}[\boldsymbol{R}(t,1)]^{(1)} &= \boldsymbol{R}_{2;60,60}(t,1) = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{60}[R_{ij}^{(1)}(t,1)]^{(1)}] \\ &= 1 - \prod_{i=1}^{2}[1 - \exp[-\sum_{j=1}^{60}[\lambda_{ij}^{(1)}(1)]^{(1)}t]] \\ &= 1 - [1 - \exp[-[0.0008 \cdot 40 + 0.0011 \cdot 20]t]]^2 \\ &= 1 - [1 - \exp[-0.054t]]^2 \\ &= 2\exp[-0.054t] - \exp[-0.108t],\end{aligned} \tag{41}$$

$$\begin{aligned}[\boldsymbol{R}(t,2)]^{(1)} &= \boldsymbol{R}_{2;60,60}(t,2) = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{60}[R_{ij}^{(1)}(t,2)]^{(1)}] \\ &= 1 - \prod_{i=1}^{2}[1 - \exp[-\sum_{j=1}^{60}[\lambda_{ij}^{(1)}(2)]^{(1)}t]] \\ &= 1 - [1 - \exp[-[0.0009 \cdot 40 + 0.0012 \cdot 20]t]]^2 \\ &= 1 - [1 - \exp[-0.060t]]^2 \\ &= 2\exp[-0.060t] - \exp[-0.120t],\end{aligned} \tag{42}$$

$$\begin{aligned}[\boldsymbol{R}(t,3)]^{(1)} &= \boldsymbol{R}_{2;60,60}(t,3) = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{60}[R_{ij}^{(1)}(t,3)]^{(1)}] \\ &= 1 - \prod_{i=1}^{2}[1 - \exp[-\sum_{j=1}^{60}[\lambda_{ij}^{(1)}(3)]^{(1)}t]]\end{aligned}$$

$$= 1-[1-\exp[-[0.0010\cdot 40+0.0013\cdot 20]t]]^2$$
$$= 1-[1-\exp[-0.066t]]^2$$
$$= 2\exp[-0.066t]-\exp[-0.132t].$$
(43)

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$ at the operation state z_1, calculated from the results given by (40)-(43), according to (27), respectively are:

$$M_1(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(1)}\,dt = 2/0.054 - 1/0.108 \cong 27.78, \quad (44)$$

$$M_1(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(1)}dt = 2/0.060 - 1/0.120 = 25.00, \quad (45)$$

$$M_1(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(1)}dt = 2/0.066 - 1/0.132 \cong 22.73.$$
(46)

At the system operation state z_2, the system is composed of the series-parallel subsystem S_2 with the structure showed in Figure 3, containing four identical series subsystems ($k^{(2)} = 4$), each composed of eighty components ($l_1^{(2)} = 80$, $l_2^{(2)} = 80$, $l_3^{(2)} = 80$, $l_4^{(2)} = 80$) with the exponential reliability functions. In all series subsystems of the subsystem S_2 there are respectively:

- the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 1,2,...,40,$ with the conditional reliability function co-ordinates

$$[R_{ij}^{(2)}(t,1)]^{(2)} = \exp[-0.0014t],\ [R_{ij}^{(2)}(t,2)]^{(2)} = \exp[-0.0015t],$$
$$[R_{ij}^{(2)}(t,3)]^{(2)} = \exp[-0.0016t],\ i = 1,2,3,4,\ \ j = 1,2,...,40;$$

- the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 21,22,...,40,$ with the conditional reliability function co-ordinates

$$[R_{ij}^{(2)}(t,1)]^{(2)} = \exp[-0.0018t],\ [R_{ij}^{(2)}(t,2)]^{(2)} = \exp[-0.0020t],$$
$$[R_{ij}^{(2)}(t,3)]^{(2)} = \exp[-0.0022t],\ i = 1,2,3,4,\ j = 41,42,...,80.$$

Thus, at the operation state z_2, the system is identical with the subsystem S_2 that is a four-state series-parallel system with its structure shape parameters $k^{(2)} = 4$), $l_1^{(2)} = 80$, $l_2^{(2)} = 80$, $l_3^{(2)} = 80$, $l_4^{(2)} = 80$, and according to the formulae appearing after Definition 3.11 in (Kołowrocki, Soszyńska-Budny, 2011) and (34)-(36), its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(2)} = [1, [\boldsymbol{R}(t,1)]^{(2)}, [\boldsymbol{R}(t,2)]^{(2)}, [\boldsymbol{R}(t,3)]^{(2)}],\ t \geq 0, \quad (47)$$

where

$$[\boldsymbol{R}(t,1)]^{(2)} = \boldsymbol{R}_{4;80,80,80,80}(t,1) = 1-\prod_{i=1}^{4}[1-\prod_{j=1}^{80}[R_{ij}^{(2)}(t,1)]^{(2)}]$$
$$= 1-\prod_{i=1}^{4}[1-\exp[-\sum_{j=1}^{80}[\lambda_{ij}^{(2)}(1)]^{(2)}t]]$$
$$= 1-[1-\exp[-[0.0014\cdot 40+0.0018\cdot 40]t]^{4}$$
$$= 1-[1-\exp[-0.128t]]^{4}$$
$$= 4\exp[-0.128t]-6\exp[-0.256t]+4\exp[-0.384t]-\exp[-0.512t], \quad (48)$$

$$[\boldsymbol{R}(t,2)]^{(2)} = \boldsymbol{R}_{4;80,80,80,80}(t,2) = 1-\prod_{i=1}^{4}[1-\prod_{j=1}^{80}[R_{ij}^{(2)}(t,2)]^{(2)}]$$
$$= 1-\prod_{i=1}^{4}[1-\exp[-\sum_{j=1}^{80}[\lambda_{ij}^{(2)}(2)]^{(2)}t]]$$
$$= 1-[1-\exp[-[0.0015\cdot 40+0.0020\cdot 40]t]^{4}$$
$$= 1-[1-\exp[-0.140t]]^{4}$$
$$= 4\exp[-0.140t]-6\exp[-0.280t]+4\exp[-0.420t]-\exp[-0.560t], \quad (49)$$

$$[\boldsymbol{R}(t,3)]^{(2)} = \boldsymbol{R}_{4;80,80,80,80}(t,3) = 1-\prod_{i=1}^{4}[1-\prod_{j=1}^{80}[R_{ij}^{(2)}(t,3)]^{(2)}]$$
$$= 1-\prod_{i=1}^{4}[1-\exp[-\sum_{j=1}^{80}[\lambda_{ij}^{(2)}(3)]^{(2)}t]]$$
$$= 1-[1-\exp[-[0.0016\cdot 40+0.0022\cdot 40]t]^{4}$$
$$= 1-[1-\exp[-0.152t]]^{4}$$
$$= 4\exp[-0.152t]-6\exp[-0.304t]+4\exp[-0.456t]-\exp[-0.608t]. \quad (50)$$

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$ at the operation state z_1, calculated from the results given by (47)-(50), according to (27), respectively are:

$$M_2(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(2)}\,dt = 4/0.128 - 6/0.256 + 4/0.384 - 1/0.512 \cong 16.27, \quad (51)$$

$$M_2(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(2)}dt = 4/0.140 - 6/0.280 + 4/0.420 - 1/0.560 \cong 14.88, \quad (52)$$

$$M_2(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(2)}dt = 4/0.152 - 6/0.304 + 4/0.456 - 1/0.608 \cong 13.71. \quad (53)$$

At the system operation state z_3, the system is a series system with the structure showed in Figure 1, composed of two series-parallel subsystems S_1 and S_2 illustrated respectively in Figure 2 and Figure 3.

The subsystem S_1 with the structure showed in Figure 2, consists of two identical series subsystems ($k^{(3)} = 2$), each composed of sixty components ($l_1^{(3)} = 60$, $l_2^{(3)} = 60$) with the exponential reliability functions. In both series subsystems of the subsystem S_1 there are respectively:

- the components $E_{ij}^{(1)}$, $i = 1,2$, $j = 1,2,...,40$, with the conditional reliability function co-ordinates

$$[R_{ij}^{(1)}(t,1)]^{(3)} = \exp[-0.0009t], [R_{ij}^{(1)}(t,2)]^{(3)} = \exp[-0.0010t],$$
$$[R_{ij}^{(1)}(t,3)]^{(3)} = \exp[-0.0011t],\ i = 1,2,\ j = 1,2,...,40;$$

- the components $E_{ij}^{(1)}$, $i = 1,2$, $j = 41,42,...,60$, with the conditional reliability function co-ordinates

$$[R_{ij}^{(1)}(t,1)]^{(3)} = \exp[-0.0012t],\ [R_{ij}^{(1)}(t,2)]^{(3)} = \exp[-0.0014t],$$
$$[R_{ij}^{(1)}(t,3)]^{(3)} = \exp[-0.0016t],\ i = 1,2,\ j = 41,42,...,60.$$

Thus, at the operation state z_3, the subsystem S_1 is a four-state series-parallel system with its structure shape parameters $k^{(3)} = 2$, $l_1^{(3)} = 60$, $l_2^{(3)} = 60$, and according to the formulae appearing after Definition 3.11 in [18] and (34)-(36), its conditional reliability function is given by

$$[\boldsymbol{R}^{(1)}(t,\cdot)]^{(3)} = [1, [\boldsymbol{R}^{(1)}(t,1)]^{(3)}, [\boldsymbol{R}^{(1)}(t,2)]^{(3)}, [\boldsymbol{R}^{(1)}(t,3)]^{(3)}],\ t \geq 0,$$
(54)

where

$$[\boldsymbol{R}^{(1)}(t,1)]^{(3)} = \boldsymbol{R}_{2;60,60}(t,1) = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{60}[R_{ij}^{(1)}(t,1)]^{(3)}]$$
$$= 1 - \prod_{i=1}^{2}[1 - \exp[-\sum_{j=1}^{60}[\lambda_{ij}^{(1)}(1)]^{(3)}t]]$$
$$= 1 - [1 - \exp[-[0.0009 \cdot 40 + 0.0012 \cdot 20]t]^2$$
$$= 1 - [1 - \exp[-0.060t]]^2$$
$$= 2\exp[-0.060t] - \exp[-0.120t], \quad (55)$$

$$[\boldsymbol{R}^{(1)}(t,2)]^{(3)} = \boldsymbol{R}_{2;60,60}(t,2) = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{60}[R_{ij}^{(1)}(t,2)]^{(3)}]$$
$$= 1 - \prod_{i=1}^{2}[1 - \exp[-\sum_{j=1}^{630}[\lambda_{ij}^{(1)}(2)]^{(3)}t]]$$
$$= 1 - [1 - \exp[-[0.0010 \cdot 40 + 0.0014 \cdot 20]t]^2$$
$$= 1 - [1 - \exp[-0.068t]]^2$$
$$= 2\exp[-0.068t] - \exp[-0.136t], \quad (56)$$

$$[\boldsymbol{R}(t,3)]^{(1)} = \boldsymbol{R}_{2;60,60}(t,3) = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{60}[R_{ij}^{(1)}(t,3)]^{(1)}]$$
$$= 1 - \prod_{i=1}^{2}[1 - \exp[-\sum_{j=1}^{60}[\lambda_{ij}^{(1)}(3)]^{(1)}t]]$$
$$= 1 - [1 - \exp[-[0.0011 \cdot 40 + 0.0016 \cdot 20]t]^2$$
$$= 1 - [1 - \exp[-0.076t]]^2$$
$$= 2\exp[-0.076t] - \exp[-0.152t]. \quad (57)$$

The subsystem S_2 with the structure showed in Figure 3, consists of four identical series subsystems ($k^{(3)} = 4$), each composed of eighty components ($l_1^{(3)} = 80$, $l_2^{(3)} = 80$, $l_3^{(3)} = 80$,

$l_4^{(3)} = 80$) with the exponential reliability functions given below. In all series subsystems of the subsystem S_2 there are respectively:

- the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 1,2,...,40$, with the conditional reliability function co-ordinates

$$[R_{ij}^{(2)}(t,1)]^{(3)} = \exp[-0.0010t],\ [R_{ij}^{(2)}(t,2)]^{(3)} = \exp[-0.0011t],$$
$$[R_{ij}^{(2)}(t,3)]^{(3)} = \exp[-0.0012t],\ i = 1,2,3,4,\ \ j = 1,2,...,40;$$

- the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 41,42,...,80$, with the conditional reliability function co-ordinates

$$[R_{ij}^{(2)}(t,1)]^{(3)} = \exp[-0.0014t],\ [R_{ij}^{(2)}(t,2)]^{(3)} = \exp[-0.0016t],$$
$$[R_{ij}^{(2)}(t,3)]^{(3)} = \exp[-0.0018t],\ i = 1,2,3,4,\ j = 41,42,...,80.$$

Thus, at the operation state z_3, the subsystem S_2 is a four-state series-parallel system with its structure shape parameters $k^{(3)} = 4$, $l_1^{(3)} = 80$, $l_2^{(3)} = 80$, $l_3^{(3)} = 80$, $l_4^{(3)} = 80$, and according to the formulae appearing after Definition 3.11 in (Kołowrocki, Soszyńska-Budny, 2011) and (34)-(36), its conditional reliability function is given by

$$[\boldsymbol{R}^{(2)}(t,\cdot)]^{(3)} = [1,\ [\boldsymbol{R}^{(2)}(t,1)]^{(3)}, [\boldsymbol{R}^{(2)}(t,2)]^{(3)}, [\boldsymbol{R}^{(2)}(t,3)]^{(3)}],\ t \geq 0,$$
(58)

where

$$[\boldsymbol{R}^{(2)}(t,1)]^{(3)} = \boldsymbol{R}_{4;80,80,80,80}(t,1) = 1 - \prod_{i=1}^{4}[1 - \prod_{j=1}^{80}[R_{ij}^{(2)}(t,1)]^{(3)}]$$
$$= 1 - \prod_{i=1}^{4}[1 - \exp[-\sum_{j=1}^{80}[\lambda_{ij}^{(2)}(1)]^{(3)}t]]$$
$$= 1 - [1 - \exp[-[0.0010 \cdot 40 + 0.0014 \cdot 40]t]^4$$
$$= 1 - [1 - \exp[-0.096t]]^4$$
$$= 4\exp[-0.096t] - 6\exp[-0.192t] + 4\exp[-0.288t] - \exp[-0.384t],$$
(59)

$$[\boldsymbol{R}^{(2)}(t,2)]^{(3)} = \boldsymbol{R}_{4;80,80,80,80}(t,2) = 1 - \prod_{i=1}^{4}[1 - \prod_{j=1}^{80}[R_{ij}^{(2)}(t,2)]^{(3)}]$$
$$= 1 - \prod_{i=1}^{4}[1 - \exp[-\sum_{j=1}^{80}[\lambda_{ij}^{(2)}(2)]^{(3)}t]]$$
$$= 1 - [1 - \exp[-[0.0011 \cdot 40 + 0.0016 \cdot 40]t]^4$$
$$= 1 - [1 - \exp[-0.108t]]^4$$
$$= 4\exp[-0.108t] - 6\exp[-0.216t] + 4\exp[-0.324t] - \exp[-0.432t], \quad (60)$$

$$[\boldsymbol{R}^{(2)}(t,3)]^{(3)} = \boldsymbol{R}_{4;80,80,80,80}(t,3) = 1 - \prod_{i=1}^{4}[1 - \prod_{j=1}^{80}[R_{ij}^{(2)}(t,3)]^{(3)}]$$
$$= 1 - \prod_{i=1}^{4}[1 - \exp[-\sum_{j=1}^{80}[\lambda_{ij}^{(2)}(3)]^{(3)}t]]$$

$$= 1-[1-\exp[-[0.0012\cdot 40+0.0018\cdot 40]t]^4$$
$$=1-[1-\exp[-0.120t]]^4$$
$$= 4\exp[-0.120t]-6\exp[-0.240t]+4\exp[-0.360t]-\exp[-0.480t]. \quad (61)$$

Considering that the system at the operation state z_3 is a four-state series system composed of subsystems S_1 and S_2, after applying the formulae appearing after Definition 3.4 in (Kołowrocki, Soszyńska-Budny, 2011) and (54)-(57) and (58)-(61), its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(3)} = [1, [\boldsymbol{R}(t,1)]^{(3)}, [\boldsymbol{R}(t,2)]^{(3)}, [\boldsymbol{R}(t,3)]^{(3)}],\ t \ge 0, \quad (62)$$

where

$$[\boldsymbol{R}(t,1)]^{(3)} = \overline{\boldsymbol{R}}_2(t,1) = [\boldsymbol{R}^{(1)}(t,1)]^{(3)}\,[\boldsymbol{R}^{(2)}(t,1)]^{(3)}$$
$$= 8\exp[-0.156t]-12\exp[-0.252t]+8\exp[-0.348t]$$
$$-2\exp[-0.424t]-4\exp[-0.216t]+6\exp[-0.312t]$$
$$-4\exp[-0.408t]+\exp[-0.504t],$$
(63)

$$[\boldsymbol{R}(t,2)]^{(3)} = \overline{\boldsymbol{R}}_2(t,2) = [\boldsymbol{R}^{(1)}(t,2)]^{(3)}\,[\boldsymbol{R}^{(2)}(t,2)]^{(3)}$$
$$= 8\exp[-0.176t]-12\exp[-0.284t]+8\exp[-0.392t]$$
$$-2\exp[-0.500t]-4\exp[-0.236t]+6\exp[-0.344t]$$
$$-4\exp[-0.452t]+\exp[-0.560t], \quad (64)$$

$$[\boldsymbol{R}(t,3)]^{(3)} = \overline{\boldsymbol{R}}_2(t,3) = [\boldsymbol{R}^{(1)}(t,3)]^{(3)}\,[\boldsymbol{R}^{(2)}(t,3)]^{(3)}$$
$$= 8\exp[-0.196t]-12\exp[-0.316t]+8\exp[-0.436t]$$
$$-2\exp[-0.556t]-4\exp[-0.256t]+6\exp[-0.376t]$$
$$-4\exp[-0.496t]+\exp[-0.616t].$$
(65)

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$ at the operation state z_3, calculated from the results given by (62)-(65), according to (27), respectively are:

$$M_3(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(3)}dt = 8/0.156 - 12/0.252 + 8/0.348 - 2/0.424 - 4/0.216$$
$$+ 6/0.312 - 4/0.408 + 1/0.504 \cong 14.82, \quad (66)$$

$$M_3(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(3)}dt = 8/0.176 - 12/0.284 + 8/0.392 - 2/0.500 - 4/0.236$$
$$+ 6/0.344 - 4/0.452 + 1/0.560 \cong 13.04, \quad (67)$$

$$M_3(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(3)}dt = 8/0.196 - 12/0.316 + 8/0.436 - 2/0.556 - 4/0.256$$
$$+ 6/0.376 - 4/0.496 + 1/0.616 \cong 11.48. \quad (68)$$

At the system operation state z_4, the system is a series system with the scheme showed in Figure 1, composed of the subsystem S_1 and S_2 illustrated respectively in Figure 2 and Figure 3, whereas the subsystem S_1 is a series-parallel system and the subsystem S_2 is a series-“2 out of 4” system.
The subsystem S_1 consists of two identical series subsystems ($k^{(4)} = 2$), each composed of sixty components ($l_1^{(4)} = 60$, $l_2^{(4)} = 60$) with the exponential reliability functions the same as at the operation state z_1. Thus, according to (54)-(57), the subsystem S_1 conditional reliability function at the operation state z_4, is given by

$$[\boldsymbol{R}^{(1)}(t,\cdot)]^{(4)} = [1, [\boldsymbol{R}^{(1)}(t,1)]^{(4)}, [\boldsymbol{R}^{(1)}(t,2)]^{(4)}, [\boldsymbol{R}^{(1)}(t,3)]^{(4)}], \; t \geq 0, \quad (69)$$

where

$$[\boldsymbol{R}^{(1)}(t,1)]^{(4)} = 2\exp[-0.054t] - \exp[-0.108t], \quad (70)$$

$$[\boldsymbol{R}^{(1)}(t,2)]^{(4)} = 2\exp[-0.060t] - \exp[-0.120t],$$
(71)

$$[\boldsymbol{R}(t,3)]^{(4)} = 2\exp[-0.066t] - \exp[-0.132t]. \quad (72)$$

The subsystem S_2 consists of four identical series subsystems ($k^{(4)} = 4$), each composed of eighty components ($l_1^{(4)} = 80$, $l_2^{(4)} = 80$, $l_3^{(4)} = 80$, $l_4^{(4)} = 80$) with the exponential reliability functions the same as at the operation state z_2 and is a series-“2 out of 4” system ($m = 2$). Thus, at the operation state z_4, the subsystem S_2 is a four-state series-“2 out of 4” system, with its structure shape parameters ($k^{(4)} = 4$), each composed of eighty components $l_1^{(4)} = 80$, $l_2^{(4)} = 80$, $l_3^{(4)} = 80$, $l_4^{(4)} = 80$, and according to the formulae appearing after Definition 8.1 in (Kołowrocki, Soszyńska-Budny, 2011) and (37)-(39), its conditional reliability function is given by

$$[\boldsymbol{R}^{(2)}(t,\cdot)]^{(4)} = [1, \; [\boldsymbol{R}^{(2)}(t,1)]^{(4)}, [\boldsymbol{R}^{(2)}(t,2)]^{(4)}, [\boldsymbol{R}^{(2)}(t,3)]^{(4)}], \; t \geq 0,$$
(73)

where

$$[\boldsymbol{R}^{(2)}(t,1)]^{(4)} = \boldsymbol{R}^{2}_{4;80,80,80,80}(t,1) = 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0 \\ r_1+r_2+r_3+r_4 \leq 1}}^{1} \prod_{i=1}^{4} [\prod_{j=1}^{80} [R_{ij}^{(2)}(t,1)]^{(4)}]^{r_i} [1 - \prod_{j=1}^{80} [R_{ij}^{(2)}(t,1)]^{(4)}]^{1-r_i}$$

$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0 \\ r_1+r_2+r_3+r_4 \leq 1}}^{1} \prod_{i=1}^{4} \exp[-r_i \sum_{j=1}^{80} [\lambda_{ij}^{(2)}(1)]^{(4)} t][1 - \exp[-\sum_{j=1}^{80} [\lambda_{ij}^{(2)}(1)]^{(4)} t]]^{1-r_i}$$

$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0 \\ r_1+r_2+r_3+r_4 \leq 1}}^{1} \prod_{i=1}^{4} \exp[-r_i[0.0014 \cdot 40 + 0.0018 \cdot 40]t]$$

$$\cdot [1 - \exp[-[0.0014 \cdot 40 + 1.0018 \cdot 40]t]]^{1-r_i}$$

$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0 \\ r_1+r_2+r_3+r_4 \leq 1}}^{1} \prod_{i=1}^{4} \exp[-r_i 0.128t][1 - \exp[-0.128t]]^{1-r_i}$$

$$= 1 - \sum_{i=0}^{1} \binom{4}{i} \exp[-i \cdot 0.128t][1 - \exp[-0.128t]]^{4-i}$$

$$= 1 - \exp[-0 \cdot 0.128\,t]\,[1 - \exp[-0.128t]^4 - 4\exp[-1 \cdot 0.128\,t]\,[1 - \exp[-0.128t]^3$$
$$= 6\exp[-0.256t] - 8\exp[-0.384t] + 3\exp[-0.512t], \quad (74)$$

$$[\boldsymbol{R}^{(2)}(t,2)]^{(4)} = \boldsymbol{R}^{2}_{4;80,80,80,80}(t,2) = 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4}[\prod_{j=1}^{80}[R^{(2)}_{ij}(t,2)]^{(4)}]^{r_i}[1 - \prod_{j=1}^{80}[R^{(2)}_{ij}(t,2)]^{(4)}]^{1-r_i}$$
$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4} \exp[-r_i \sum_{j=1}^{80}[\lambda^{(2)}_{ij}(2)]^{(4)}t][1 - \exp[-\sum_{j=1}^{80}[\lambda^{(2)}_{ij}(2)]^{(4)}t]]^{1-r_i}$$
$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4} \exp[-r_i[0.0015 \cdot 40 + 0.0020 \cdot 40]t]$$
$$\cdot[1 - \exp[-[0.0015 \cdot 40 + 1.0020 \cdot 40]t]]^{1-r_i}$$
$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4} \exp[-r_i 0.140t][1 - \exp[-0.140t]]^{1-r_i}$$
$$= 1 - \sum_{i=0}^{1}\binom{4}{i}\exp[-i \cdot 0.140t][1 - \exp[-0.140t]]^{4-i}$$
$$= 1 - \exp[-0 \cdot 0.140\,t]\,[1 - \exp[-0.140t]^4 - 4\exp[-1 \cdot 0.140\,t]\,[1 - \exp[-0.140t]^3$$
$$= 6\exp[-0.280t] - 8\exp[-0.420t] + 3\exp[-0.560t], \quad (75)$$

$$[\boldsymbol{R}^{(2)}(t,3)]^{(4)} = \boldsymbol{R}^{2}_{4;80,80,80,80}(t,3) = 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4}[\prod_{j=1}^{80}[R^{(2)}_{ij}(t,3)]^{(4)}]^{r_i}[1 - \prod_{j=1}^{80}[R^{(2)}_{ij}(t,3)]^{(4)}]^{1-r_i}$$
$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4} \exp[-r_i \sum_{j=1}^{80}[\lambda^{(2)}_{ij}(3)]^{(4)}t][1 - \exp[-\sum_{j=1}^{80}[\lambda^{(2)}_{ij}(3)]^{(4)}t]]^{1-r_i}$$
$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4} \exp[-r_i[0.0016 \cdot 40 + 0.0022 \cdot 40]t]$$
$$\cdot[1 - \exp[-[0.0016 \cdot 40 + 1.0022 \cdot 40]t]]^{1-r_i}$$
$$= 1 - \sum_{\substack{r_1,r_2,r_3,r_4=0\\ r_1+r_2+r_3+r_4\le 1}}^{1} \prod_{i=1}^{4} \exp[-r_i 0.152t][1 - \exp[-0.152t]]^{1-r_i}$$
$$= 1 - \sum_{i=0}^{1}\binom{4}{i}\exp[-i \cdot 0.152t][1 - \exp[-0.152t]]^{4-i}$$
$$= 1 - \exp[-0 \cdot 0.152\,t]\,[1 - \exp[-0.152t]^4 - 4\exp[-1 \cdot 0.152\,t]\,[1 - \exp[-0.152t]^3$$
$$= 6\exp[-0.304t] - 8\exp[-0.456t] + 3\exp - 0.608t]. \quad (76)$$

Considering that the system at the operation state z_4 is a four-state series system composed of subsystems S_1 and S_2, after applying the formulae appearing after Definition 3.4 in (Kołowrocki, Soszyńska-Budny, 2011) and (69)-(72) and (73)-(76), its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(4)} = [1, [\boldsymbol{R}(t,1)]^{(4)}, [\boldsymbol{R}(t,2)]^{(4)}, [\boldsymbol{R}(t,3)]^{(4)}],\ t \ge 0, \quad (77)$$

where

$$[\boldsymbol{R}(t,1)]^{(4)} = \overline{\boldsymbol{R}}_2(t,1) = [\boldsymbol{R}^{(1)}(t,1)]^{(4)}\,[\boldsymbol{R}^{(2)}(t,1)]^{(4)}$$
$$= 12\exp[-0.310t] - 6\exp[-0.364t] - 16\exp[-0.438t]$$

$$+ 8\exp[-0.492t] + 6\exp[-0.566t] - 3\exp[-0.620t], \quad (78)$$

$$[\boldsymbol{R}(t,2)]^{(4)} = \overline{\boldsymbol{R}}_2(t,2) = [\boldsymbol{R}^{(1)}(t,2)]^{(4)} [\boldsymbol{R}^{(2)}(t,2)]^{(4)}$$
$$= 12\exp[-0.340t] - 6\exp[-0.400t] - 16\exp[-0.480t]$$
$$+ 8\exp[-0.540t] + 6\exp[-0.620t] - 3\exp[-0.680t], \quad (79)$$

$$[\boldsymbol{R}(t,3)]^{(4)} = \overline{\boldsymbol{R}}_2(t,3) = [\boldsymbol{R}^{(1)}(t,3)]^{(4)} [\boldsymbol{R}^{(2)}(t,3)]^{(4)}$$
$$= 12\exp[-0.370t] - 6\exp[-0.436t] - 16\exp[-0.522t]$$
$$+ 8\exp[-0.588t] + 6\exp[-0.674t] - 3\exp[-0.740t]. \quad (80)$$

The mean values of the system sojourn times $T(u)$ in the reliability state subsets after applying the formula (77)-(80) and (27), are:

$$M_4(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(4)} dt = 12/0.310 - 6/0.364 - 16/0.438 + 8/0.492 + 6/0.566 - 3/0.620$$
$$\cong 7.72, \quad (81)$$

$$M_4(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(4)} dt = 12/0.340 - 6/0.400 - 16/0.480 + 8/0.540 + 6/0.620 - 3/0.680$$
$$\cong 7.04, \quad (82)$$

$$M_4(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(4)} dt = 12/0.370 - 6/0.436 - 16/0.522 + 8/0.588 + 6/0.674 - 3/0.740$$
$$\cong 6.47. \quad (83)$$

In the case when the system operation time is large enough its unconditional four-state reliability function is given by the vector

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1), \boldsymbol{R}(t,2), \boldsymbol{R}(t,3)], \; t \geq 0, \quad (84)$$

where according to (25) and considering the exemplary system operation process transient probabilities at the operation states determined by (18), the vector co-ordinates are given respectively by

$$\boldsymbol{R}(t,1) = p_1[\boldsymbol{R}(t,1)]^{(1)} + p_2[\boldsymbol{R}(t,1)]^{(2)} + p_3[\boldsymbol{R}(t,1)]^{(3)} + p_4[\boldsymbol{R}(t,1)]^{(4)}$$
$$= 0.190 \cdot [\boldsymbol{R}(t,1)]^{(1)} + 0.043 \cdot [\boldsymbol{R}(t,1)]^{(2)} + 0.312 \cdot [\boldsymbol{R}(t,1)]^{(3)} + 0.455 \cdot [\boldsymbol{R}(t,1)]^{(4)} \quad (85)$$
for $t \geq 0$,

$$\boldsymbol{R}(t,2) = p_1[\boldsymbol{R}(t,2)]^{(1)} + p_2[\boldsymbol{R}(t,2)]^{(2)} + p_3[\boldsymbol{R}(t,2)]^{(3)} + p_4[\boldsymbol{R}(t,2)]^{(4)}$$
$$= 0.190 \cdot [\boldsymbol{R}(t,2)]^{(1)} + 0.043 \cdot [\boldsymbol{R}(t,2)]^{(2)} + 0.312 \cdot [\boldsymbol{R}(t,2)]^{(3)} + 0.455 \cdot [\boldsymbol{R}(t,2)]^{(4)} \quad (86)$$
for $t \geq 0$,

$$\boldsymbol{R}(t,3) = p_1[\boldsymbol{R}(t,3)]^{(1)} + p_2[\boldsymbol{R}(t,3)]^{(2)} + p_3[\boldsymbol{R}(t,3)]^{(3)} + p_4[\boldsymbol{R}(t,3)]^{(4)}$$
$$= 0.190 \cdot [\boldsymbol{R}(t,3)]^{(1)} + 0.043 \cdot [\boldsymbol{R}(t,3)]^{(2)} + 0.312 \cdot [\boldsymbol{R}(t,3)]^{(3)} + 0.455 \cdot [\boldsymbol{R}(t,3)]^{(4)} \quad (87)$$
for $t \geq 0$,

where coordinates $[\boldsymbol{R}(t,1)]^{(1)}$, $[\boldsymbol{R}(t,1)]^{(2)}$, $[\boldsymbol{R}(t,1)]^{(3)}$, $[\boldsymbol{R}(t,1)]^{(4)}$ are given by (41), (48), (62), (76), $[\boldsymbol{R}(t,2)]^{(1)}$, $[\boldsymbol{R}(t,2)]^{(2)}$, $[\boldsymbol{R}(t,2)]^{(3)}$, $[\boldsymbol{R}(t,2)]^{(4)}$ are given by (42), (49), (63), (77) and $[\boldsymbol{R}(t,3)]^{(1)}$, $[\boldsymbol{R}(t,3)]^{(2)}$, $[\boldsymbol{R}(t,3)]^{(3)}$, $[\boldsymbol{R}(t,3)]^{(4)}$ are given by (43), (50), (64), (80).
The graph of the four-state exemplary system reliability function is illustrated in Figure 4.

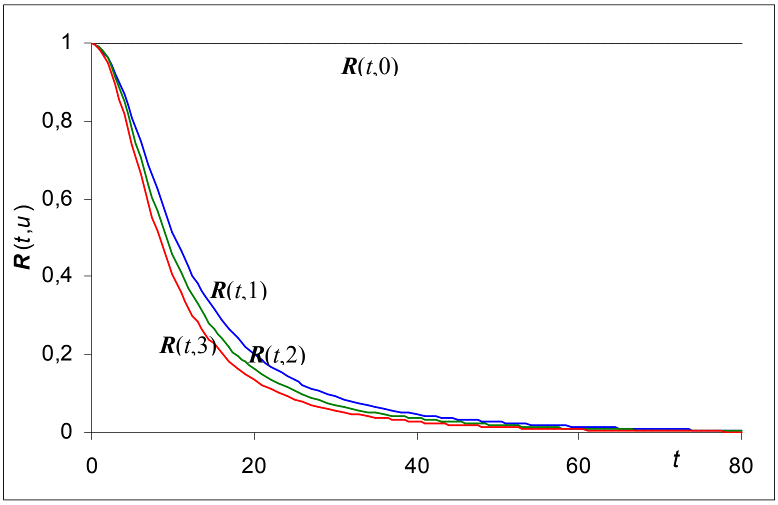

Fig. 4. The graph of the exemplary system reliability function $\boldsymbol{R}(t,\cdot)$ coordinates

The expected values and standard deviations of the system unconditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, calculated from the results given by (84)-(87), according to (27) and considering (18), (44)-(46), (51)-(53), (66)-(68) and (81)-(83), respectively are:

$$M(1) = p_1 M_1(1) + p_2 M_2(1) + p_3 M_3(1) + p_4 M_4(1)$$
$$= 0.190 \cdot 27.78 + 0.043 \cdot 16.27 + 0.312 \cdot 14.82 + 0.455 \cdot 7.72 \cong 14.11,$$
(88)

$$M(2) = p_1 M_1(2) + p_2 M_2(2) + p_3 M_3(2) + p_4 M_4(2)$$
$$= 0.190 \cdot 25.00 + 0.043 \cdot 14.88 + 0.312 \cdot 13.04 + 0.455 \cdot 7.04 \cong 12.66, \quad (89)$$

$$M(3) = p_1 M_1(3) + p_2 M_2(3) + p_3 M_3(3) + p_4 M_4(3)$$
$$= 0.190 \cdot 22.73 + 0.043 \cdot 13.71 + 0.312 \cdot 11.48 + 0.455 \cdot 6.47 \cong 11.43. \quad (90)$$

Farther, considering (29) and (88), (89) and (90), the mean values of the system unconditional lifetimes in the particular reliability states 1, 2, 3, respectively are:

$$\overline{M}(1) = M(1) - M(2) = 1.45, \ \overline{M}(2) = M(2) - M(3) = 1.23, \ \overline{M}(3) = M(3) = 11.43. \quad (91)$$

Since the critical reliability state is $r = 2$, then the system risk function, according to (30), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}(t,2) \text{ for } t \geq 0, \quad (92)$$

where $\boldsymbol{R}(t,2)$ is given by (86).

Hence, by (31), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 2.255. \tag{93}$$

The graph of the risk function $\boldsymbol{r}(t)$ of the exemplary four-state system operating at the variable conditions is given in Figure 5.

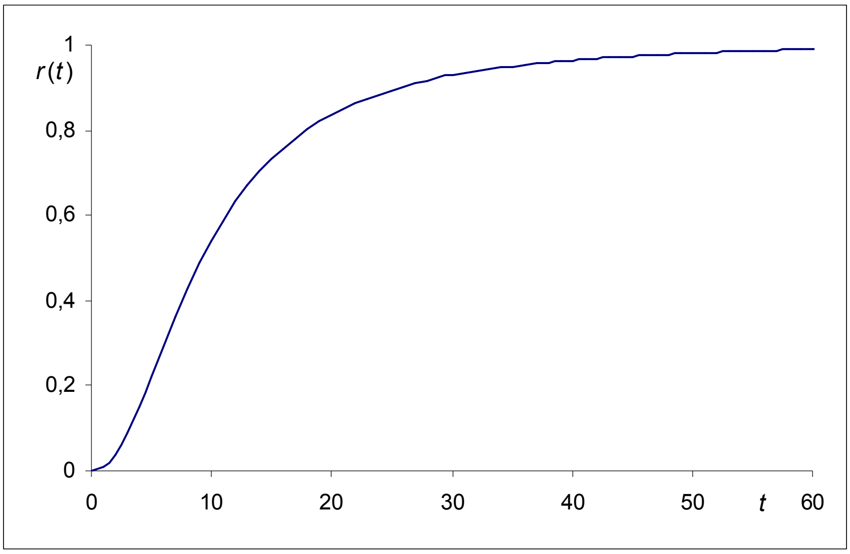

Fig. 5. The graph of the exemplary system risk function $r(t)$

4 CONCLUSION

The integrated general model of complex systems' reliability, linking their reliability models and their operation processes models and considering variable at different operation states their reliability structures and their components reliability parameters is constructed and applied to the reliability evaluation of the exemplary system composed of a series-parallel and a series-"m out of l" subsystems linked in series. The predicted reliability characteristics of the exemplary system operating at the variable conditions are different from those determined for this system operating at constant conditions. This fact justifies the sensibility of considering real systems at the variable operation conditions that is appearing out in a natural way from practice. This approach, upon the good accuracy of the systems' operation processes and the systems' components reliability parameters identification, makes their reliability prediction more precise.

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