ALGORITHMS OF ISOMETRIC SURFACES CONSTRUCTIONS

G. Tsitsiashvili, M. Osipova

•

Russia, Vladivostok, Radio str. 7, IAM FEB RAS
e-mail: guram@iam.dvo.ru , mao1975@list.ru

In this paper a problem of a construction of isometric surfaces is considered. It is important in different technical applications. This problem usually is solved for surfaces with constant Gauss curvature using so called "gold" Gauss theorem. But in practical applications it is interesting to consider more wide class of surfaces. Here this problem is solved for surfaces of cylinder, conical and spherical types and their generalizations using a criterion of isometric based on equalities of coefficients of first quadratic forms.

1. INTRODUCTION

In the differential geometry a concept of the isometric is introduced as follows. Define a surface in three dimensional space E^3 as a set described by equalities

$$X = x(u_1,u_2),\ Y = y(u_1,u_2),\ Z = z(u_1,u_2),\ (u_1,u_2) \in G \subseteq E^2,$$

where the functions x, y, z on the domain G are smooth. Say that the surfaces

$$S_1 = \{X = x_1(u_1,u_2),\ Y = y_1(u_1,u_2),\ Z = z_1(u_1,u_2),\ (u_1,u_2) \in G\},$$

$$S_2 = \{X = x_2(u_1,u_2),\ Y = y_2(u_1,u_2),\ Z = z_2(u_1,u_2),\ (u_1,u_2) \in G\}$$

are isometric if for any pair of points on the surface S_1 (a length of a geodesic curve connected these points) coincides with a distance between appropriate points on the surface S_2. So the surface S_1 may be transformed into the surface S_2 without a dilation and a compression. Last condition has important significance in a solution of modern technical problems including reliability theory also [2].

From the Gauss theorem [1] the surfaces S_1, S_2 are isometric if and only if coefficients of their first quadratic forms coincide:

$$A^1_{i,j} = A^2_{i,j},\ i,j = 1,2, \tag{1}$$

where

$$A^k_{i,j} = \left(\frac{\partial x_k}{\partial u_i}, \frac{\partial y_k}{\partial u_i}, \frac{\partial z_k}{\partial u_i}\right) \cdot \left(\frac{\partial x_k}{\partial u_j}, \frac{\partial y_k}{\partial u_j}, \frac{\partial z_k}{\partial u_j}\right),\ i,j,k = 1,2.$$

Here "$\cdot$" is the scalar product operation. But in a literature a problem of the isometric is solved mainly for surfaces with constant Gauss curvature [3, p.126] using so called the "gold" Gauss theorem. This restriction is sufficiently inconvenient in an applied sense and it is worthy to use a direct application of the condition (1) and a search of more general relations for isometric surfaces. In this paper these relations are constructed for surfaces of cylinder, conical and spherical types and for some their generalizations.

2. SURFACE OF CYLINDER TYPE

Define the first quadrant S_1 in the space E^2 by the equalities $x_1 = u_1$, $y_1 = 0$, $z_1 = u_2$ and a surface of a cylinder type S_2 by equalities

$$x_2 = a(u_1),\ y_2 = b(u_1),\ z_2 = u_2.$$

Coefficients of the first quadratic form for the surface S_1 satisfy the relations $A^1_{i,j} = \delta_{i,j}$ where $\delta_{i,j}$, $i, j = 1,2$, is the Kronecker symbol. For the surface S_2 it is not difficult to obtain the equalities

$$A^2_{1,2} = A^2_{2,1} = 0,\ A^2_{2,2} = 1,\ A^2_{1,1} = (a(u_1))^2 + (b(u_1))^2 = 1.$$

So for the isometric of the surfaces S_1, S_2 it is sufficient to fulfill relations

$$(a'(u_1))^2 \le 1,\ (b'(u_1))^2 = 1 - (a'(u_1))^2. \tag{2}$$

Consequently to solve considered problem for the cylinder surface S_2 it is sufficient using the function $a(t)$ to find the function $b(t)$ from the equality (2).

Example 1. A partial case of this problem solution is the pair of functions

$$a(u_1) = \cos u_1,\ b(u_1) = \sin u_1. \tag{3}$$

If in the first quadrant of E^2 we separate a stripe with sides not parallel to coordinate axes u_1, u_2, then from the equalities (3) it is possible to transform this stripe into spiral surface enveloped around the cylinder S_2.

3. SURFACE OF CONICAL TYPE

Assume that the angle S_1 in the space E^2 is described by the equalities $x_1 = u_2 \cos u_1$, $y_1 = u_2 \sin u_1$. Define the surface of a conical type S_2 by the formulas

$$x_2 = u_2 a(u_1),\ y_2 = u_2 b(u_1),\ z_2 = u_2 c(u_1).$$

Coefficients of the first quadratic form for the surface S_1 satisfy the formulas

$$A^1_{1,1} = u_2^2,\ A^1_{1,2} = A^1_{2,1} = 0,\ A^1_{2,2} = 1.$$

And coefficients of the first quadratic form for the surface S_2 satisfy the relations

$$A^2_{1,1} = u_2^2\left((a'(u_1))^2 + (b'(u_1))^2 + (c'(u_1))^2\right),\ A^2_{1,2} = A^2_{2,1} = \frac{u_2}{2}\left(a^2(u_1) + b^2(u_1) + c^2(u_1)\right)',$$

$$A^2_{2,2} = a^2(u_1) + b^2(u_1) + c^2(u_1).$$

So to fulfill the equalities in (1) it is sufficient to satisfy the formulas

$$a^2(u_1) + b^2(u_1) + c^2(u_1) = 1, \tag{4}$$

$$(a'(u_1))^2 + (b'(u_1))^2 + (c'(u_1))^2 = 1, \tag{5}$$

Seek a solution of this problem as follows
$a(u_1) = \cos F \cos G$, $b(u_1) = \sin F \sin G$, $c(u_1) = \sin G$, $F = F(s)$, $G = G(s)$, $s = s(u_1)$

where $F(s)$, $G(s)$ are arbitrary smooth functions of an argument s and $s(u_1)$ is a smooth function of the argument u_1. Such a choice of the functions a,b,c leads to the equality (4) independently on a choice of the function s. Simple calculations show that to fulfill the formulas (5) it is sufficient to satisfy the condition

$$1=\left(s'(u_1)\right)^2\left[\left(F'(s)\right)^2+\left(G'(s)\right)^2\right],$$

that leads to the formula

$$u_1=\pm\int\sqrt{\left(F'(s)\right)^2\cos^2 G(s)+\left(G'(s)\right)^2}\,ds. \qquad (6)$$

Example 2. In a partial case it is possible to take real α and put

$$G(s)=\alpha,\ F(s)=s,\ s(u_1)=u_1$$

and so

$$x_2=u_2\cos u_1\cos\alpha,\ y_2=u_2\cos u_1\cos\alpha,\ z_2=\sin\alpha. \qquad (7)$$

The relations (7) define usual conical surface and the formulas (6) specify a wide class of surfaces isometric to an angle in the space E^2.

4. SURFACE OG SPHERICAL TYPE

Suppose that S_1 is defined as follows

$$x_1=a(u_2)\cos u_1,\ y_1=a(u_2)\sin u_1,\ z_1=b(u_2),\ \left(a'(u_2)\right)^2+\left(b'(u_2)\right)^2=1.$$
$$A^1_{1,1}=a^2(u_2),\ A^1_{1,2}=A^1_{2,1}=0,\ A^1_{2,2}=1.$$

Assume that for some real γ the condition $\left(a'(u_2)\right)^2\le 1$ is true. Define the surface S_2 by the equalities

$$x_2=\gamma a(u_2)\cos\frac{u_1}{\gamma},\ y_2=\gamma a(u_2)\sin\frac{u_1}{\gamma},\ z_2=B(u_2).$$

Then the functions

$$A^2_{1,1}=a^2(u_2),\ A^2_{1,2}=A^2_{2,1}=0,\ A^2_{2,2}=\left(B'(u_2)\right)^2+\gamma^2\left(a'(u_2)\right)^2.$$

So the surfaces S_1, S_2 are isometric if

$$\left(B'(u_2)\right)^2+\gamma^2\left(a'(u_2)\right)^2=1,$$

consequently

$$B(u_2)=\int\sqrt{1-\gamma^2\left(a'(u_2)\right)^2}\,du_2.$$

Example 3. Assume that the equality $a(u_2)=\cos u_2$ is true and so the surface S_1 is a sphere with unity radius and with a cut along a meridian. Then the surface S_2 may coincide with a bending of a sphere by a type [3,Figure 3]: spindled surface of a rotating with a self overlap and etc
Consider possible generalizations of isometric surface with a spherical type. Suppose that smooth functions $F(s)$, $G(s)$ satisfy the condition

$$\left(F'(s)\right)^2+\left(G'(s)\right)^2=1 \qquad (8)$$

and $f(u_1,u_2)$ is smooth function. Define the surface S_1 as follows

$$x_1 = F\left(f(u_1,u_2)\right),\ y_1 = G\left(f(u_1,u_2)\right),\ z_1 = R(u_2), \tag{9}$$

then the functions

$$A_{1,1}^1 = \left(\frac{\partial f(u_1,u_2)}{\partial u_1}\right)^2,\ A_{1,2}^1 = A_{2,1}^1 = \frac{\partial f(u_1,u_2)}{\partial u_1}\cdot\frac{\partial f(u_1,u_2)}{\partial u_2},\ A_{2,2}^1 = \left(\frac{\partial f(u_1,u_2)}{\partial u_2}\right)^2 + \left(R'(u_2)\right)^2.$$

Take a real number γ and define the surface S_2 by the equalities

$$x_2 = \gamma F\left(\frac{f(u_1,u_2)}{\gamma}\right),\ y_2 = \gamma G\left(\frac{f(u_1,u_2)}{\gamma}\right),\ z_2 = R(u_2). \tag{10}$$

Then the functions

$$A_{1,1}^2 = \left(\frac{\partial f(u_1,u_2)}{\partial u_1}\right)^2,\ A_{1,2}^2 = A_{2,1}^2 = \frac{\partial f(u_1,u_2)}{\partial u_1}\cdot\frac{\partial f(u_1,u_2)}{\partial u_2},\ A_{2,2}^2 = \left(\frac{\partial f(u_1,u_2)}{\partial u_2}\right)^2 + \left(R'(u_2)\right)^2$$

and so the relation (1) is true. Consequently the surfaces S_1, S_2 defined by the formulas (8) - (10) are isometric.

REFERENCES

1. Andreev G.N. 2007. Additional chapters of geometry. Differential geometry of curved surfaces. Moscow.: MGIU. (In Russian).
2. Ushakov I.A. 2009. History of science through prism of inspiration. Glamour of geometry. Book 3. Moscow: URSS. (In Russian).
3. Fomenko V.T. 1998. Bending of surfaces// Soros educational journal. Vol. 5. P. 122-127. (In Russian).