"COLORING" OF MAP BY FINITE NUMBER OF COLORED POINTS USING FUZZY RECTANGLES

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ABSTRACT

In this paper an algorithm of a map "coloring" by a finite number of color points is constructed. This algorithm is based on the procedure of the interval images recognition and on the algorithm of a construction of a fuzzy rectangle. It is significantly simpler and compact than the triangulation procedure using in the mapping.

Keywords: a fuzzy rectangle, an interval images recognition, a polygon triangulation, a classification algorithm, a mapping problem.

1. INTRODUCTION

In this paper an algorithm of a map "coloring" by a finite number of color points is constructed. This algorithm is based on a concept of fuzzy rectangles [1-7]. For any color point surrounding internal and external rectangles are built. The rectangles containing points with same colors coincide or do not intersect. If these points have different colors then their internal rectangles do not intersect and their external rectangles may intersect only if they correspond to points with different colors. But membership functions connected with intersecting rectangles are conformed. Conformation algorithm is based on the interval images recognition procedure [8]. It is much simpler and compact than the triangulation procedure using in the mapping.

2. INTERNAL AND EXTERNAL RECTANGLES

A main idea of [1] in a construction of multidimensional segments (rectangles) is a dividing of recognized class of objects into subclasses. An each subclass is accorded to some external and embedded in it internal multidimensional segments. Segments accorded to different subclasses of a same class do not intersect (or intersect on a set with zero Lebesgue measure). Internal segments accorded to different classes also do not intersect (or intersect on a set with zero Lebesgue measure).

Assume that there is a finite set of vectors on a plane

$$Z_1 = \big((x_{11},\ x_{21}), j_1\big), \ldots, Z_n = \big((x_{1n},\ x_{2n}), j_n\big),$$

$$-\infty < x_{ik} < \infty,\ i = 1,2,\ j_k \in \{1, \ldots, m\},\ k = 1, \ldots, n.$$

If real numbers c, d coincide then the interval $(c, d) = \emptyset$. The coordinates x_{ik} characterize a displacement of the point Z_k on the plane and the coordinate j_k is the number of the class to which this point belong (the color of the point). Assume that the set $\{x_{i1}, \ldots, x_{in}\}$, $i = 1,2$, consists of different numbers. Denote $P_s = \{k\colon\ j_k = s\}$, $Q_s = \{k\colon\ j_k \neq s\}$.

Internal and external one dimensional segments. Fix the index i and put $k \in P_s$, denote

$$a_{ik} = \min\big(x_{it}\colon\ x_{it} \le x_{ik}\ ,\ \ t \in P_s\ ,\ x_{it} \le x_{iq} \le x_{ik} \Longrightarrow q \in P_s\big),$$

$$b_{ik} = \max\left(x_{it}:\ x_{it} \ge x_{ik}\ ,\ \ t \in P_s\ ,\ x_{it} \ge x_{iq} \ge x_{ik} \Rightarrow q \in P_s\right),\ a_{ik} \le b_{ik},$$

$$A_{ik} = \max(x_{it}:\ t \in Q_s\ ,\ x_{it} < a_{ik}),\ B_{ik} = \min(x_{it}:\ t \in Q_s\ ,\ x_{it} > b_{ik}), A_{ik} \le B_{ik}\ . \qquad (1)$$

Among the segments which contain the point x_{ik} and do not contain points x_{it}, $t \in Q_s$, the segment $[a_{ik}\ , b_{ik}]$ is maximal. Call it the internal segment.

Theorem 1. The segments $[a_{ik}\ , b_{ik}]$, $[a_{ik'}\ , b_{ik'}]$, $k \ne k'$, coincide or do not intersect.

The number A_{ik} can not be defined by the formula (1) if $x_{ik} = \min(x_{it}{:}\ 1 \le t \le n)$. Then put $A_{ik} = \min(x_{it}{:}\ 1 \le t \le n) = A_i$. Analogously the number B_{ik} cannot be defined by the formula (1) if $x_{ik} = \max(x_{it}{:}\ 1 \le t \le n)$. Then put $B_{ik} = \max(x_{it}{:}\ 1 \le t \le n) = B_i$.

Among segments which contain the point x_{ik} and do not contain points x_{it}, $t \in Q_s$, the segment $[A_{ik}\ , B_{ik}]$ is maximal. Call $[A_{ik}\ , B_{ik}]$ the external segment containing x_{ik} and $[a_{ik}\ , b_{ik}] \subseteq [A_{ik}\ , B_{ik}]$.

Theorem 2. If $j_k = j_{k'}$, $k \ne k'$ then the segments $[A_{ik}\ , B_{ik}]$, $[A_{ik'}\ , B_{ik'}]$ coincide or have general boundary point. If $j_k \ne j_{k'}$ then these segments cannot coincide but may intersect and each point of this intersection contains no more than in two different segments.

Proofs of Theorems 1, 2 are based on elementary logic-geometric considerations.

Rectangles surrounding dedicated points. Define internal and external rectangles surrounding the point x_{ik} by the equalities

$$[a_{1k}\ , b_{1k}]\otimes[a_{2k}\ , b_{2k}],\ [A_{1k}\ , b_{1k}]\otimes[A_{2k}\ , B_{2k}].$$

Theorem 3. The rectangles $[a_{1k}\ , b_{1k}]\otimes[a_{2k}\ , b_{2k}]$, $[a_{1k'}\ , b_{1k'}]\otimes[a_{2k'}\ , b_{2k'}]$, $k \ne k'$, coincide or do not intersect.

Theorem 4. The rectangles $[A_{1k}\ , B_{1k}]\otimes[A_{2k}\ , B_{2k}]$, $[A_{1k'}\ , B_{1k'}]\otimes[A_{2k'}\ , B_{2k'}]$, $j_k = j_{k'}$, $k \ne k'$ coincide or intersect on a set with zero Lebesgue measure (by pieces of their boundaries).

The statements of Theorems 3, 4 directly follow Theorems 1, 2.

Theorem 5. The rectangles $[A_{1k}\ , B_{1k}]\otimes[A_{2k}\ , B_{2k}]$, $[A_{1k'}\ , B_{1k'}]\otimes[A_{2k'}\ , B_{2k'}]$, $j_k \ne j_{k'}$, $k \ne k'$, cannot coincide but may intersect. An each point of such intersections may belong no more than to two different rectangles.

Proof. Fix k and put $s = j_k$. By a definition in the sets

$$[a_{1k}\ , b_{1k}]\otimes[a_{2k}\ , b_{2k}],\ [A_1\ , A_{1k}]\otimes[a_{2k}\ , b_{2k}],\ [B_{1k}\ , B_1]\otimes[a_{2k}\ , b_{2k}],$$

$$[a_{1k}\ , b_{1k}]\otimes[A_2\ , A_{2k}],\ [a_{1k}\ , b_{1k}]\otimes[B_{2k}\ , B_2]$$

only points $X_t = (x_{1t}, x_{2t})$ satisfying the equality $j_t = s$, may contain. In the sets

$$(A_{1k}, a_{1k}) \otimes\ [A_2, B_2],\ (b_{1k}, B_{1k}) \otimes\ [A_2, B_2],\ [A_1, B_1] \otimes (A_{2k}, a_{2k}),\ [A_1, B_1] \otimes (b_{2k}, B_{2k})$$

there are not points X_t, $1 \le t \le n$. Consequently the rectangles

$$[A_{1k}\ , a_{1k}]\otimes[a_{2k}\ , b_{2k}],\ [b_{1k}\ , B_{1k}]\otimes[a_{2k}\ , b_{2k}],\ [a_{1k}\ , b_{1k}]\otimes[A_{2k}\ , a_{2k}],\ [a_{1k}\ , b_{1k}]\otimes[b_{2k}\ , B_{2k}]$$

contain in a single external rectangle $[A_{1k}\ , B_{1k}]\otimes[A_{2k}\ , B_{2k}]$.

Define the sets

$$C_{1k}^- = [A_1\ , A_{1k}]\otimes\{A_{2k}\},\ C_{1k}^+ = [B_{1k}\ , B_1]\otimes\{A_{2k}\},\ C_{2k}^- = [A_1\ , A_{1k}]\otimes\{B_{2k}\},$$

$$C_{2k}^+ = [B_{1k}\ , B_1]\otimes\{B_{2k}\},\ \ C_{3k}^- = \{A_{1k}\}\otimes[A_2\ , A_{2k}],\ C_{3k}^+ = \{A_{1k}\}\otimes[B_{2k}\ , B_2],$$

$$C_{4k}^- = \{B_{1k}\}\otimes[A_2\ , A_{2k}],\ C_{4k}^+ = \{B_{1k}\}\otimes[B_{2k}\ , B_2],\ \ C_{pk} = C_{pk}^+ \cup C_{pk}^-,\ \text{p=1,…,4}.$$

In any set $C_{pk}, p = 1, \dots, 4$ there is only single point $X_{t_{pk}}$ from the set $\{X_1, \dots, X_n\}$. And all these points (some of them may coincide) satisfy inequalities $j_{t_{pk}} \neq s$. Construct now rectangles

$$R_{1k} = [A_1, a_{1k}] \otimes [A_2, a_{2k}], L_{1k} = [A_1, A_{1k}] \otimes [A_2, A_{2k}], R_{2k} = [A_1, a_{1k}] \otimes [b_{2k}, B_2],$$

$$L_{2k} = [A_1, A_{1k}] \otimes [B_{2k}, B_2], \ R_{3k} = [b_{1k}, B_1] \otimes [A_2, a_{2k}], L_{3k} = [B_{1k}, B_1] \otimes [A_2, A_{2k}],$$

$$R_{4k} = [b_{1k}, B_1] \otimes [b_{2k}, B_2], L_{4k} = [B_{1k}, B_1] \otimes [B_{2k}, B_2], \ L_{pk} \subseteq R_{pk}, \text{p=1,…,4},$$

$$S_{1k} = [A_{1k}, a_{1k}] \otimes [A_{2k}, a_{2k}], S_{2k} = [A_{1k}, a_{1k}] \otimes [b_{2k}, B_{2k}],$$

$$S_{3k} = [b_{1k}, B_{1k}] \otimes [A_{2k}, a_{2k}], S_{4k} = [b_{1k}, B_{1k}] \otimes [b_{2k}, B_{2k}],$$

By the definition of the external rectangle we obtain that for any point $X_t \in L_{pk}$, $j_t \neq s$, the inclusion $[A_{1t}, B_{1t}] \otimes [A_{2t}, B_{2t}] \subseteq R_{pk}$ is true. And the external rectangle $[A_{1t}, B_{1t}] \otimes [A_{2t}, B_{2t}]$ which has some internal point of the rectangle S_{pk} contains S_{pk} completely.

Prove now that internal points of the rectangle S_{pk} may belong besides of $[A_{1k}, B_{1k}] \otimes [A_{2k}, B_{2k}]$ to no more than single another external rectangle $[A_{1t}, B_{1t}] \otimes [A_{2t}, B_{2t}]$, $j_t \neq s$. Consider the case $p = 1, k = 1$ because in all other cases this statement may be verified similar.

If for all *t* so that $X_t \in L_{1k}$, $j_t \neq s$, we have that all j_t coincide. then last statement is obvious. Assume now that $X_t \in L_{1k}$, $j_t \neq s$, $S_{1k} \subseteq [A_{1t}, B_{1t}] \otimes [A_{2t}, B_{2t}]$ and there is $X_{t'} \in L_{1k}$, so that $j_{t'} \neq s$, $j_{t'} \neq j_t$. Then it is clear that $[A_{1t'}, B_{1t'}] \otimes [A_{2t'}, B_{2t'}] \cap S_{1k} = \emptyset$ because $x_{t'1} < x_{t1}$, $x_{t'2} < x_{t2}$ and $(A_{1,k}, A_{2,k}) \in [A_{1t}, B_{1t}] \otimes [A_{2t}, B_{2t}]$. Similar statements may be proved for internal points of the rectangles

$$[A_{1,k}, a_{1,k}] \otimes [b_{2,k}, B_{2,k}], \qquad [b_{1,k}, B_{1k}] \otimes [A_{2,k}, a_{2,k}], [b_{1,k}, B_{1k}] \otimes [b_{2,k}, B_{2,k}].$$

So Theorem 5 is proved.

3. CONSTRUCTION OF FUZZY SET FOR POINTS WITH IDENTICAL COLOR

Without a restriction of a generality suppose that there are numbers $0 = J_0 < J_1 < J_2 < \dots < J_m = n$ so that $P_s = \{k: J_{s-1} < k \leq J_s\}$, $1 \leq s \leq m$. From Theorems 3, 4 for fixed *s* the set of indexes $\{k \in P_s\}$ is divided into equivalence classes with elements which belong to coincident internal and external rectangles.

Consider the case $s = 1$ and suppose that appropriate equivalence classes are indexes sets $\{1, \dots, k_1\}$, $\{k_1 + 1, \dots, k_2\}$,…, $\{k_{l-1} + 1, \dots, k_l = J_1\}$. Denote $\gamma_{iq} = a_{ik_q}$, $\Gamma_{i,q} = A_{ik_q}$, $\delta_{iq} = b_{ik_q}$, $\Delta_{iq} = B_{ik_q}$. It is clear that $\Gamma_{iq} \leq \gamma_{iq} \leq \delta_{iq} \leq \Delta_{iq}$, $i = 1,2$, $1 \leq q \leq l$.

For fixed q, $1 \leq q \leq l$, define the function $\mu_q(X)$, $X \in E^2$, by conditions:

a) $X \in \mathcal{B}_q$, $\mathcal{B}_q = [\gamma_{1q}, \delta_{1q}] \otimes [\gamma_{2q}, \delta_{2q}] \Rightarrow \mu_q(X) = 1$,

b) $X \notin \mathcal{A}_q$, $\mathcal{A}_q = [\Gamma_{1q}, \Delta_{1q}] \otimes [\Gamma_{2q}, \Delta_{2q}] \Rightarrow \mu_q(X) = 0$,

c) assume that for $0 \leq \lambda \leq 1$ the inclusion $X = (x_1, x_2) \in \partial G_\lambda$ is true where ∂G_λ is the set

$$G_\lambda = \otimes_{i=1}^{2} [\gamma_{iq} + \lambda(\Gamma_{iq} - \gamma_{iq}), \ \delta_{iq} + \lambda(\Delta_{iq} - \delta_{iq})],$$

boundary then $\mu_q(X) = 1 - \lambda$, and so for $X \in \mathcal{A}_q \backslash \mathcal{B}_q = \bigcup_{0 \leq \lambda \leq 1} \partial G_\lambda$

$$\mu_q(X) = 1 - \max_{1 \leq i \leq 2} \max \left[\frac{x_i - \gamma_{iq}}{\Gamma_{iq} - \gamma_{iq}}, \frac{x_i - \delta_{iq}}{\Delta_{iq} - \delta_{iq}} \right]. \tag{2}$$

From the equality (2) obtain

$$\mu_q(X) \leq 1 - \frac{x_i-\gamma_{iq}}{\Gamma_{iq}-\gamma_{iq}},\ \Gamma_{iq} \leq x_i \leq \gamma_{iq}\ ;\ \mu_q(X) \leq 1 - \frac{x_i-\delta_{iq}}{\Delta_{iq}-\delta_{iq}},\ \delta_{iq} \leq x_i \leq \Delta_{iq}. \quad (3)$$

Define now the fuzzy set which denotes an inclusion of the point $X = (x_1, x_2)$ into one of constructed external rectangles $[\Gamma_{1q}, \Delta_{1q}] \otimes [\Gamma_{2q}, \Delta_{2q}]$, $1 \leq q \leq m$ [2]. As these external rectangles intersect only on pieces of their boundaries where appropriate functions equal zero then it is possible to define the membership function of this fuzzy set by the equality $\mu(X) = \sum_{q=1}^{l} \mu_q(X)$.

4. CONSTRUCTION OF FUZZY SETS FOR POINTS WITH DIFFERENT COLORS USING MAP BACKGROUND

For any s, $s = 1, \ldots, m$, we constructed a fuzzy set with a membership function $\mu^s(X)$ of the point X .

Theorem 6. The following inequality

$$\sum_{s=1}^{m} \mu^s(X) \leq 1\ ,\ X \in E^2, \quad (4)$$

takes place.

Proof. To prove the inequality (4) it is necessary to use Theorem 5 with its designations and proof and to estimate the function $\sum_{s=1}^{m} \mu^s(X)$ for $X \in S_{1k}$. In this case

$$\sum_{s=1}^{m} \mu^s(X) = \mu^s(X) + \mu^{j_t}(X)$$

where from the formula (3) we have

$$\mu^s(X) \leq 1 - \frac{x_1 - a_{1k}}{A_{1k} - a_{1k}},\ \mu^{j_t}(X) \leq 1 - \frac{x_1 - A_{1k}}{a_{1k} - A_{1k}}.$$

So the inequality (4) is true. Theorem 6 is proved.

At the end of this considerations denote $\mu^0(X) = 1 - \sum_{s=1}^{m} \mu^s(X)$ and call this nonnegative difference the membership function of the fuzzy set which describes a background of the map. Consequently on a base of an information about the finite set of colored points on a plane we construct fuzzy sets which define coloring of the map.

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