ASYMPTOTIC OF CYCLY EXISTENCE IN ORIENTED GRAPH WITH LOW RELIABLE EDGES

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ABSTRACT

In this paper a power asymptotic of a probability that there is a cycle in a random oriented graph with n nodes and low reliable edges is constructed. An accelerated algorithm for a calculation of asymptotic coefficients with $O(s(n)\ln n)$ products, where $s(n)$ is an amount of products in a multiplication of two matrixes with a size $n \times n$, is constructed.

Keywords: a cycle, an oriented graph, an edge, a probability.

1. INTRODUCTION

We consider a problem of power asymptotic construction for a probability of a cycle existence in a random graph with low reliable edges. A presence of cycles in a deterministic oriented graph allows to factorize it by a relation of a cycle equivalence [1], [2]. A calculation of an amount of cycles with minimal length may be applied in an investigation of free scale networks which receive large spread last years. [3, Theorems 10 - 12]. An algorithm of a calculation of power asymptotic coefficients with products amount $O(s(n)\ln n)$, where $s(n)$ is an amount of products for a multiplication of two matrixes with a size $n \times n$, is constructed.

2. ASYMPTOTIC OF CYCLE EXISTENCE

Consider an oriented graph G with nodes $1,\dots,n$, without loops and fold edges. Denote $A = \|a_{ij}\|_{i,j=1}^{n}$ its adjacency matrix, D - minimal cycle length, C - an amount of cycles with minimal length in the graph G. Construct a model of an oriented random graph G_* with nodes $1,\dots,n$ in which only edges of the graph G may enter. The edge (i,j) enters with the probability $p_{ij} = h$, $h \to 0$ (it is low reliable). Random events that different edges enter the graph G_* are independent. Denote S the event that there is a cycle in the graph G_* and put $P(S)$ its probability.

Theorem 1. The limit relation $P(S) \sim Ch^D,\ h \to 0$ is true.

Proof. As $= \bigcup_{1<k\le n} S_k$, where S_k is the event that there is simple (without repetitions of nodes) cycle with the length k in the graph G_* so *P(S)* satisfies the relation

$$P(S) = P(\bigcup_{1<k\le n} S_k) = P(\bigcup_{D\le k\le n} S_k) \sim P(S_D) \sim C(h^D),\ h \to 0.$$

Theorem 1 is proved.
Define $c_k = trA^k$ and calculate asymptotic constants *D, C*.
Theorem 2. If $min\,(k\colon c_k > 0) \le n$ then $D = min\,(k\colon c_k > 0)$, $C = \frac{c_D}{D}$.

Proof. It is well known that the element $a_{ii}^{(k)}$ of the matrix A^k equals the amount of ways $(i = i_1,\dots,i_{k-1},i_k,i)$ with the length k in the oriented graph G. If $k = D$ then all cycles with the length k

contain k different nodes. Indeed if not there is a cycle with the length k passes through some node more than one time. So this cycle has length smaller than k.

Consequently the equality $D = \min(k: c_k > 0)$ is true and all cycles with the length D are simple. So the cycle $(i = i_1, \dots, i_{k-1}, i_k, i)$ adds units in k diagonal elements of the matrix A^k and the equality $C = \frac{c_D}{D}$ takes place. Theorem 2 is proved.

Assume that the constant D is known and $k_1 = \min(k: 2^k > n)$. Represent the constant D in the binary-number system and write it in the form

$$D = 2^{l_1} + 2^{l_2} + \dots + 2^{l_r}, 0 \le l_1 < l_2 < \dots < l_r \le k_1.$$

Calculate now the matrixes $A^{2^1} = A \times A$, $A^{2^2} = A^{2^1} \times A^{2^1}$,…, $A^{2^{k_1}} = A^{2^{k_1-1}} \times A^{2^{k_1-1}}$, using $k_1 s(n) = O(s(n) \ln n)$ products. Then the constant C may be calculated by the formula

$$C = \frac{tr\left(A^{l_1} \cdot A^{l_2} \cdot \ldots \cdot A^{2^{l_r}}\right)}{D} \quad (1)$$

using $O(s(n) \ln n)$ products. The constant $D = \min(k: trA^k > 0)$ may be found by a sequential calculation of the matrixes A^k, $1 < k \le n$, using $O(s(n)n)$ products. So there is a problem to accelerate an algorithm of the constant D calculation.

3. ACCELERATED ALGORITHM OF CONSTANT *D* CALCULATION

Put $B = A + I$ where I is the unit matrix and denote $d_k = trB^k - n$.

Theorem 3. If Theorem 2 condition is true then

$$D = \min(k: b_k > 0), 0 = b_1 < b_2 < \dots < b_n . \quad (2)$$

Proof. The relation (2) is a corollary of the equality

$$b_k = tr(A + I)^k - n = \sum_{j=1}^{k} C_k^j \, trA^j ,$$

where C_k^j is a number of combination from k by j.

Using Theorem 3 and an idea of a dichotomy dividing for a search of a root of monotonically increasing and continuous function construct the following algorithm of the constant D definition. Using the formulas $B^{2^{t+1}} = B^{2^t} \cdot B^{2^t}$, $t > 0$, calculate by $s(n)$ products. If $d_{2^{k_1}} = 0$ then we stop calculation and put formally $D = \infty$, $C = 0$. If not define $q_1 = \min\left(k: d_{2^k} > 0\right)$, $q_1 < [\log_2 n] + 1$, where $[a]$ is an integer part of a real number a.

Denote $P = 2^{q_1}$, $Q = 2^{q_1-1}$ and construct the following recurrent procedure: if $d_{Q+2^{q_1-2}} > 0$ then $P := Q + 2^{q_1-2}$, else $Q := Q + 2^{q_1-2}$, if $d_{Q+2^{q_1-3}} > 0$ then $P := Q + 2^{q_1-3}$, else $Q := Q + 2^{q_1-3}$ and so on. This procedure continues $q_1 - 1$ steps till we obtain the equality $P - Q = 1$. Then the relation $D = P$ is true. To fulfill this recurrent procedure it is necessary to make $O(s(n) \ln n)$ products. Theorem 3 is proved.

Consequently the asymptotic constants D, C may be calculated by $O(s(n) \ln n)$ products.

4. CONCLUSION REMARKS

For the standard algorithm of the multiplication of two matrixes with the size $n \times n$ $s(n) = O(n^3)$, for F. Strassen algorithm $s(n) = O(n^{2.81})$, for *D*. Coppersmith and Sh. Winograd algorithm $s(n) = O(n^{2.3755})$ and for V. Williams algorithm $s(n) = O(n^{2.3727})$ [4]. But main part of calculators consider that the F. Strassen algorithm is the most applicable among algorithms accelerated in a comparison with the standard one.

Assume that elements of the matrix $V = \|v_{ij}\|_{i,j=1}^{n}$, $v_{ij} \geq 0$, characterize weights of the graph G edges and in the model of the random graph G_* the probability $p_{ij} \sim v_{ij}h$, $h \to 0$. Then it is not complicated to obtain that the probability of the cycle existence in the graph G_* satisfies the relation $P(S) \sim \frac{trV^D}{D}$,. $h \to 0$. And the matrix V^D is calculated similar to the matrix A^D by $O(s(n)\ln n)$ products.

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