
ORIENTED GRAPHS WITH UNRELIABLE NODES

G.Sh. Tsitsiashvili

IAM, FEB RAS, Vladivostok, Russia,
Far Eastern Federal University

e-mail: guram@iam.dvo.ru

ABSTRACT

In applications there are networks with unreliable nodes. To construct models of such networks and effective algorithms of their analysis in this paper some elements from the graph theory, the algebra, the sets theory and the probability theory are combined. These constructions allow to investigate oriented graphs with unreliable nodes using as accuracy so asymptotic formulas. All constructed algorithms have a linear complexity by a number of graph nodes.

Keywords: an oriented graph, an unreliable node, relations of equivalence and partial order.

INTRODUCTION

In the reliability theory usually graphs with unreliable edges are investigated [1], [2]. But networks represented by graphs with unreliable nodes appear in modern applications. [3]. In this paper a model of random graph in which a failure of some node stops work of all other nodes to which there are ways in the graph is considered. [4]. A similar model is a model of a random network controllability [5].

In a problem of the random network controllability an important role play a minimal nodes cover in the graph. A nodes cover is a set of nodes that each edge touches with this set. The nodes cover is minimal if it has a minimal number of nodes. The Konig theorem establishes an equivalence of the minimal nodes cover and the maximal matching [6]. Here the maximal matching is a set of edges which have not common nodes. A matching is maximal if it has a maximal number of edges. The Konig theorem states that in a bipartite graph a number of edges in the maximal matching equals a number of nodes in the minimal nodes cover. It gives a large number of effective algorithms of the graph theory [6] – [13].

But an analysis of stochastic networks demands not only to define a number of nodes in the minimal nodes cover but to enumerate all such covers or at least to find their amount. This problem becomes NP problem. So in a consideration of networks with unreliable nodes it is necessary to change a model of a network using some specific of the oriented graph. In this paper such a problem is solved using concepts of an equivalence and a partial order between graph nodes. This approach allows to obtain algorithms of a calculation some failures events in the graph with the complexity $O(n^2)$ by the number n of the graph nodes.

1. FAILURES IN ORIENTED GRAPHS WITH UNRELIABLE NODES

Assume that deterministic and oriented graph G have the nodes set U and the edges set V . Each node v of the graph графа G with the probability q_v fails. These failures of different nodes are independent random events. Each failure of some node leads to stops of all other nodes to which there are ways from failed node in the graph G . In such a way the random graph G_* is defined. We shall calculate probabilities that some sets of nodes in the graph G_* stop.

On the nodes set V define the binary relation $v_1 \sim v_2$ if in the graph G there are ways from the node v_1 to the node v_2 and from the node v_2 to the node v_1 . The relation " \sim " is symmetric, transitive and reflexive. So it is the equivalence relation. As a result the nodes set V may be divided into the classes of the equivalence (the clusters) $w(v) = \{v': v' \sim v\}$. On the set of the clusters $W = \{w(v): v \in V\}$ introduce the binary relation $w(v_1) \succcurlyeq w(v_2)$ if in the graph G there is a way from the node v_1 to the node v_2 . So for any $v'_1 \in w(v_1)$, $v'_2 \in w(v_2)$ in the graph G there is a way from the node v'_1 to the node v'_2 . The relation " \succcurlyeq " is antisymmetric, transitive and reflexive. So this relation is the partial order relation.

From the definition of the random graph G_* we have that in the cluster $w \in W$ all nodes fail with the probability

$$q(w) = 1 - \prod_{v \in w} (1 - q_v)$$

Or work with the probability $p(w) = 1 - q(w)$. Further denote

$$A(T) = \prod_{w \in T} p(w), B(T) = \prod_{w \in T} q(w), T \subseteq W$$

with $A(\emptyset) = B(\emptyset) = 1$.

Later we shall consider the set of clusters W in which any cluster w fails with the probability $q(w)$ or does not fail with the probability $p(w)$. The cluster w failure leads to a stop of all clusters $w' \in W$ which satisfy the condition $w \succcurlyeq w'$ independently that they fail or do not fail.

Assume that \tilde{W} is the set of maximal (in a sense of the partial order " \succcurlyeq ") elements from W . Calculate the probability Q that all clusters from the set W stop. It is not complicated to obtain that this event is a stop of all clusters $w \in \tilde{W}$. Consequently we have:

$$Q = B(\tilde{W}). \quad (1)$$

Consider the probability $Q(w)$ that the cluster $w \in W$ stops. For this aim define the set of clusters $F(w) = \{w' : w' \succcurlyeq w\}$. It is not difficult to check that

$$Q(w) = 1 - A(F(w)). \quad (2)$$

Calculate the probability $Q(R)$ that at least a single cluster from the set R stops. Define the sets $F_i = F(w_i)$, $i = 1, \dots, r$ and put

$$F(R) = \bigcup_{i=1}^r F_i$$

then

$$Q(R) = 1 - A(F(R)). \quad (3)$$

Calculate now the probability $S(R, L)$ that in the random graph G_* all clusters from the set L work and there is at least a single cluster in the set R which stops. Introduce the sets $W_1 = F(L)$, $W_2 = F(R) \setminus W_1$ then

$$S(R, L) = A_1(W_1)(1 - A_1(W_2)). \quad (4)$$

2. ASYMPTOTIC ANALYSIS OF FAILURES PROBABILITY FOR SET OF CLUSTERS

Consider a problem of a calculation of the probability $Q'(R)$ that all clusters from the set $R = \{w_1, \dots, w_r\} \subseteq W$ stop their work. Accuracy calculation of the probability $Q'(R)$ demands an amount of arithmetical operations which geometrically increases by the number n of the graph G nodes. So this problem will be solved in an assumption that there are positive numbers $c(w)$ which satisfy the relation

$$p(w) = \exp(-c(w)h), h \rightarrow 0, w \in W. \quad (5)$$

Theorem 1. Assume that $c(w) \equiv 1$ then for any integer r , for $h \rightarrow 0$ and for $n(1, \dots, r)$ equal to a number of clusters in the intersection T of the sets $F_i, i = 1, \dots, r$, we have:

$$Q'(R) = hn(1, \dots, r) + o(h). \quad (6)$$

Proof. Assume that $p = \exp(-h)$ and denote $D(k)$ the random event that all clusters of the set F_k work. Then from the known formula for a probability of a finite set of events aggregation we have

$$Q'(R) = 1 - P(\cup_{k=1}^r D(k)) = 1 - \sum_{k=1}^r (-1)^{k-1} \sum_{1 \leq i(1) < \dots < i(k) \leq r} P(D(i(1)) \dots D(i(k))). \quad (7)$$

A number of clusters in the set $G(i(1), \dots, i(k)) = F_{i(1)} \cup \dots \cup F_{i(k)}$ similar to Formula (7) equals

$$\sum_{s=1}^k (-1)^{s-1} \sum_{1 \leq j(1) < \dots < j(s) \leq k} n(i(j(1)) \dots i(j(s)))$$

Consequently

$$P(D(i(1)) \dots D(i(k))) = \exp\left(-h \left(\sum_{s=1}^k (-1)^{s-1} \sum_{1 \leq j(1) < \dots < j(s) \leq k} n(i(j(1)), \dots, i(j(s)))\right)\right). \quad (8)$$

Substituting the relations (8) into Formula (7) and using the exponent expansion into the Taylor series we obtain:

$$\begin{aligned} Q'(R) &= 1 - \sum_{k=1}^r (-1)^{k-1} \sum_{1 \leq i(1) < \dots < i(k) \leq r} \\ &\times \left(1 - h \left(\sum_{s=1}^k (-1)^{s-1} \sum_{1 \leq j(1) < \dots < j(s) \leq k} n(i(j(1)), \dots, i(j(s)))\right)\right) + o(h) = \\ &= l_1 + \sum_{k=1}^r \sum_{1 \leq i(1) < \dots < i(k) \leq r} n(i(1), \dots, i(k)) l(i(1), \dots, i(k)) + o(h), h \rightarrow 0. \end{aligned} \quad (9)$$

As any commutation of the indexes $1, \dots, r$ in Formula (9) does not change $Q'(R)$ so the following equalities are true:

$$l(i(1), \dots, i(k)) = l(1, \dots, k), 1 \leq i(1) < \dots < i(k) \leq r, 1 \leq k \leq r. \quad (10)$$

From Formula (9) we have:

$$l_1 = 1 - \sum_{k=1}^r (-1)^{k-1} = 1 + (\sum_{k=0}^r (-1)^k - 1) = 0. \quad (11)$$

$$\begin{aligned} l(1) &= \sum_{k=1}^r (-1)^{k-1} C_{r-1}^{k-1} = \sum_{k=0}^{r-1} (-1)^k C_{r-1}^k = 0, \quad l(1, 2) = -\sum_{k=1}^r (-1)^{k-1} C_{r-2}^{k-2} = 0, \dots \\ l(1, \dots, r-1) &= (-1)^{r-2} \sum_{k=r-1}^r (-1)^{k-1} C_1^{k-r+1} = 0, \quad l(1, \dots, r) = 1. \end{aligned} \quad (12)$$

Theorem 1 is proved.

Corollary 1. Denote $c(1, \dots, r)$ the sum of $c(w)$ by clusters w from the set T then for $h \rightarrow 0$ we have:

$$Q'(R) = hc(1, \dots, r) + o(h). \quad (13)$$

Proof. Indeed in Corollary 1 assumption Formula (9) may be rewritten as follows:

$$Q'(R) = l_1 + \sum_{k=1}^r \sum_{1 \leq i(1) < \dots < i(k) \leq r} c(i(1), \dots, i(k)) l(i(1), \dots, i(k)) + o(h)$$

Using Formulas (11), (12) we obtain Formula (13). Corollary 1 is proved.

Remark 1. Corollary 1 allows to analyze essential cooperative effects in the oriented graph with unreliable nodes because the characteristic $c(1, \dots, r)$ may differ significantly from the sum

$\sum_{k=1}^r c(k)$. The formulas (1) – (4), (13) show that procedures to calculate probabilities in these formulas have the complexity $O(n^2)$ by the number n of the graph nodes.

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