TOTAL TIME ON TEST TRANSFORMS ORDERING OF SEMI-MARKOV SYSTEM

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ABSTRACT

First passage time of semi-Markov performance process of a multistate system are considered. TTT (Total time on Test) transform ordering is discussed.

1. INTRODUCTION

First passage times of appropriate stochastic process have often been used to represent *times to failure* of devices or systems which are subject to shocks and wear, random repair time and random interruptions during their operations. The life distribution properties of these processes have therefore been widely investigated in reliability and maintenance literature. When the performance process of a multistate reliability system is Markov or semi-Markov process, we need to study the ageing properties of the first passage time distribution from up states to down states. Identification of failure rate model in statistical lifetime data analysis is major problem in the field of reliability and survival analysis. The total time on test (TTT) transform is used as a tool for identification of failure distribution model in binary system. When a system is modeled by Markov or semi-Markov process it is quiet interesting to get a procedure for the ordering of the failure distribution using its TTT which based on transition probability function.

Use of TTT transform for the identification of failure rate models (IFR/DFR/ Bathtub shaped/constant) in the binary system case is discussed by Barlow and Campo (1975). Later, Klefsjo (1982) presented some relationship between the TTT transform and other ageing properties (with their duals) of random variable. Barlow and Proschan (1996) discussed a wide application of IFR/DFR distributions in maintenance and replacement policies of a binary reliability system. Nair et al. (2008) studied the properties of TTT transforms of order n and examined their applications in reliability analysis.

But when we consider a complex system whose performance process is Markov or semi-Markov, we need the knowledge of ordering for applying suitable maintenance and repair/replacement policies. The ordering of distribution of performance process is Markov/semi-Markov will be helpful to the engineers and designers for applying suitable maintenance and repair or replacement policies.

In this paper, we consider a semi-Markov system whose first passage time distribution and the reliability function based on the transition probability function in the up states. We define the TTT using transition probability function. The conditions of ordering are discussed.

This paper is arranged as follows. Section 2 recall the existing results for identification of failure rate model of random variables based on TTT. In section 3, we discuss TTT transform ordering based on transition probability function of a semi-Markov process. In Section 4, we introduce some conditions for ordering properties of the semi-Markov system based on TTT built from transition probability function. Conclusions are given at the last section.

2. TTT TRANSFORM OF A LIFETIME RANDOM VARIABLE

Many parametric lifetime models such as Gamma, Weibull, and Truncated Normal distributions have monotone failure rate. The failure rate function λ(*t*) will be continuous and twice differentiable for all $t > 0$ with the exception of the exponential distribution. Total time on test (TTT) transform is a fundamental tool in reliability investigation.

Given a random sample of size *n* from a nonnegative random variable with distribution *F*, let $X_{n1}, X_{n2}, ..., X_{nr}, ..., X_{nn}$ be the order statistic corresponding to the sample. The total time on test to the *r*-th failure is

$$T(X_{nr}) = nX_{n1} + (n-1)(X_{n2} - X_{n1}) + ... + (n-r+1)(X_{nr} - X_{n(r-1)}).$$

It is the sum of all observed lifetimes that can be expressed as $T(X_{nr}) = \sum_{k=1}^{r} X_{nk} + (n-r)X_{nr}$.

If *F* is exponential with mean *θ* and we observe the first *r* ordered values, then it is well known that the maximum likelihood, minimum variance, unbiased estimator of *θ* is, $\hat{\theta}(r,n) = \frac{1}{r}T(X_{nr})$.

Define $H_n^{-1}(r/n) = \frac{1}{n}T(X_{rn})$, then $H_n^{-1}(r/n) = \int_0^{F_n^{-1}(r/n)} [1 - F_n(u)]du$, where $F_n(u)$ is the empirical distribution function defined as

$$F_n(u) = \begin{cases} 0, & u < X_{n1} \\ i/n, & X_{in} \le u < X_{i+1,n} \\ 1, & u \ge X_{nn} \end{cases} \quad and \;\; F_n^{-1}(u) = \inf\{x : F_n(x) \ge u\}.$$

The fact that $F_n(x) \to F(x)$ uniformly at continuity points of *F*, by Glivenko Cantelli theorem, implies $\lim_{n\to\infty, r/n\to t} \int_0^{F_n^{-1}(r/n)} [1 - F_n(u)]du = \int_0^{F^{-1}(t)} [1 - F(u)]du$ uniformly in $t \in [0,1]$.

Define $H_F^{-1}(t) = \int_0^{F^{-1}(t)} [1 - F(u)]du, \quad t \in [0,1]$ to be the TTT of the distribution *F*.
Barlow and Campo (1975) proved the following result.

Theorem 2.1 *There is a 1-1 correspondence between distributions F and their transform* H_F^{-1}.

Note that H_F is a distribution with support on $[0, \mu]$, where μ is the mean of *F*, since $H_F^{-1}(1) = \int_0^{F^{-1}(1)} [1 - F(u)]du = \mu$ when $F(0^-) = 0$.

Then $\phi(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)}$ is a continuous increasing function on [0*;* 1], which is 0 at *t* = 0 and 1 at *t* = 1.

Model Identification

Let $G(x) = 1 - \exp(-x/\theta), \quad x, \theta \ge 0$ be the exponential distribution with mean μ. Then $H_G^{-1}(t) = \int_0^{G^{-1}(t)} e^{-x/\theta} dx = \int_0^{G^{-1}(t)} \theta dG(x) = \theta t$ and scaled TTT,

$$\phi(t) = \frac{H_G^{-1}(t)}{H_G^{-1}(1)} = t, \quad t \in [0,1]. \tag{3.1}$$

The scaled TTT of the Exponential distribution is a 45^0 line on $[0,1]$. The normalized total time on test is the boundary between the corresponding transforms of IFR and DFR distributions. TTT that permits to classify distributions according to their failure rate is that its slope evaluated at *t* = *F*(*x*) is the reciprocal of the failure rate at *X*.

$$\frac{d}{dt}H_F^{-1}(t)|_{t=F(x)}=\frac{(1-t)}{f[F^{-1}(t)]}|_{t=F(x)}=\frac{1-F(x)}{f(x)}=\frac{1}{\lambda(x)}, \qquad (3.2)$$

where λ is the failure rate of *F*.

Semi-Markov system

We are concerned with a multistate system (MSS) having $M+1$ states 0,1,…,M where '0' is the best state and '*M*' is the worst state, see Barlow and Wu (1978) for details of MSSs. At time zero the system begins at its best state and as time passes the system begins to deteriorate. It is assumed that the time spent by the system in each state is random with arbitrary sojourn time distribution. The system stays in some acceptable states for some time and then it moves to unacceptable (down) state. The first time at which the MSS enters the down state after spending a random amount of time in acceptable states is termed as the first passage time (failure time) to the down state of the MSS. We study the aging properties of the first passage time distribution of the MSS modeled by the semi-Markov process $\{Y_t, t\geq 0\}$. In the MSS with states *{0,1,…,k,k+1,…,M}* where *{0,1,…,k}* is the acceptable states, the sojourn time between state '*i*' to state '*j*' is assumed to be distributed with arbitrary distribution F_{ij}.

First Passage Time And Reliability Function

Let *E = {0,1,…,M}* be a set representing the state of the MSS and probability space with probability function *P*, on which we define a bivariate time homogeneous Markov chain $(X,T)=\{X_n,T_n, n\in\{0,1,2,...\}\}$, X_n takes values of *E* and T_n on the half real line $R^+=[0,\infty)$, with $0\leq T_1\leq T_2\leq ...\leq T_n\leq ...$ Put $U_n=T_n-T_{n-1}$ for all $n\geq 1$. This Markov process is called a Markov renewal process (MRP) with transition function, the semi-Markov kernel, Q = [Q_{ij}], where $Q_{ij}(t)=P[X_{n+1}=j, U_n\leq t\,|\,X_n=i], i,j\in E, t\geq 0$ and $Q_{ii}(t)=0, i\in E, t\geq 0$.

Now we consider the semi-Markov process (SMP), as defined in Pyke (1961). It is the generalization of Markov process with countable state space. SMP is a stochastic process which moves from one state to another of a countable number of states with successive states visiting form a Markov chain, and that the process stays in a given state a random length of time, the distribution of which may depend on this state as well as on the one to be visited in the next. Define $Z_t=X_{N_t}, N_t=\sup\{n, T_n=U_1+U_2+...+U_n\leq t\}$, it is the semi-Markov process associated with the MRP defined above. In terms of *Z*, the times $T_1,T_2,...$ are successive times of transitions for *Z*, and $X_0,X_1,...$ are successive states visited. If Q has the form $Q_{ij}(t)=P[X_{n+1}=j\,|\,X_n=i][1-e^{-\lambda(i)t}], i,j\in E, t\geq 0,$ for some function λ(*i*), $j\in E$ then the process Z_t is a Markov process. That is, in a Markov process, the distributions of the sojourn times are all exponential independent of the next state. The word *semi*-Markov comes from the somewhat limited Markov property which *Z* enjoys, namely, that the future of *Z* is independent of its past given the present state provided the "present" is the time of jump. Let I_{ij} =indicator function of $\{i=j\}$. Define the transition probability that system occupied state $j\in E$ at time *t* > 0, given that it is started at state *i* at time zero, as, $i,j\in E, t\geq 0$

$$p_{ij}(t)=P[Z_t=j\mid Z_0=i]=P[X_{N_t}=j\mid X_0=i]=h_i(t)I_{ij}+Q*P(t)(i,j),$$

where $h_i(t)=1-\sum_k Q_{ik}(t)$, $P(t)=[p_{ij}(t)]$ and $Q*P(t)(i,j)=\sum_{k\in E}\int_0^t Q_{ik}(dx)p_{kj}(t-x)$

To obtain the reliability function of the semi-Markov system described above, we must define a new process, *Y* with state space $U\cup\nabla$, where *U* denotes set of all up states *{*0*;* 1*; :::; k}* and ∇ is the absorbing state in which all the states *{k +* 1*, ...;M}* of the system is united. Let T_D denote the time of first entry to the down states of *Z* process.

That is, $Y_t=Z_t(\omega)$ if $t<T_D(\omega)$ and $Y_t=\nabla$ if $t\geq T_D(\omega)$.

Let $1=(1,1,...,1)^1$, a unit row vector with appropriate dimension. The process Y_t is a semi-Markov process with semi-Markov kernel

$$\begin{bmatrix}\overbrace{Q_{11}(t)}^{Up} & \overbrace{Q_{12}(t)}^{Down}\\ 0 & 0\end{bmatrix}$$

We denote $\alpha=(\overbrace{\alpha(0),...,\alpha(k)}^{Up,\alpha_1},\overbrace{\alpha(k+1),...,\alpha(M)}^{Down,\alpha_2})$ where $\alpha(i)=P(Y_0=i)$.

The reliability function is

$$\begin{aligned}R(t)&=P[\forall u\in[0,t],Z_u\in U]=P[Y_t\in U]=\sum_{j\in U}P[Y_t=j]\\&=\sum_{i\in U}\sum_{j\in U}P[Y_t=j,Y_0=i]\\&=\sum_{i\in U}\sum_{j\in U}p_{ij}(t)\alpha(i).\end{aligned}\tag{4.3}$$

3. TTT ORDERING OF SEMI-MARKOV SYSTEM

In order to identify the failure rate behavior of a semi-Markov system based on the transition probability function, we define the TTT based on transition probability function in up states as follows. Let *F* be the first passage time distribution of a semi-Markov system, define

$$H_{p_{ij}}^{-1}(t)=\int_0^{F^{-1}(t)}p_{ij}(u)du,\forall i,j\in U,t\in[0,1]\tag{4.4}$$

where

$$\begin{aligned}F^{-1}(t)&=\inf\{(1-R(t))\geq t\}\\&=\inf\left\{x:\left(1-\sum_{i\in U}\sum_{j\in U}p_{ij}(x)\alpha(i)\right)\geq t\right\}\\&=\inf\left\{x:\left(\sum_{i\in U}\sum_{j\in U}p_{ij}(x)\alpha(i)\right)\leq 1-t\right\}\end{aligned}$$

But

$$\begin{aligned}H_F^{-1}(t)&=\int_0^{F^{-1}(t)}\sum_{i\in U}\sum_{j\in U}p_{ij}(u)du,\forall i,j\in U,t\in[0,1]\\&=\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)}p_{ij}(x)dx\\&=\sum_{i\in U}\sum_{j\in U}\alpha(i)H_{p_{ij}}^{-1}(t)\end{aligned}$$

Then

$$H_F^{-1}(1)=\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx=\sum_{i\in U}\sum_{j\in U}\alpha(i)H_{p_{ij}}^{-1}(1)\quad \frac{H_F^{-1}(t)}{H_F^{-1}(1)}=\frac{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(t)} p_{ij}(x)dx}{\sum_{i\in U}\sum_{j\in U}\alpha(i)\int_0^{F^{-1}(1)} p_{ij}(x)dx},\quad t\in[0,1].$$

Let F and G be the distributions of first passage time of two semi-Markov processes with transition probability functions $p_{ij}(u)\quad and\quad q_{ij}(u)\ \forall i,j\in U, t\in[0,1]$

Theorem 3.1. IF $F^{-1}(t)\le G^{-1}(t), \forall i,j\in U, t\in[0,1]$ and $p_{ij}(u)\ge q_{ij}(u)\forall i,j\in U, t\in[0,1]$ then $H_{p_{ij}}^{-1}(t)=\int_0^{F^{-1}(t)} p_{ij}(u)du\le \int_0^{G^{-1}(t)} q_{ij}(u)du = H_{q_{ij}}^{-1}(t)$.

When interarrival time of semi-Markov p-system is less that of q-system, we have TTT of F is less than that of G.

Let $H_F^{-1}(t)=\sum_{i\in U}\sum_{j\in U}\alpha(i)H_{p_{ij}}^{-1}(t)\le\sum_{i\in U}\sum_{j\in U}\beta(i)H_{q_{ij}}^{-1}(t)=H_G^{-1}(t), t\in[0,1]$. When this holds we say that the first passage time of semi-Markov p-system is less that that of q-system in TTT order. But this is possible only when $F^{-1}(t)\le G^{-1}(t), \forall i,j\in U, t\in[0,1]$ and $p_{ij}(u)\ge q_{ij}(u)\forall i,j\in U, t\in[0,1]$.

As the rate of transition from state *i* to state *j* is greater in p-sysetm than in q-system then the failure of system occur rapidly. So that TTT of the p-system will be smaller than that of q-system.

Also $F^{-1}(t)\le G^{-1}(t)$ indicate that chance of occurrence of p-system early failure is greater than that of the q-system.

In a semi-Markov system it is very difficult to compute mean or variance or expected value of any convex function of first passage time random variable. In such situation TTT is found to be good ordering tool.

4. TTT ORDERING OF CONTINUOUS TIME MARKOV PROCESSES

Consider a Markov process in continuous time and discrete state space {1,2,…,M}, Doob (1953), p.241. The system starts in state '1' at time zero and as it enters 'M', it remains there. Consider the intensity matrix, $H=[h_{ij}]$ with entries

$$h_{ij}=0,\quad i\in\{1,2,...,M-1\}, j\ne i+1, h_{ii+1}=h,\quad h_{iM}=0.$$

The Kolmogorov's system of differential equation becomes,
for $p_{ij}(t-u)=P[Y_t=j\mid Y_u=i], 0\le u<t$ and we take $u=0$,

$$p_{ik}^1(t)=-hp_{ik}(t)+hp_{i+1k}(t), i<M,\quad p_{Mk}(t)=0$$

with initial conditions, $p_{ik}(0)=\delta_{ik}$, the indicator of *{i=k}*. Then,

$$p_{Mk}(t)=0,\quad k\ne M, p_{MM}(t)=1$$

and it is easily verified that the solution is

$$p_{ik}(t)=\begin{cases}0, \quad k<i\\ \dfrac{(ht)^{k-i}e^{-ht}}{\Gamma(k)}, \quad i\le k<M\\ e^{-ht}[e^{ht}-1-ht-...-\dfrac{(ht)^{M-i-1}}{\Gamma(M-i)}], \quad k=M.\end{cases}$$

Here the process is of monotone paths. Now consider $\forall i,j\in\{0,1,...,M-1\}$

$$H^{-1}_{p_{ij}}(1) \quad =\int_0^\infty \frac{(ht)^{k-i}e^{-ht}}{\Gamma(k)}dt=\frac{h^{k-i}}{\Gamma(k)}\int_0^\infty t^{k-i}e^{-ht}dt=\frac{\Gamma(k-i+1)}{\Gamma(k)h}.$$

Therefore

$$\frac{H^{-1}_{p_{ij}}(t)}{H^{-1}_{p_{ij}}(1)}=\frac{\Gamma(k)h}{\Gamma(k-i+1)}\int_0^{F^{-1}(t)}\frac{(hu)^{k-i}e^{-hu}}{\Gamma(k)}du$$
$$=\frac{h^{k-i+1}}{\Gamma(k-i+1)}\int_0^{F^{-1}(t)}u^{k-i}e^{-hu}du.$$

It is the scaled TTT transform for q-system with *r* and *k*.
Similarly, let

$$\frac{H^{-1}_{q_{ij}}(t)}{H^{-1}_{q_{ij}}(1)}=\frac{\Gamma(k)r}{\Gamma(k-i+1)}\int_0^{G^{-1}(t)}\frac{(ru)^{k-i}e^{-ru}}{\Gamma(k)}du$$
$$=\frac{r^{k-i+1}}{\Gamma(k-i+1)}\int_0^{G^{-1}(t)}u^{k-i}e^{-ru}du.$$

be the scaled TTT transform for q-system with *r* and *k*.

Since $u^{k-i}e^{-ru}$ is unimodel curve, $\dfrac{H^{-1}_{p_{ij}}(t)}{H^{-1}_{p_{ij}}(t)}\le\dfrac{H^{-1}_{q_{ij}}(t)}{H^{-1}_{q_{ij}}(1)}$ if $\int_0^{F^{-1}(t)}u^{k-i}e^{-hu}\le\int_0^{G^{-1}(t)}u^{k-i}e^{-ru}du, t\in[0,1], h\le r.$

This is possible only when $F^{-1}(t)\le G^{-1}(t), t\in[0,1]$. But $\overline{F}(x)\le\overline{G}(x)$ imply $F^{-1}(t)\le G^{-1}(t), t\in[0,1]$. Thus $Z_p\le_{st}Z_q$ would imply $Z_p\le_{TTT}Z_q$ order of semi-Markov systems with restrictions.

5. CONCLUSION

The TTT transform ordering of first passage time distribution of a semi-Markov system is discussed. This ordering is applicable for Multi-state systems whose performance process is modeled using semi-Markov process.

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