OPTIMAL DESIGN OF STEP STRESS PARTIALLY ACCELERATED LIFE TEST UNDER PROGRESSIVE TYPE-II CENSORED DATA WITH RANDOM REMOVAL FOR FRECHET DISTRIBUTION

Sana Shahab, Sadia Anwar, Arif Ul Islam

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Department of Statistics & Operations Research
(Aligarh Muslim University, Aligarh-202002, India)

e-mail: sana.shahb@gmail.com

ABSTRACT

In this article, progressive censoring and step stress partially accelerated life test are combined to develop a step-stress PALT with Progressively type-II Censored Data with the random removal. The removals from the test are assumed to have binomial distribution and uniform distribution and the life time of the testing products are considered to follow Frechet distribution. The parameters are estimated by using the maximum likelihood method and asymptotic confidence interval estimates of the model parameters are also evaluated by using Fisher information matrix. Statistically optimal PALT plans are developed such that the Generalized Asymptotic Variance (GAV) of the Maximum Likelihood Estimators (MLEs) of the model parameters at design stress is minimized. At the end, simulation study is performed to illustrate the statistical properties of the parameters.

KEYWORDS: Partially Accelerated Life Tests; Binomial Removal; Uniform Removal; Progressive Censoring; Maximum Likelihood Estimator; Generalized Asymptotic Variance

1 INTRODUCTION

When the product of high reliability is tested, the result of the some commonly used life test gives no or very few failures by the end of the test. In these types of the testing, the accelerated life testing (ALT) is used to obtain failures quickly. In such cases the testing is done at higher than usual use conditions. Three types of testing such as constant-stress, step-stress and progressive-stress are commonly used. In ALT, the mathematical model relating the lifetime of the unit and the stress is known or can be assumed. For detailed study of ALT see Nelson [1]. So as to, ALT data cannot be extrapolated to normal use condition. So, in such cases, partially accelerated life testing (PALT) is a more appropriate test to be used to estimate the statistical model parameters. Ismail et al. [2] introduced the Optimum Simple Time-Step Stress Plans for Partially Accelerated Life Testing with Censoring.

In many life tests, the experiment does not observe the failure times of all components. In such cases, the censored sampling arises. The most common censoring schemes are type-I censoring and type-II censoring. These two censoring schemes do not allow for units to be removed from the test at the points other than the final termination point. Moreover, there are some cases in which components are lost or removed from the test before failure. This would lead to progressive censoring. For progressive censoring see Balakrishnan and Aggarwala [3] and Balakrishnan [4]. Under the progressive type II censoring scheme, the experimenter puts n components on test at time zero. The first failure is observed at Y_1 and then R_1 of surviving components is randomly selected and removed. When the second failure occurs at time Y_2, R_2 of surviving components is randomly selected and removed and when *(m-1)*th failure is observed at the time Y_{m-1}, R_{m-1} of the surviving units are randomly selected and removed from the experiment, the experiment terminates when the m^{th} failure component is observed at X_m and $R_m = n - m - \sum_{i=1}^{m-1} R_i$ all removed. In this censoring scheme $R_1, R_2, ..., R_m$ are all prefixed. However, in some practical experiments, these numbers cannot be pre-fixed and they occur at random. Inference based on progressively Type II censored data is discussed by many authors. Yuen and Tse [5] considered the estimation problem for Weibull distribution under progressive Censoring with random removals. Yang et al. [6] statistically analyzed the Weibull Distributed Lifetime Data under Type-II Progressive Censoring with Binomial Removals. Wu [7] used progressively Type-II censored data with uniform removals to estimate the parameters of Pareto distribution. Ismail et al. [8] introduced the Optimal Design of Step-Stress Life Test with Progressively type-II Censored Exponential Data with binomial removals. Bander [9] estimated the maximum likelihood for Generalized Pareto Distribution under Progressive Censoring with Binomial Removals. Chang et al. [10] studied the progressive censoring with Random Removals for the Burr Type XII Distribution.

2 THE MODEL AND TEST METHOD

2.1 The Frechet Distribution

The Frechet distribution is a special case of the generalized extreme value distribution. The generalized extreme value (GEV) distribution is a family of continuous probability distributions developed within extreme value theory to combine the Gumbel, Fréchet and Weibull families also known as type I, II and III extreme value distributions. The lifetimes of the test items are assumed to follow a Frechet distribution. The probability density function (pdf) of the Gompertz distribution is given by

$$f(t) = \alpha\theta^{\alpha}\, t^{-\alpha-1} \exp\left(-\left(\frac{t}{\theta}\right)^{-\alpha}\right) \tag{1}$$

And the cumulative distribution function is given by

$$F(t) = \exp\left(-\left(\frac{t}{\theta}\right)^{-\alpha}\right) \tag{2}$$

The survival function of the Frechet distribution is given by

$$\overline{F}(t) = 1 - \exp\left(-\left(\frac{t}{\theta}\right)^{-\alpha}\right)$$

2.2 Assumptions

- n identical and independent units are put on the life used condition and the lifetime of each testing unit follows Frechet distribution.
- The test is terminated at the m^{th} failure, where m is prefixed ($m \le n$).
- Each of the n units is first run under normal use condition. If it does not fail or remove from the test by a pre-specified time τ, it is put under accelerated condition.

- At the i^{th} failure a random number of the surviving units, $R_i, i = 1,2,......m-1$, are randomly selected and removed from the test. Finally, at the m^{th} failure the remaining surviving units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are all removed from the test and the test is terminated.
- The lifetime, say Y, of a unit under SS-PALT can be written as

$$Y = \begin{cases} T & \text{if} \quad T \geq \tau \\ \tau + (T - \tau)/\beta & \text{if} \quad T < \tau \end{cases} \tag{3}$$

where T is the lifetime of the unit under normal use condition, τ is the stress change time and β is the acceleration factor; $\beta > 1$. Therefore, the pdf of Y can be written as in the following form

Therefore probability density function (pdf) of Y can be written as

$$f(y) = \begin{cases} 0 & y \leq 0 \\ f_1(y) & 0 < y \leq \tau \\ f_2(y) & y > \tau \end{cases}$$

$$f(y) = \begin{cases} 0 & y \leq 0 \\ \alpha\theta^{\alpha} y^{-\alpha-1} \exp\left(-\left(\dfrac{y}{\theta}\right)^{-\alpha}\right) & 0 < y \leq \tau \\ \alpha\theta^{\alpha} \beta\left(\tau + \beta(y-\tau)\right)^{-\alpha-1} \exp\left(-\left(\dfrac{\tau + \beta(y-\tau)}{\theta}\right)^{-\alpha}\right) & y > \tau \end{cases} \tag{4}$$

$$F(y) = \begin{cases} 0 & y \leq 0 \\ \exp\left(-\left(\dfrac{y}{\theta}\right)^{-\alpha}\right) & 0 < y \leq \tau \\ \exp\left(-\left(\dfrac{\tau + \beta(y-\tau)}{\theta}\right)^{-\alpha}\right) & y > \tau \end{cases} \tag{5}$$

3 MAXIMUM LIKELIHOOD ESTIMATION

3.1 Parameter Estimation with the Binomial Removals

The number of units removed from the test at each failure time follows a binomial distribution and any individual unit being removed is independent of the others but with the same probability *p*. That is, $R_1 \sim bino(n-m, p)$ and for i = 2, 3,......, m −1, $R_i \sim bino\left(n-m-\sum_{j=1}^{i-1} r_j, p\right)$ and $r_m = n-m-r_1-r_2-...........-r_{m-1}$.

Let $(y_i, r_i, \delta_{1i}, \delta_{2i}), i=1,2,.......m$ denote the observation obtained form a progressively type-II censored sample with random removals in a step-stress PALT. Here $y_{(1)} \le y_{(2)} \le .... \le y_{(m)}$.

Thus for the progressive censoring with the pre determined number of the removals $R = (R_1 = r_1, ..., R_{m-1} = r_{m-1})$, the conditional likelihood function of the observations $y = \{(y_i, r_i, \delta_{1i}, \delta_{2i}), i = 1,2,........m.\}$ can be defined as follow

$$L(y_i; \alpha, \beta, \theta, \delta_{1i}, \delta_{2i} \mid R = r) = \prod_{i=1}^{m} \left\{ \left[f_1(y_i)(\overline{F}_1(y_i))^{r_i} \right]^{\delta_{1i}} \left[f_2(y_i)(\overline{F}_2(y_i))^{r_i} \right]^{\delta_{2i}} \right\} \tag{6}$$

$$L(y; \theta, \alpha, \beta, \delta_{1i}, \delta_{2i} / R = r) = \prod_{i=1}^{m} \left\{ \left[\alpha\theta^{\alpha} y_i^{-\alpha-1} \exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha} \right) \left(1 - \exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha} \right) \right)^{r_i} \right]^{\delta_{1i}} \right.$$
$$\left. \left[\alpha\theta^{\alpha} \beta\left(\tau + \beta(y_i - \tau)\right)^{-\alpha-1} \exp\left(-\left(\frac{\tau + \beta(y_i - \tau)}{\theta} \right)^{-\alpha} \right) \left(\exp\left(-\left(\frac{\tau + \beta(y_i - \tau)}{\theta} \right)^{-\alpha} \right) \right)^{r_i} \right]^{\delta_{2i}} \right\} \tag{7}$$

The number of units removed at each failure time follows a binomial distribution such that

$$P(R_1 = r_1) = \binom{n-m}{r_1} P^{r} (1-P)^{n-m-r_1}$$

And for *i=2, 3,…….,m-1*

$$P(R_i = r_i \mid R_{i-1} = r_{i-1}, ..., R_1 = r_1) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} P^{r_i} (1-P)^{n-m-\sum_{j=1}^{i} r_j}$$

where $0 \le r_i \le n-m-(r_1+r_2+........+r_{i-1})$. Furthermore, suppose that R_i is independent of Y_i for all *i*. Then the joint likelihood function can be found as

$$L(y_i; \alpha, \beta, \theta, p, \delta_{1i}, \delta_{2i}) = L_1(y_i; \alpha, \beta, \theta, p, \delta_{1i}, \delta_{2i} \mid R = r) P(R, p) \tag{8}$$

where $P(R, p)$ is the joint probability distribution of $R = (r_1, r_1, r_1, ...r_m)$ and is given by

$$P(R,p) = P(R_{m-1}=r_{m-1}, R_{m-2}=r_{m-2},...R_1=r_1)$$
$$= P(R_{m-1}=r_{m-1}/R_{m-2}=r_{m-2},...R_1=r_1) \times P(R_{m-2}=r_{m-2}/R_{m-3}=r_{m-3},...R_1=r_1)$$
$$\times P(R_{m-3}=r_{m-3}/R_{m-4}=r_{m-2},...R_1=r_1) \times ... P(R_2=r_2/R_1=r_1) P(R_1=r_1)$$

$$= \frac{(n-m)!}{\left(n-m-\sum_{i=1}^{m-1} r_i\right)! \sum_{i=1}^{m-1} r_i!} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i} \qquad (9)$$

Now by substituting $L_1(y_i;\alpha,\beta,\theta,p,\delta_{1i},\delta_{2i} \mid R=r)$ and $P(R,p)$ from the equation (7) and (9) in (8) we get the likelihood function

$$L(y,\theta,\beta,p,\delta_{1i},\delta_{2i}) = \prod_{i=1}^{m} \left\{ \left[\alpha\theta^{\alpha} y_i^{-\alpha-1} \exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right) \left(1-\exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)\right)^{r_i} \right]^{\delta_{1i}} \right.$$
$$\left. \left[\alpha\theta^{\alpha}\beta\left(\tau+\beta(y_i-\tau)\right)^{-\alpha-1} \exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right) \left(\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\right)^{r_i} \right]^{\delta_{2i}} \right\} \qquad (10)$$
$$\times \frac{(n-m)!}{\left(n-m-\sum_{i=1}^{m-1} r_i\right)! \prod_{i=1}^{m-1} r_i!} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_1}$$

The log-likelihood of the above equation is given by

$$\log L = \left[m\log\alpha + m\alpha\log\theta - (\alpha+1)\sum_{i=1}^{m_u}\log y_i - \sum_{i=1}^{m_u}(y_i/\theta)^{-\alpha} + r_i\sum_{i=1}^{m_u}\log\left(1-\exp\left(-(y_i/\theta)^{-\alpha}\right)\right) \right]$$
$$\left[m_a\log\beta - (\alpha+1)\sum_{i=1}^{m_a}\log\left(\left(\tau+\beta(y_i-\tau)\right)\right) - \sum_{i=1}^{m_a}\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha} \right.$$
$$\left. + r_i\sum_{i=1}^{m_a}\log\left(1-\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\right) \right] + \log c_1 + \sum_{i=1}^{m_i} r_i \log p$$
$$+ \left[(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i \right] \log(1-p)$$

The maximum likelihood estimators of β and θ can be derived directly by maximizing the equation (7) instead of (10) because $P(R,p)$ does not involves the parameter β and θ. Similarly the binomial parameter *p* does not depend on $L_1(y_i;\alpha,\beta,\theta,p,\delta_{1i},\delta_{2i} \mid R=r)$, hence the MLE of *p* can be found by maximizing P(R; p) directly. Thus, the maximum likelihood estimates (MLEs), of β and θ can be found by solving the following equations:

$$\frac{\partial \log L}{\partial \theta}=\frac{m\alpha}{\beta}-\alpha\theta^{\alpha-1}\sum_{i=1}^{m_u} y_i^{-\alpha}+\alpha\theta^{\alpha-1}\sum_{i=1}^{m_u}\frac{r_i y_i^{-\alpha}\exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)}{1-\exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)}$$

$$-\alpha\theta^{\alpha-1}\sum_{i=1}^{m_a}\left(\tau+\beta\left(y_i-\tau\right)\right)^{-\alpha}+\alpha\theta^{\alpha-1}\sum_{i=1}^{m_a}\frac{r_i\left(\tau+\beta\left(y_i-\tau\right)\right)^{-\alpha}\exp\left(-\left(\frac{\tau+\beta\left(y_i-\tau\right)}{\theta}\right)^{-\alpha}\right)}{1-\exp\left(-\left(\frac{\tau+\beta\left(y_i-\tau\right)}{\theta}\right)^{-\alpha}\right)} \tag{11}$$

$$\frac{\partial \log L}{\partial \beta}=\frac{m_a}{\beta}-\left(\alpha+1\right)\sum_{i=1}^{m_a}\frac{y_i-\tau}{\tau+\beta\left(y_i-\tau\right)}+\alpha\theta^{\alpha}\sum_{i=1}^{m_a}\frac{y_i-\tau}{\left(\tau+\beta\left(y_i-\tau\right)\right)^{\alpha+1}}$$

$$-\alpha\theta^{\alpha}\sum_{i=1}^{m_a}\frac{r_i\left(y_i-\tau\right)\left(\tau+\beta\left(y_i-\tau\right)\right)^{-\alpha-1}\exp\left(-\left(\frac{\tau+\beta\left(y_i-\tau\right)}{\theta}\right)^{-\alpha}\right)}{\left[1-\exp\left(-\left(\frac{\tau+\beta\left(y_i-\tau\right)}{\theta}\right)^{-\alpha}\right)\right]} \tag{12}$$

Independently, the MLE of the binomial parameter p can be obtained by solving the following equation:

$$\frac{\partial \log L}{\partial p}=\sum_{i=1}^{m-1}\frac{r_i}{p}-\frac{\left[\left(m-1\right)\left(n-m\right)-\sum_{i=1}^{m-1}\left(m-i\right)r_i\right]}{1-p} \tag{13}$$

Therefore we get $\hat{p}$ from equation (13)

$$\hat{p}=\frac{\sum_{i=1}^{m-1} r_i}{\sum_{i=1}^{m-1} r_i+\left(m-1\right)\left(n-m\right)-\sum_{i=1}^{m-1}\left(m-i\right)r_i}$$

3.2 Estimation with the Uniform Removal

The number of units removed from the test at each failure time follows a uniform discrete distribution. That is, $R_1 \sim Unif\left(0,n-m\right)$ and for i = 2, 3,......, m −1, $R_i \sim Unif\left(0,n-m-\sum_{j=1}^{i-1} r_j\right)$ and $r_m=n-m-r_1-r_2-............-r_{m-1}$ .Such that,

$$P(R_1 = r_1) = \frac{1}{n-m+1}$$

And for *i=2, 3 …m-1*.

$$P(R_i = r_i \mid R_{i-1} = r_{i-1}, ..., R_1 = r_1) = \frac{1}{n-m-\sum_{j=1}^{i-1} r_j + 1}$$

where $P(R)$, the joint probability distribution of $R = (r_1, r_1, r_1, ...r_m)$ and is given by

$$P(R = r) = \frac{1}{n-m-\sum_{i=1}^{m-1} r_i + 1} \tag{14}$$

where $0 \le r_i \le n-m-(r_1 + r_2 + ... + r_{i-1}), i = 1,2,.............m-1$.

It is clear that *P(R=r)* does not depend on the parameters β and θ and, hence the maximum likelihood estimators can be derived directly by maximizing the equations (7) and then solving the equations (11)

4 FISHER INFORMATION MATRIX & ASYMPTOTIC CONFIDENCE INTERVAL

The asymptotic variance-covariance matrix of the ML estimators of the parameters can be approximated by numerically inverting the Fisher-information matrix F and The Fisher information matrix is obtained by taking the negative second partial derivatives of the log-likelihood function and for the binomial removal it can be written

$$F = \begin{bmatrix} -\frac{\partial^2 l}{\partial \theta^2} & -\frac{\partial^2 l}{\partial \theta \partial \beta} & -\frac{\partial^2 l}{\partial \theta \partial p} \\ -\frac{\partial^2 l}{\partial \beta \partial \theta} & -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial p} \\ -\frac{\partial^2 l}{\partial p \partial \theta} & -\frac{\partial^2 l}{\partial p \partial \beta} & -\frac{\partial l}{\partial p^2} \end{bmatrix}$$

And for the uniform removal, fisher information matrix can be written as

$$F = \begin{bmatrix} -\frac{\partial^2 l}{\partial \theta^2} & -\frac{\partial^2 l}{\partial \theta \partial \beta} \\ -\frac{\partial^2 l}{\partial \beta \partial \theta} & -\frac{\partial^2 l}{\partial \beta^2} \end{bmatrix}$$

Elements of Fisher Information matrix are

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{m\alpha}{\theta^2} - \alpha(\alpha-1)\sum_{i=1}^{m_u} y_i^{-\alpha}\theta^{\alpha-2} \quad + \alpha(\alpha-1)\theta^{\alpha-2}\sum_{i=1}^{m_a}\left(\tau+\beta(y_i-\tau)\right)^{-\alpha}$$

$$+\alpha\sum_{i=1}^{m_u}\frac{r_i y_i^{-\alpha}\exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)\left[(\alpha-1)\theta^{\alpha-2} - y_i^{-\alpha}\right]}{\left[1-\exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)\right]} - \alpha^2\theta^{2\alpha-2}\sum_{i=1}^{m_u}\frac{r_i y_i^{-2\alpha}\exp\left(-2\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)}{\left[1-\exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)\right]^2}$$

$$+\alpha\sum_{i=1}^{m_a}\frac{r_i\left(\tau+\beta(y_i-\tau)\right)^{-\alpha}\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\left[(\alpha-1)\theta^{\alpha-2}+\alpha\theta^{2\alpha-2}\left(\tau+\beta(y_i-\tau)\right)^{-\alpha}\right]}{\left[1-\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\right]}$$

$$-\alpha^2\theta^{2\alpha-2}\sum_{i=1}^{m_a}\frac{r_i\left(\tau+\beta(y_i-\tau)\right)^{-2\alpha-1}\exp\left(-2\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)}{\left[1-\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{m_a}{\beta^2} + (\alpha+1)\sum_{i=1}^{m_a}\frac{(y_i-\tau)^2}{\left[\tau+\beta(y_i-\tau)\right]^2} - \alpha(\alpha+1)\theta^{\alpha}\sum_{i=1}^{m_a}\frac{(y_i-\tau)^2}{\left(\tau+\beta(y_i-\tau)\right)^{\alpha+2}}$$

$$-\alpha\theta^{\alpha}\sum_{i=1}^{m_a}\frac{r_i(y_i-\tau)^2\left(\tau+\beta(y_i-\tau)\right)^{-\alpha-2}\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\left[-(\alpha+1)+\alpha\theta^{\alpha}\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\right]}{1-\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)}$$

$$-\alpha\sum_{i=1}^{m_a}\frac{r_i(y_i-\tau)^2\left(\tau+\beta(y_i-\tau)\right)^{-2\alpha-2}\exp\left(-2\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)}{\left[1-\exp\left(-\left(\frac{\tau+\beta(y_i-\tau)}{\theta}\right)^{-\alpha}\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial p^2} = -\sum_{i=1}^{m_i}\frac{r_i}{p^2} - \frac{\left[(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i\right]}{(p-1)^2}$$

$$\frac{\partial^2 \log L}{\partial\theta\partial\beta} = \frac{\partial^2 \log L}{\partial\beta\partial\theta} = \alpha^2\theta^{\alpha-1}\sum_{i=1}^{m_a}\left(\tau+\beta\left(y_i-\tau\right)\right)^{-\alpha-1}\left(y_i-\tau\right)$$

$$-\alpha^2\sum_{i=1}^{m_u}\frac{r_i\left(y_i-\tau\right)\left(\tau+\beta\left(y_i-\tau\right)\right)^{-\alpha-1}\theta^{\alpha-1}\exp\left(-\left(\frac{\tau+\beta\left(y_i-\tau\right)}{\theta}\right)^{-\alpha}\right)\left[1+\theta^{\alpha}\left(\tau+\beta\left(y_i-\tau\right)\right)^{-\alpha}\right]}{\left[1-\exp\left(-\left(\frac{\tau+\beta\left(y_i-\tau\right)}{\theta}\right)^{-\alpha}\right)\right]}$$

$$+\alpha^2\sum_{i=1}^{m_a}\frac{r_i\theta^{2\alpha-1}\left(y_i-\tau\right)\left(\tau+\beta\left(y_i-\tau\right)\right)^{-2\alpha-1}\exp\left(-2\left(\frac{\tau+\beta\left(y_i-\tau\right)}{\theta}\right)^{-\alpha}\right)}{\left[1-\exp\left(-\left(\frac{y_i}{\theta}\right)^{-\alpha}\right)\right]^2}$$

$$\frac{\partial^2 \ln L}{\partial p\partial\theta} = \frac{\partial^2 \ln L}{\partial\theta\partial p} = \frac{\partial^2 \ln L}{\partial p\partial\beta} = \frac{\partial^2 \ln L}{\partial\beta\partial p} = 0$$

The variance covariance and covariance matrix of the parameter for the binomial removal can be written

$$\Sigma = \begin{bmatrix} -\frac{\partial^2 l}{\partial\beta^2} & -\frac{\partial^2 l}{\partial\beta\partial\theta} & -\frac{\partial^2 l}{\partial\beta\partial p} \\ -\frac{\partial^2 l}{\partial\theta\partial\beta} & -\frac{\partial^2 l}{\partial\theta^2} & -\frac{\partial^2 l}{\partial\theta\partial p} \\ -\frac{\partial^2 l}{\partial p\partial\beta} & -\frac{\partial^2 l}{\partial p\partial\theta} & -\frac{\partial l}{\partial p^2} \end{bmatrix}^{-1} = \begin{bmatrix} AVar\left(\hat{\beta}\right) & ACov\left(\hat{\beta}\hat{\theta}\right) & ACov\left(\hat{\beta}\hat{p}\right) \\ ACov\left(\hat{\theta}\hat{\beta}\right) & AVar\left(\hat{\theta}\right) & ACov\left(\hat{\theta}\hat{p}\right) \\ ACov\left(\hat{p}\hat{\beta}\right) & ACov\left(\hat{p}\hat{\theta}\right) & AVar\left(\hat{p}\right) \end{bmatrix}$$

And for the uniform removal case it can be written as

$$\Sigma = \begin{bmatrix} -\frac{\partial^2 l}{\partial\beta^2} & -\frac{\partial^2 l}{\partial\beta\partial\theta} \\ -\frac{\partial^2 l}{\partial\theta\partial\beta} & -\frac{\partial^2 l}{\partial\theta^2} \end{bmatrix}^{-1} = \begin{bmatrix} AVar\left(\hat{\beta}\right) & ACov\left(\hat{\beta}\hat{\theta}\right) \\ ACov\left(\hat{\theta}\hat{\beta}\right) & AVar\left(\hat{\theta}\right) \end{bmatrix}$$

The $100(1-\xi)\%$ asymptotic confidence interval for θ, β and p can be written as

$$\left[\hat{\theta} \pm Z_{1-\frac{\xi}{2}}\sqrt{AVar\left(\hat{\theta}\right)}\right], \left[\hat{\beta} \pm Z_{1-\frac{\xi}{2}}\sqrt{AVar\left(\hat{\beta}\right)}\right] \text{and} \left[\hat{p} \pm Z_{1-\frac{\xi}{2}}\sqrt{AVar\left(\hat{p}\right)}\right]$$

5 OPTIMUM TEST PLAN

The present criterion by which one can choose the optimal value of τ is based on the determinant of the Fisher's information matrix. Maximization of that that determinant is equivalent to minimization of the generalized asymptotic variance (GAV) of the MLE of the model parameters. The GAV is the reciprocal of the determinant of the Fisher's information matrix *F* that is

$$GAV = \frac{1}{|F|}$$

So, the optimal value of τ is chosen in such a way that the determinant of the Fisher's information matrix *F* is maximized and then the GAV is minimized. This is called the D-optimality criterion.

6 SIMULATION STUDY

In order obtain MLEs of β, θ and p and to study the properties of these estimates through Mean squared errors (MSEs),) and the coverage rate of asymptotic confidence intervals for different sample sizes, a simulation study is performed. Moreover, we will determine the optimal stress change time which minimizes the generalized asymptotic variance of the MLE of parameters. To perform the simulation study, we used the following steps

a) First specify the value of n and m.
b) The value of the parameters are chosen to be $\alpha = 2.87, \theta = 3.02, \beta = 2.62, p = 0.67, \tau = 3.5.$
c) Generate a random sample with size n and censoring size m with random removals, $r_i, i = 1,2,.........m-1$ from the random variable Y given by (4).
d) Generate a group value $R_i \sim bino\left(n - m - \sum_{j=1}^{i-1} r_j, p\right)$ and also $R_i \sim Unif\left(0, n - m - \sum_{j=1}^{i-1} r_j\right)$ where, $r_m = n - m - r_1 - r_2 - .... - r_{m-1.}$
e) For different sample sizes n= 20, 60, 80,100 and 120, compute the ML estimates.
f) The mean squared error (MSE), the coverage rate of the 95% confidence interval of parameters and Bias are obtained associated with the MLE of the parameters, optimal value of τ and also the Optimal GAV of the MLEs of the model parameters are obtained numerically for each sample size.

Table 1(i); Simulation study results with Binomial Removals for
$\alpha = 2.87, \theta = 3.02, \beta = 2.62, p = 0.67, \tau = 3.5.$

n	m	Binomial case			95% Confidence interval coverage			τ	$\lvert F^{-1} \rvert$
		$\hat{\theta}$	$\hat{\beta}$	$\hat{p}$	$CP_{\hat{\theta}}$	$CP_{\hat{\beta}}$	$CP_{\hat{P}}$		
20	9	2.983612	4.889763	0.897212	0.92039	0.90121	0.90313	3.8746	1.336
	19	2.977351	4.886342	0.895871	0.92141	0.90341	0.90583	3.8786	1.465
60	9	2.953811	4.867830	0.847492	0.92156	0.90547	0.90876	3.7424	1.493
	19	2.907351	4.858361	0.808313	0.92183	0.90645	0.90963	3.7413	1.502
	29	2.895634	4.888907	0.804721	0.92190	0.90673	0.90991	3.7409	1.573
	39	2.893631	4.683670	0.800838	0.92199	0.90843	0.91234	3.7289	1.638
	49	2.890731	4.642846	0.799743	0.92213	0.90863	0.91425	3.7263	1.693
	59	2.865341	4.619843	0.795982	0.92254	0.91633	0.91473	3.7084	1.699
80	9	2.862563	4.983741	0.769371	0.92261	0.91740	0.91533	3.7052	1.712
	19	2.860726	4.738421	0.766932	0.92275	0.91834	0.91642	3.5566	1.734
	29	2.846535	4.597361	0.759826	0.93280	0.91876	0.91735	3.5503	1.782
	39	2.818732	4.55836	0.685821	0.93289	0.91899	0.91841	3.4371	1.791
	49	2.815721	4.387461	0.588763	0.93385	0.92934	0.91893	3.5778	1.832
	59	2.687631	4.334524	0.559831	0.93481	0.92997	0.91934	3.0766	1.854
	69	2.665434	4.284712	0.530841	0.93541	0.93013	0.91953	3.0355	1.871
	79	2.646213	4.097361	0.508349	0.94753	0.93084	0.91979	2.7009	1.889
100	9	2.619736	3.869763	0.487354	0.95130	0.93099	0.91991	2.7987	1.920
	19	2.605531	3.898731	0.379421	0.95353	0.93194	0.92421	2.7354	1.943
	29	2.576435	3.757365	0.339741	0.95365	0.93245	0.92632	2.7354	2.132
	39	2.557261	3.728371	0.336821	0.95475	0.93385	0.92713	2.7047	2.223
	49	2.397251	3.686510	0.309431	0.95573	0.93642	0.92795	2.7028	2.264
	59	2.307360	3.428761	0.304814	0.95752	0.93752	0.92846	2.7011	2.349
	69	2.152841	3.087361	0.233193	0.95883	0.93840	0.92896	2.7006	2.382
	79	2.119423	2.787361	0.230341	0.95992	0.94671	0.92888	2.6937	2.467
	89	2.094381	2.629834	0.178287	0.96862	0.94689	0.92913	2.6795	2.484
	99	2.007378	2.198347	0.145931	0.97432	0.94778	0.92999	2.6654	2.961
120	9	2.003841	2.007973	0.089831	0.97652	0.95032	0.93252	2.6473	2.999
	19	1.997763	1.775983	0.089720	0.97743	0.95075	0.93419	2.5531	3.012
	29	1.947345	1.999631	0.084566	0.97832	0.95174	0.93555	2.5139	3.058
	39	1.917371	1.929831	0.059741	0.97865	0.95195	0.93860	2.4961	3.184
	49	1.886351	1.909832	0.057631	0.97921	0.95348	0.94642	2.4741	3.452
	59	1.807363	1.905987	0.005574	0.98134	0.95534	0.94875	2.3961	3.872
	69	1.743251	1.889874	0.005174	0.98353	0.95613	0.95875	2.3756	3.891
	79	1.797356	1.899642	0.003752	0.98463	0.95732	0.95999	2.3367	3.928
	89	1.586352	1.858943	0.001734	0.98561	0.95822	0.96641	2.3205	3.963
	99	1.559736	2.81874	0.001538	0.98673	0.95913	0.96831	2.1858	3.971
	109	1.5372651	1.77321	0.009634	0.98751	0.95989	0.97654	2.0751	3.984
	119	1.386345	1.79731	0.002752	0.98462	0.96143	0.97943	2.0356	3.991

Table 1(ii); Simulation study results with Binomial Removals for
$\alpha = 2.87, \theta = 3.02, \beta = 2.62, p = 0.67, \tau = 3.5.$

n	m	$\text{Bias}_{\hat{\theta}}$	$\text{Bias}_{\hat{\beta}}$	$\text{Bias}_{\hat{P}}$
20	9	0.006691	0.009879	0.089431
	19	0.006687	0.009773	0.067909
60	9	0.005982	0.005928	0.063989
	19	0.005791	0.005721	0.047298
	29	0.005194	0.004823	0.045901
	39	0.003791	0.004594	0.028432
	49	0.003913	0.003909	0.018931
	59	0.003492	0.003791	0.015986
80	9	0.003389	0.003588	0.014982
	19	0.002780	0.002791	0.011955
	29	0.002678	0.002279	0.011577
	39	0.002569	0.002254	0.008793
	49	0.002378	0.002093	0.003985
	59	0.002354	0.002056	0.003416
	69	0.002334	0.001973	0.001567
	79	0.001682	0.001671	0.001391
100	9	0.001494	0.001498	0.001198
	19	0.001475	0.001289	0.001203
	29	0.000971	0.001182	0.001982
	39	0.000849	0.000678	0.000689
	49	0.000692	0.000451	0.000486
	59	0.000578	0.000381	0.000198
	69	0.000387	0.000078	0.000139
	79	0.000234	0.000029	0.000116
	89	0.000209	7.35×10^{-5}	0.000104
	99	0.000209	3.74×10^{-5}	0.000101
120	9	9.87×10^{-5}	9.59×10^{-6}	0.000094
	19	9.31×10^{-5}	9.38×10^{-6}	0.000047
	29	7.52×10^{-5}	9.27×10^{-7}	0.000029
	39	4.39×10^{-5}	5.62×10^{-7}	0.000016
	49	2.58×10^{-5}	5.39×10^{-7}	8.79×10^{-5}
	59	5.81×10^{-7}	2.76×10^{-7}	5.91×10^{-5}
	69	4.79×10^{-7}	5.73×10^{-8}	4.13×10^{-5}
	79	3.35×10^{-7}	4.79×10^{-8}	9.95×10^{-6}
	89	2.79×10^{-7}	3.79×10^{-8}	8.88×10^{-6}
	99	2.12×10^{-7}	1.87×10^{-8}	6.83×10^{-6}
	109	2.07×10^{-7}	8.67×10^{-9}	4.67×10^{-6}
	119	6.87×10^{-8}	5.19×10^{-9}	1.39×10^{-6}

Table 2; Simulation study results with uniform removals for
$\alpha = 2.87, \theta = 3.02, \beta = 2.62, \mathrm{p} = 0.67$ and $\tau = 3.5$

n	m	MLE		95%Confidence Interval coverage		$Bias_{\hat{\theta}}$	$Bias_{\hat{\beta}}$	τ^*	$\lvert F^{-1} \rvert$
		$\hat{\theta}$	$\hat{\beta}$	$CP_{\hat{\theta}}$	$CP_{\hat{\beta}}$				
20	9	3.94710	4.98997	0.91018	0.87329	0.097631	0.009959	5.9829	1.009
	19	3.92741	4.98975	0.91317	0.87440	0.093859	0.009936	5.9693	1.119
60	9	3.91931	4.99742	0.91712	0.87489	0.073185	0.009368	5.8746	1.239
	19	3.91673	4.95836	0.91738	0.87511	0.068166	0.008489	5.7837	1.265
	29	3.91391	4.98890	0.91740	0.87546	0.057217	0.008299	5.4728	1.363
	39	3.91832	4.99888	0.91765	0.87599	0.043608	0.006943	5.4643	1.371
	49	3.67721	4.99817	0.91785	0.87632	0.041735	0.006509	5.4489	1.398
	59	3.65360	4.98736	0.91889	0.87790	0.030360	0.006297	5.4098	1.403
80	9	3.48721	4.95742	0.91940	0.87793	0.018429	0.006098	5.4071	1.412
	19	3.46831	4.92646	0.91985	0.87888	0.009588	0.006024	5.3064	1.425
	29	3.47974	4.90896	0.91990	0.89999	0.005981	0.005949	5.1984	1.451
	39	3.29346	4.90693	0.91998	0.91354	0.005945	0.005439	5.1697	1.463
	49	3.25312	4.78931	0.92011	0.91616	0.005674	0.005190	5.0983	1.470
	59	3.21038	4.75726	0.92042	0.91659	0.005395	0.004987	5.0582	1.623
	69	3.09531	4.73842	0.92086	0.91923	0.005194	0.004956	5.0193	1.674
	79	3.09836	4.73571	0.92090	0.91987	0.004004	0.004547	4.9875	1.680
100	9	3.05647	4.59831	0.92119	0.92156	0.003598	0.003757	4.8572	1.731
	19	3.05563	4.56828	0.92187	0.92181	0.003283	0.002899	4.6365	2.643
	29	3.03844	4.29784	0.92319	0.92615	0.003093	0.002875	4.5324	2.684
	39	3.03573	4.27641	0.92355	0.90842	0.003062	0.002598	4.5084	2.299
	49	3.01963	4.25989	0.92488	0.92476	0.000999	0.002429	4.3948	2.384
	59	2.98450	4.23791	0.92556	0.92589	0.000739	0.002125	4.2874	2.715
	69	2.95741	4.09912	0.92580	0.92757	0.000721	0.001356	4.2683	2.764
	79	2.93474	4.06983	0.92666	0.92783	0.000699	0.000896	4.2543	2.754
	89	2.91093	4.06728	0.92691	0.92791	0.000570	0.000597	4.1974	2.794
	99	2.90983	4.01734	0.92921	0.92798	0.000398	0.000496	4.0746	2.790
120	9	2.90657	3.93837	0.92957	0.92799	0.000096	0.000063	3.9973	2.917
	19	2.90633	3.90973	0.93421	0.93523	0.000068	0.000039	3.8374	2.932
	29	2.90435	3.68347	0.93511	0.92645	0.000036	0.000019	3.6467	2.938
	39	2.90313	3.65531	0.92585	0.92685	0.000019	0.000013	3.5621	2.972
	49	2.86414	3.48327	0.92881	0.92985	9.96×10^{-5}	9.02×10^{-5}	3.4788	2.977
	59	2.84531	3.45177	0.93183	0.93145	9.28×10^{-7}	8.84×10^{-5}	3.1845	2.999
	69	2.84313	3.42641	0.93371	0.93351	9.09×10^{-7}	8.36×10^{-5}	3.1477	3.031
	79	2.75443	3.28421	0.93612	0.93831	8.45×10^{-7}	7.79×10^{-5}	3.1248	3.074
	89	2.73249	3.26947	0.93722	0.935145	8.33×10^{-7}	7.34×10^{-5}	3.0983	3.187
	99	2.71734	3.24874	0.94513	0.93690	7.84×10^{-7}	6.78×10^{-5}	3.0387	3.191
	109	2.59931	3.19834	0.94721	0.93800	6.69×10^{-7}	6.34×10^{-5}	3.0276	3.284
	119	2.51677	3.15893	0.94882	0.93641	4.05×10^{-7}	5.99×10^{-6}	2.9975	3.291

7 CONCLUSION

This paper considers the SS-PALT under type-II progressive censoring with Binomial and uniform removals assuming frechet distribution**.** Comparison between both removal are shown. The Newton-Raphson method is applied to obtain the optimal stress-change time τ^* which minimizes the GAV.

The numerical study for obtaining the optimum plan for binomial removal is tabulated in table 1 for different sample size and table 2 describes uniform removal for possible values of scale and shape parameters. From the above results it is easy to find that for the fixed values of the parameters, the error and optimal time decrease with increasing sample size n.
Performance of testing plans and model assumptions are usually evaluated by the properties of the maximum likelihood estimates of model parameters. Hence from the numerical result we can conclude that estimates of binomial are more stable with relatively small error with increasing sample size. Therefore, the test design obtained here is robust design and work well for binomial removal.

As a future work, this study can be extended to explore the situation under type-I progressive censoring

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