ASYMPTOTIC ANALYSIS OF FEW NODES FAILURE IN ORIENTED RANDOM GRAPH

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ABSTRACT

In this paper an oriented graph with high reliable nodes is considered. A node stops its work if there is failed node which has a way to this node. Asymptotic formulas for a calculation of a probability that some nodes in the graph stop their work are obtained for different conditions on a graph structure and on a number of the nodes. These formulas allow to obtain conditions on reserve in considered graphs if nodes have different failure probabilities.

Keywords: an oriented graph, a high reliable node, a failure probability.

INTRODUCTION

In the reliability theory there is a problem of an investigation of graphs with unreliable edges and absolutely reliable nodes [1], [2], graphs with unreliable nodes and absolutely reliable edges [3], [4], [5]. Graphs analyzed in cited articles are not oriented.

In this paper a model of a random graph in which a node stops its work if there is failed node which has a way to this node. This model is considered in the monograph [6] devoted to failures trees in technological systems. Such model also is connected with an idea of random network controllability which is used in medical - biological applications [7].

In this paper asymptotic formulas are obtained for a calculation of a stop probability of few nodes in a condition that graph nodes are high reliable but have different failure probabilities. First results in this direction have been obtained in a case of identical failure probabilities of nodes [8]. The formulas allow to analyze a problem of a reservation in considered graphs. One of approaches to produce such formulas is a construction of incompatible events sequence [9]. But in our case this approach leads to calculations with geometrical complexity by a number of graph nodes.Obtained result is based on an asymptotic expansion in the Poincare inclusion and exclusion formula. Specifics of the obtained result is an inclusion of nodes failures weights into asymptotic formulas.

MAIN RESULTS

Consider the oriented graph G with the finite set of nodes I. On the set I define a relation of a partial order: $i' \geqslant i''$, if in the graph G there is a way from the node i' to the node i''. For each node $i \in I$ contrast the set of nodes $F_i = \{ i' : i' \geqslant i\}$. Assume that the graph G nodes work independently with the probabilities $p_i, i \in I$ and if in F_i there is a failed node then the node i stops its work.

Consider the subset of nodes $R = \{1, \dots, r\} \subseteq I$ and calculate the probability $Q(R)$ that all nodes in R stop their work. Denote $c(i_1, \dots, i_k)$ the sum of all c_i such that $i \in T(i_1, \dots, i_k) = \bigcap_{j=1}^{k} F_{i_j}, 1 \le i_1 < \dots < i_k \le r, 1 \le k \le r$.

Theorem 1. If $p_i = exp(-c_i h),\ c_i > 0, i \in I$, then for any natural r we have the relation $Q(R) = h\, c(1, \dots, r) + O(h^2), h \to 0$.

Proof. Denote $D(k)$ the random event that there are not fails in the set F_k. Then from the Poincare formula of inclusions and exclusions we have

$$Q(R) = 1 - P\left(\bigcup_{k=1}^{r} D(k)\right) = 1 - \sum_{k=1}^{r} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le r} P\left(\bigcap_{j=1}^{k} D(i_j)\right), \tag{1}$$

$$P\left(\bigcap_{j=1}^{k} D(i_j)\right) = exp(-h\overline{N}),\ \overline{N} = \sum_{s=1}^{k} (-1)^{s-1} \sum_{1 \le j(1) < \dots < j(s) \le k} c(i_{j(1)}, \dots, i_{j(s)}).$$

Substituting the relation (2) into Formula (1) and using the Taylor expansion of the exponent we obtain for $h \to 0$:

$$Q(R) = 1 - \sum_{k=1}^{r} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le r} \left(1 - h\overline{N}\right) + O(h^2) =$$
$$= l_0 + h \sum_{k=1}^{r} \sum_{1 \le i_1 < \dots < i_k \le r} c(i_1, \dots, i_k)\, l(i_1, \dots, i_k) + O(h^2). \tag{2}$$

Here $l(i_1, \dots, i_k)$ are some integer coefficients. As each transposition of indexes $i_1, \dots, i_k$, $1 \le i_1 < \dots < i_k \le r, 1 \le k \le r,$ in Formula (2) does not change $Q(R)$, so we have the equalities

$$l(i_1, \dots, i_k) = l(1, \dots, k),\ l_0 = 1 - \sum_{k=1}^{r} (-1)^{k-1} C_r^k = \sum_{k=0}^{r} (-1)^k C_r^k = 0,$$
$$l(1, \dots, s) = \sum_{k=s}^{r} (-1)^{k-s} C_r^{k-s} = 0,\ 1 \le s \le r-1,\ l(1, \dots, r) = 1.$$

Theorem 1 is proved.

For $c(1, \dots, r) = 0$ from Theorem 1 we obtain that $Q(R) = O(h^2)$. But a problem is to formulate sufficient conditions of the relation $Q(R) = O(h^{t+1}),\ t \ge 2$. Such conditions may be obtained from the following statement.

Theorem 2. If $p_i = exp(-c_i h),\ c_i > 0, i \in I,$ and for any subset of different nodes $i, j, k \in R$ we have the relation $F_i \cap F_j \cap F_k = \emptyset$ then for $2 < 2t < r$ the equality $Q(R) = O(h^{t+1})$ is true.

Proof. Denote $\overline{C}(i_1, \dots, i_k)$ the sum of all c_i such that $i \in \bigcup_{j=1}^{k} F_{i_j}$ and put

$$S_l = \sum_{k=1}^{r} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le r} \overline{C}^{\,l}(i_1, \dots, i_k), \tag{3}$$
$$\overline{C}(i_1) = c(i_1), \overline{C}(i_1, \dots, i_k) = \sum_{j=1}^{k} c(i_j) - \sum_{1 \le j < l \le k} c(i_j, i_l), \qquad 2 \le k \le r.$$

From Formula (1) and Theorem 2 conditions we have that $Q(R) = -\sum_{l=2}^{k} (-1)^l \frac{h^l}{l!} S_l + O(h^{t+1})$.

Prove the equalities $S_l = 0, 2 \le l \le r$, for a simplicity bounding ourselves by the case $l = 2$. Denote $b(i_1, \dots, i_4)$ the number of different nodes in the set $\{i_1, \dots, i_4\}$, $i_1, \dots, i_4 \in R$. From Formula (3) we have that for some coefficients $a(i_1, \dots, i_4)$

$$S_2 = \sum_{k=1}^{r} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le r} \sum_{1 \le j(1), \dots, j(4) \le k} a\left(i_{j(1)}, \dots, i_{j(4)}\right) c\left(i_{j(1)}, i_{j(2)}\right) c\left(i_{j(3)}, i_{j(4)}\right)$$

with $a\left(i_{j(1)}, \dots, i_{j(4)}\right)=\sum_{k=b\left(i_{j(1)},\dots,i_{j(4)}\right)}^{r}(-1)^{k-b\left(i_{j(1)},\dots,i_{j(4)}\right)} C_{r-b\left(i_{j(1)},\dots,i_{j(4)}\right)}^{k-b\left(i_{j(1)},\dots,i_{j(4)}\right)}=0$.

From Formula (3) the equality $S_2 = 0$ takes place. Theorem 2 is proved.

Corollary1. If $p_i = exp(-c_i h)$, $c_i > 0, i \in I$, and for any set of different nodes $i_1, \dots, i_{m+1} \in R$ the relation $\bigcap_{s=1}^{m+1} F_{i_s} = \emptyset$ is true then for $mt < r$ the equality $Q(R) = O(h^{t+1})$ takes place.

Remark 1. Theorem 2 gives conditions when it is possible to form a reserve in the network with high reliable nodes.

Remark 2. All calculations are made without any restrictions on an oriented graph structure. But for a convenience of the graph representation it is possible to factorize oriented graph nodes by a relation of a cyclic equivalence and to introduce an edge between two factors if there is edge between some nodes of these factors. Then a graph structure becomes more simple (it is acyclic) but a factorization procedure is sufficiently complicated.

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