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BOUNDS FOR THE RELIABILITY OF DIFFERENT REDUNDANT SYSTEMS

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ABSTRACT

In many cases, reliability can be improved by using redundant components. This is an approach that is applied especially in information networks. In this paper we study redundant systems with imperfect switches. We show that there exists a limit as the number of redundant components tends to infinity. This limit is computed for components with exponential life time distributions, which is the typical distribution for digital equipment used in information systems. For components with distributions belonging to the NBUE or HNBUE classes, bound are derived.

1. INTRODUCTION

In order to improve the reliability of a system there are mainly two possibilities. The first one is to improve the reliability of the components, the second is to implement redundancy. Mainly this is done by using more than one component to fulfill the same function, see e.g. Barlow & Proschan (1976). Redundancy means that in a technical system there are more possibilities present to ensure a function, than the necessary minimum. If one discards influences as costs and needed space, one might come to the conclusion that using redundant items, one could improve system reliability up to an arbitrarily high level. In this paper we will discuss the problem whether it is possible to improve reliability up to an arbitrary high level. Using redundant components is an approach used mainly in networks, especially in telecommunication networks. If a certain link or node fails, traffic is rerouted to other nodes and links.

In this paper we will show that, under several assumptions, reliability cannot be improved further than to a certain limit.

In section 2 we will describe the main assumptions of our model. In the next two chapters we consider two extremal modes of standby, hot standby and cold standby. Hot standby means that the load on the standby component is the same as on the main component and that no load sharing between the redundant components occurs. Cold standby describes a situation, where the redundant devices do not age at all during their standby phase, i.e. when the main component provides the service. All other modes of standby will describe modes of ageing that are between these two situations of load on the redundant components.

In the third section we describe the situation of hot standby, the worst case regarding ageing.

In the fourth section, we discuss the situation of cold standby, no ageing of the standby components.

Section five provides an example and in section six we give a summary and conclusion.

2. MAIN ASSUMPTIONS

For the model the following assumptions shall hold

a) Detection and switching to another component is not perfect but fails. Here the probability of failure of switching from the failed component to the redundant one includes the failure of the switch itself in case of detection of the failure, the failure of the detection mechanisms when the switch is working as well as failure of both switch and detection mechanisms. This resulting probability is denoted by γ
b) The lifetime of the components is random and follows the lifetime distribution F(x) with F(0) = 0 and

$$\lim_{x\to\infty} F(x) = 1$$

c) The failure times of all redundant components are completely statistically independent from each other.
d) The number of redundant components is not limited.
e) All redundant components have the same lifetime distribution.
f) The lifetime distribution of the components is continuous, differentiable and has a finite mean.

The model has been described in more detail in Shubinsky (2012).

Parallel systems with imperfect switching to redundant components will be called imperfect systems in this paper.

The following figure shows an example of a system with redundant components. Each of the m, possibly different, components has n redundant replications. We will study this type of systems for $n \to \infty$

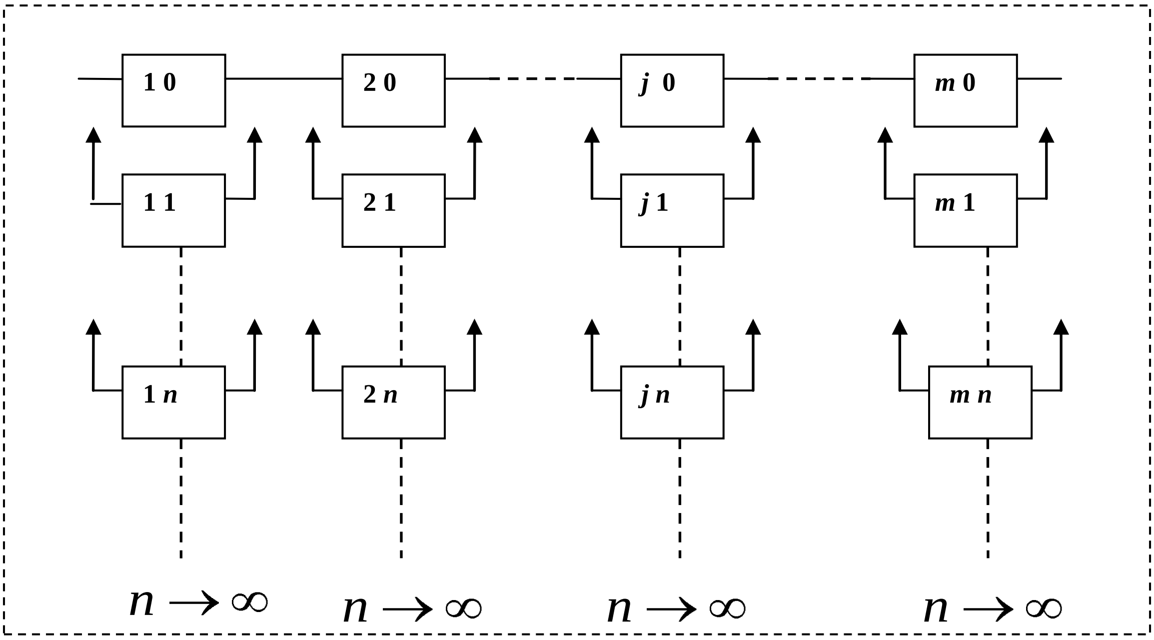

Figure 1. System with redundant components

In the following subsections we will simplify the system in figure 1 by considering only one component with its redundant replications.

3. HOT STANDBY

For hot standby, all components are under full load from the beginning. So this is in fact a situation of a simple parallel system. Assume that a component with lifetime distribution F(x) is connected in parallel with all its replications. The following figure 2 shows the reliability block diagram of the system. Assume that n components are connected in parallel.

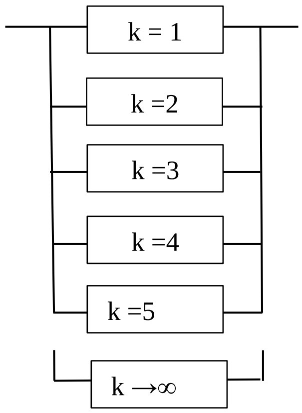

Figure 2. System with parallel structure of components

The lifetime distribution of the parallel system with hot standby can now be computed as follows.

In order to have achieve a redundancy of level k, i.e. that k are components functioning, k-1 successful switchovers are necessary with a failure on the k-th switch-over.

The probability of this event is $(1-\gamma)^{k-1}\gamma$. The distribution function of k identical units with lifetime distribution F(x) and connected in parallel is

$$1-(1-F(x))^{k}. \tag{1}$$

Combining both expressions and summing up we arrive at

$$\sum_{i=1}^{k}\gamma(1-\gamma)^{i-1}(1-(1-F(x))^{i}) \tag{2}$$

If now k tends to infinity, this gives

$$\sum_{i=1}^{\infty}\gamma(1-\gamma)^{i-1}F(x)^{i} = \frac{\gamma F(x)}{1-(1-\gamma)F(x)} = G(x), \tag{3}$$

where G(x) denotes the distribution function of the lifetime of the redundant system.

Note that, the lifetime distribution of the parallel system is given by an analytic expression. Moreover, one can observe that

$$G(x) = \frac{\gamma F(x)}{1-(1-\gamma)F(x)} \le F(x) \tag{4}$$

which follows easily from

$$\gamma F(x) \le F(x) - (1-\gamma)F(x)^{2} \text{ and}$$
$$(1-\gamma)F(x) \ge (1-\gamma)F(x)^{2}.$$

The latter is obvious since $F(x) \ge F(x)^{2}$.

Considering (4) one can see that (4) is smaller than the distribution of a single component, but even in the limiting case, the failure probability does not vanish. This is only possible for

perfect switching, i.e. γ=0. For all positive values of γ which means imperfect switching, G(x) will form a lower bound for all systems with a large but finite number of redundant elements.
Now we can compute the mean lifetime by

$$m_G = \int_0^\infty (1-G(x))dx = \int_0^\infty \frac{1-F(x)}{1-(1-\gamma)F(x)}\, dx \tag{5}$$

For an exponential distribution, one computes

$$m_G = \int_0^\infty \frac{\exp(-\lambda x)}{1-(1-\gamma)(1-\exp(-\lambda x))}\, dx = \int_0^\infty \frac{\exp(-\lambda x)}{\gamma+(1-\gamma)\exp(-\lambda x)})\, dx = -(1/\lambda)\ln(\gamma) / (1-\gamma). \tag{6}$$

For γ=1 this gives $1/\lambda$, which is the result for the exponential distribution without redundancy. Again, for imperfect switching, m_G always stays bounded and its value is determined by m_F and γ.

Now, for a function that belongs to the NBUE (new better than used in expectation) or NWUE (new worse than used in expectation) family we can show that an expression as (1) is an upper (lower) bound on the mean value of the distribution function G.

A lifetime distribution function belongs to the class NBUE (NWUE) if it satisfies

$$\int_x^\infty (1-F(t))dt \le(\ge)\ m_F(1-F(x)) ,$$

where m_F is the mean of F(x), see e.g. Barlow and Proschan (1976)

If now F(x) belongs to the class NBUE (or NWUE) the following inequality holds

$$m_G \le (\ge) -m_F\ln(\gamma)/(1-\gamma). \tag{7}$$

This result can be proven as follows.

We rewrite (6) in the following form:

$$m_G = \int_0^\infty \frac{1-F(x)}{1-(1-\gamma)F(x)}\, dx = -\int_0^\infty \frac{d \int_x^\infty (1-F(t))dt}{1-(1-\gamma)F(x)} \tag{8}$$

Integrating this expression by parts, we arrive at

$$m_G = m_F/(1-(1-\gamma)) + \int_0^\infty \int_x^\infty (1-F(t))dt\, d\frac{1}{1-(1-\gamma)F(x)} \tag{9}$$

Using the NBUE (NWUE) property this can be rewritten as

$$m_G \le (\ge)\ m_F/(1-(1-\gamma)) - \int_0^\infty m_F\,(1-F(x))\, d\frac{1}{1-(1-\gamma)F(x)} \tag{10}$$

and integrating by parts again

$$m_G \le (\ge)\ m_F \int_0^\infty \frac{(1-F(x))dx}{1-(1-\gamma)F(x)}) = -m_F \ln(\gamma)/(1-\gamma) \tag{11}$$

This proves (7).

Using the expression (3), we can derive an inequality for the residual life function T_{RL}. The latter is defined by

$$T_{RL} = \int_x^\infty (1-G(t))dt\ .$$

Using (3) we arrive at

$$T_{RL} \quad = \int_x^\infty (\frac{1-F(t)}{1-(1-\gamma)F(t)})dt = -\int_x^\infty (\frac{1}{1-(1-\gamma)F(t)})d\int_t^\infty (1-F(s))ds\ .$$

Integrating by parts, we get

$$T_{RL} = \frac{1}{1-(1-\gamma)F(x)}\int_x^\infty (1-F(t))dt + \int_x^\infty \int_t^\infty (1-F(s))ds\ d(\frac{1}{1-(1-\gamma)F(t)}).$$

For a NBUE (NWUE) distribution this leads to

$$-\frac{m_F(1-F(x))}{1-(1-\gamma)F(x)} - \int_x^\infty m_F(1-F(t))\ d(\frac{1}{1-(1-\gamma)F(t)}).$$

Integrating by parts again, this expression equals

$$T_{RL} \le (\ge)\ -m_F \int_x^\infty \frac{d(1-F(t))}{1-(1-\gamma)F(t)} = m_F \int_x^\infty \frac{dF(t)}{1-(1-\gamma)F(t)} = (m_F/\gamma)\ln(\frac{\gamma}{1-(1-\gamma)F(x)})$$

Putting everything together, we arrive at

$$T_{RL} \le (\ge)\ (m_F/\gamma)\ln(\frac{\gamma}{1-(1-\gamma)F(x)})$$

For the exponential distribution, the equality holds.

4. COLD STANDBY

The case of cold standby is the other extremal case. Here, the lifetime distribution of a parallel system is computed by

$$G(x) = \sum_{i=1}^{\infty} \gamma(1-\gamma)^{i-1}F^{(i)}(x), \tag{12}$$

where $F^{(i)}(x)$ denotes the i-fold convolution of the distribution function F(x). The convolution is defined by

$F^{(1)}(x) = F(x)$

for the first order convolution, all higher orders are defined iteratively by

$$F^{(k+1)}(x) = \int_0^x F^{(k)}(x-t)dF(t) . \qquad (13)$$

Formula (12) is derived from the probability $(1-\gamma)^{i-1}\gamma$ for a failure of the system when the switching to the i-th redundant component and the lifetime distribution $F^{(i)}(x)$ of i successively used components .

For the type of distributions given by (12), a general analytical solution does not exist. However, the following results can easily be obtained.

For an exponential distribution with density $f(x) = \lambda \exp(-\lambda x)$ one obtains (see /Shubinski/)

$$G(x) = 1-\exp(-\lambda\gamma x). \qquad (14)$$

If $\gamma=1$ (switching fails always), we arrive at the usual exponential distribution of a single component. The result (9) can be easily derived by using

$$f^{(k)}(x) = \lambda^{k}\text{-}1\exp(-\lambda x) /(k-1)! \qquad (15)$$

and computing the density g(x).

Using results of Schäbe (1986), we can also derive other analytical results for special Gamma distributions that have the following form

$$F(x) = \lambda^{\alpha}x^{\alpha-1}\exp(-\lambda x)/\Gamma(\alpha) \qquad (16)$$

The results are given in the following table.

Table 1. density functions g(x) for special types of gamma densities for f(x).

Parameters	**density g(x) of the parallel system**
$\alpha=1/2$	$\gamma\sqrt{\frac{\lambda}{\pi x}}\exp(-\lambda x)+\lambda\gamma(1-\gamma)\exp(-\lambda(1-\gamma)^2/2)\mathrm{erfc}(-\lambda(1-\gamma)\sqrt{x})$
$\alpha = 1$	$\lambda\gamma \exp(-\lambda\gamma x)$
$\alpha = 2$	$\frac{\gamma\lambda}{2\sqrt{1-\gamma}}(\exp(-(1-\sqrt{1-\gamma})\lambda x) - \exp(-(1+\sqrt{1-\gamma})\lambda x)$
$\alpha = 3$	$\frac{\lambda\gamma}{(1-\gamma)^{2/3}}\left(\frac{1}{3}\exp(\lambda x(1-\gamma)^{1/3})-\frac{2}{3}\exp(-\lambda x(1-\gamma)^{1/3})\cos(\frac{3}{2}\lambda x(1-\gamma)^{1/3}-\pi/3)\right)$
$\alpha = 4$	$\frac{\lambda\gamma}{2(1-\gamma)^{3/4}}\exp(-\lambda x)(\sinh(\lambda(1-\gamma)^{1/4}x- \sin(\lambda(1-\gamma)^{1/4}x)$

Also, it has been shown in Schäbe (1986), that

$$m_G = m_F/\gamma. \qquad (17)$$

Therefore, no approximation for m_G needs to be given.

One may note, that the mean is limited, even if the number of redundant devices becomes infinite. The distribution function G(x) has no closed form expression in the general case. So, it is worthwhile to have a bound on it. In Schäbe (1986) it has been shown in theorem 3.2 that if F belongs to the class NBUE (NWUE), the same holds for G. An analogous result has been proven for the class HNBUE (harmonic new better than used in expectation) and HNWUE (harmonic worse than used in expectation) in theorem 3.4. The latter result can be used to give a bound on G. If F is HNBUE (HNWUE), we have for the distribution G the following inequality for the residual life function, see Klefsö (1982)

$$\int_x^\infty (1-G(t))dt \le (\ge)\; m_G \exp(-x/m_G) = (m_F/\gamma) \exp(-\gamma x/m_F) \quad (18).$$

Also this expression shows, that an infinite number of redundant devices is not able to improve the residual life function further than to a certain value. For HNBUE distributions, we derived an upper bound on an infinitely increasing number of redundant devices.

5. EXAMPLE

In this section we will show how the mean lifetime depends on the number of components used for redundancy and how it depends on the probability γ of failure of switching for a cold standby system.
From (5) we have.

$$G(x) = 1-\exp(-\lambda\gamma x).$$

For a system as in figure 1 consisting of m components connected in series each having k redundant replications this gets

$$G(x) = 1-\exp(-\lambda\gamma kx).$$

This distribution has mean $1/(\lambda\gamma k)$. Now the relative mean of the system with redundancy over a system consisting of one element with failure rate λ is

$$R = 1/(\gamma k).$$

Let us now denote by $\alpha=1-\gamma$ the probability that detection of a fault and switching to the redundant component is successful.

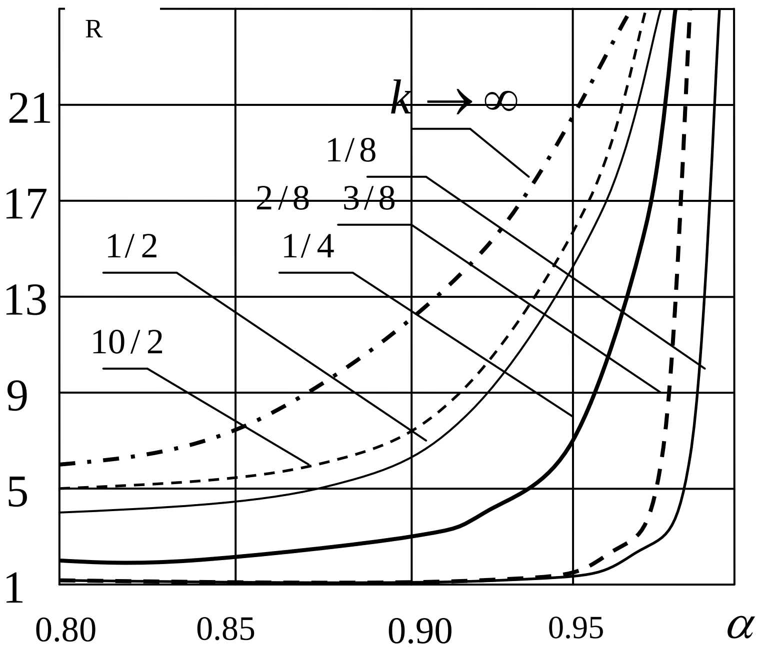

Figure 3. Relation of means 1/(γk) depending on α.

For k=1 the mean life time is plotted by a simple line. One can observe that with increasing degree of redundancy (k) the mean lifetime grows. Also, with increasing α, i.e. with increasing quality of switching, the mean lifetime also increases.

6. DISCUSSION AND CONCLUSIONS

Now we can provide the following limits for the different types of systems.

Table 2. Overview of the limit values for parallel systems with an independent number of components.

Characteristics	**Limit for hot standby**	**Limit for cold standby**
G(x)	$\frac{\gamma F(x)}{1-(1-\gamma)F(x)}$	$G(x) = \sum_{i=1}^{\infty} \gamma(1-\gamma)^{i-1} F^{(i)}(x)$
m_G	$\leq (\geq)\ -m_F \ln(\gamma)/(1-\gamma)$ For F being NBUE (NWUE), equality for the exponential distribution	$m_G = m_F/\gamma$
Residual life $\int_x^{\infty}(1-G(t))dt$	$\leq (\geq)\ (m_F/\gamma) \ln\left(\frac{\gamma}{1-(1-\gamma)F(x)}\right)$ For F being NBUE (NWUE), equality holds for the exponential distribution	$\leq (\geq)\ (m_F/\gamma) \exp(-\gamma x/m_F)$ For F being HNBUE (HNWUE), equality holds for the exponential distribution

Note that, the limit itself is an upper bound for systems with a finite number of redundant components. So the upper bounds for real systems with a finite number of components is given by the NBUE / HNBUE limits. This is given in table 3

Table 3 upper bounds for imperfect parallel systems.

Characteristics	**Limit for hot standby**	**Limit for cold standby**
$G(x)$	$\dfrac{\gamma F(x)}{1-(1-\gamma)F(x)}$	$G(x) = \sum_{i=1}^{\infty} \gamma(1-\gamma)^{i-1} F^{(i)}(x)$
m_G	$-m_F \ln(\gamma)/(1-\gamma)$ For F being NBUE	$m_G = m_F/\gamma$
Residual life $\int_x^{\infty}(1-G(t))dt$	$(m_F/\gamma)\ln\left(\dfrac{\gamma}{1-(1-\gamma)F(x)}\right)$ For F being NBUE	$(m_F/\gamma)\exp(-\gamma x/m_F)$ For F being HNBUE

An imperfect system cannot achieve better values than given in the table above for components that satisfy the NBUE or HNBUE property.

In this paper we have obtained distribution functions for parallel systems in the case that switching to redundant devices is not perfect. It has turned out that there exists a limit and reliability cannot be improved up to 1. This can only be reached if switching is perfect.

This implies that at a certain stage of system development it is worthwhile to improve the reliability of the switching algorithm that to implement further additional redundant devices.

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