A NEW BATHTUB SHAPED FAILURE RATE MODEL

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Abstract

In this paper, we introduce a new Bathtub shaped failure rate model named as x-Exponantial Model and present a comparative study with Generalized Lindley, Generalized Gamma and Exponentiated Weibull distributions.

I. INTRODUCTION

There are many distributions for modeling lifetime data. Among the known parametric models, the most popular are the Lindley, Gamma, log Normal, Exponentiated Exponential and the Weibull distributions. These five distributions are suffer from a number of drawbacks. None of them exhibit bathtub shape for their failure rate functions. The distributions exhibit only monotonically increasing, montonically decreasing or constant failure rates. This is a major weakness because most real life system exhibit bathtub shapes for their failure rate functions. Atleast three of the four distributions exhibit constant failure rates. This is a very unrealistic feature because there are hardly any real life systems that have constant failure rates. This is a major weakness because most real life system exhibit bathtub shapes for their failure rate functions. Secondly atleast three of the four distributions exhibit constant failure rates. This is a very unrealistic feature because there are hardly any real life systems that have constant failure rates.

Generalized Lindley, Generalized Gamma and Exponentiated Weibull distributions are proposed for modeling Lifetime data having bathtub shaped failue rate model. In this paper we introduce a simple model but exhibiting bathtub shaped failure rate and discuss the failure rate behavior of these distributions. A comparative study is carried out.

Section 2, discussed the Lindley Distribution, Section 3 discussed Generalized Lindley distribution, section 4 discussed Generalised Weibull distribution, section 5 discussed Generalized Gamma distribution, section 6 introduced new model, called x-Exponential and conclusions are given at the final section.

II. LINDLEY DISTRIBUTION

Lindley distribution was introduced by Lindley (1958) in the context of Bayesian statistics, as a counter example of fudicial statistics. Ghitany et al. (2008) observed that this distribution can be quite effectively used in lifetime experiments, particularly as an alternative of exponential distribution, as it also has only scale parameter. More so, in real world, we rarely encounter the engineering systems which have constant failure rate through their life span. Therefore, it seems practical to assume failure rate as a function of time. Lindley distribution is one of the distributions, having time-dependent failure rate.

The probability density function (pdf) of a Lindley random variable X, with scale parameter λ is given by

$$f(x) = \frac{\lambda}{1+\lambda}(1+x)e^{-\lambda x}, x > 0, \lambda > 0$$

The cumulative distribution function is

$$F(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}, x > 0, \lambda > 0$$

Lindley distribution is positively skewed distribution.

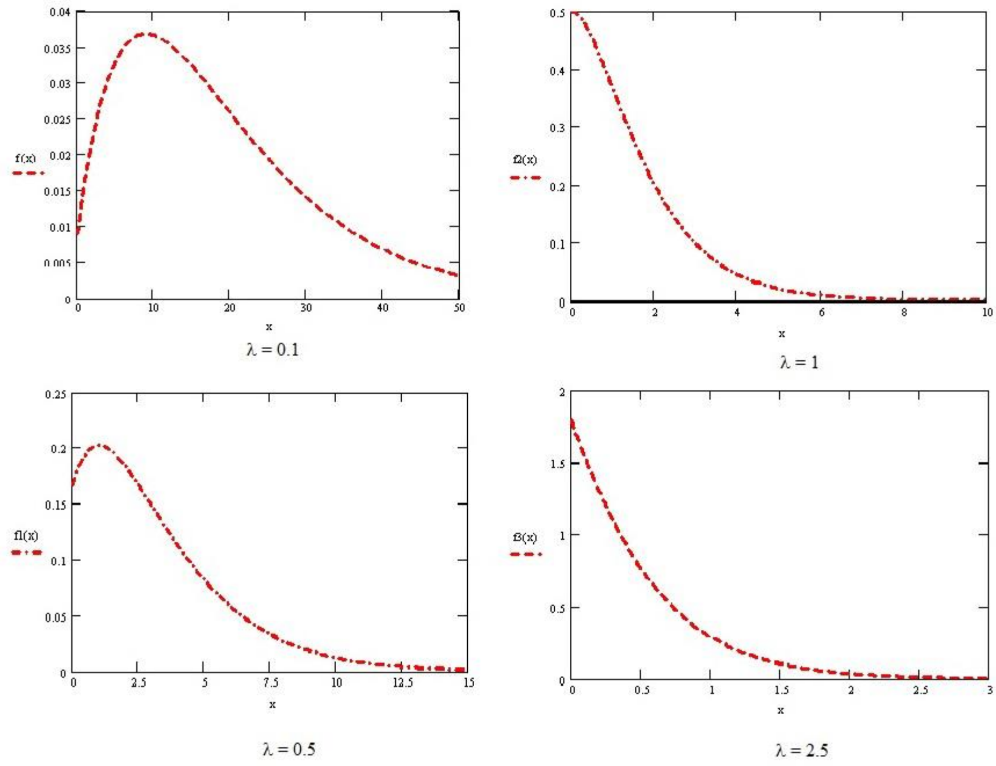

Figure 2.1: Probability density function of Lindley for λ= 0:1; 0:5; 1:0 and 2.5.

The Failure Rate Function of Lindley distribution is

$$h(x) = (\lambda^2(1+x))/(1+\lambda(1+x)), x > 0, \lambda > 0.$$

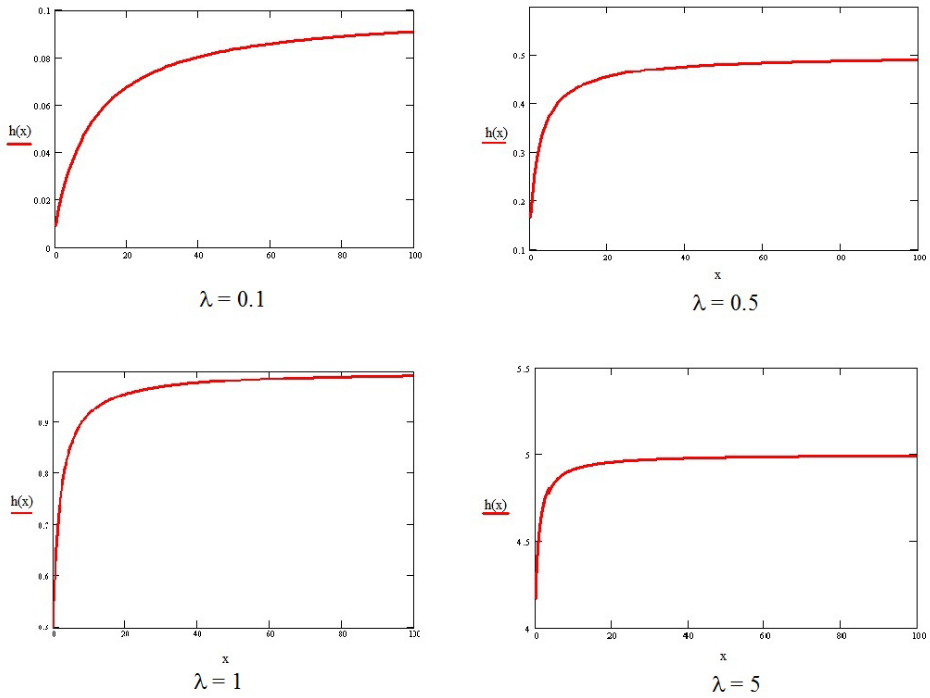

Figure 2.2: Failure rate function of Lindley distribution for λ = 0:1; 0:5; 1:0 and 5.0.

III. GENERALIZED LINDLEY DISTRIBUTION

Suppose $X_1, X_2, \dots, X_n$ are independent random variables distributed according to Lindley distribution and $T = \min(X_1, X_2, \dots, X_n)$ represent the failure time of the components of a series system, assumed to be independent. Then the probability that the system will fail before time x is given by

$$F(x) = [1 - (1 + \lambda + \lambda x)/(1 + \lambda)\ e^{\wedge}(-\lambda x)]^n, x > 0, \lambda > 0.$$

It is the distribution of the failure of a series system with independent components. The cumulative distribution function and pdf of Generalized Lindley distribution are

$$F(x) = [1 - (1 + \lambda + \lambda x)/(1 + \lambda)\ e^{\wedge}(-\lambda x)]^\alpha, x > 0, \lambda > 0, \alpha > 0$$

$$f(x) = \frac{\alpha\lambda(1 + x)}{1 + \lambda}[1 - (1 + \lambda + \lambda x)/(1 + \lambda)\ e^{\wedge}(-\lambda x)]^{\alpha - 1} e^{-\lambda x}, x > 0, \lambda > 0, \alpha > 0$$

The equation has two parameters, λ and ◎ just like the Gamma, log Normal, Weibull and exponentiated Exponential distribution. For = 1 it reduces to Lindley distribution.

The failure rate function is

$$h(x) = \frac{〚(\alpha\lambda(1 + x))/(1 + \lambda)\ [1 - (1 + \lambda + \lambda x)/(1 + \lambda)\ e^{\wedge}(-\lambda x)]〛\ {}^{\wedge}(\alpha - 1)\ e^{\wedge}(-\lambda x)}{1 - [1 - (1 + \lambda + \lambda x)/(1 + \lambda)\ e^{\wedge}(-\lambda x)]^\alpha}, x > 0, \lambda > 0, \alpha > 0$$

The shape of the failure rate function appears monotonically decreasing or to initially decrease and then increase, a bathtub shape if ◎< 1; the shape appears monotonically increasing if ◎≥1. So the Generalized Lindley distribution allows for monotonically decreasing, monotonically increasing and bathtub shapes for its failure rate function.

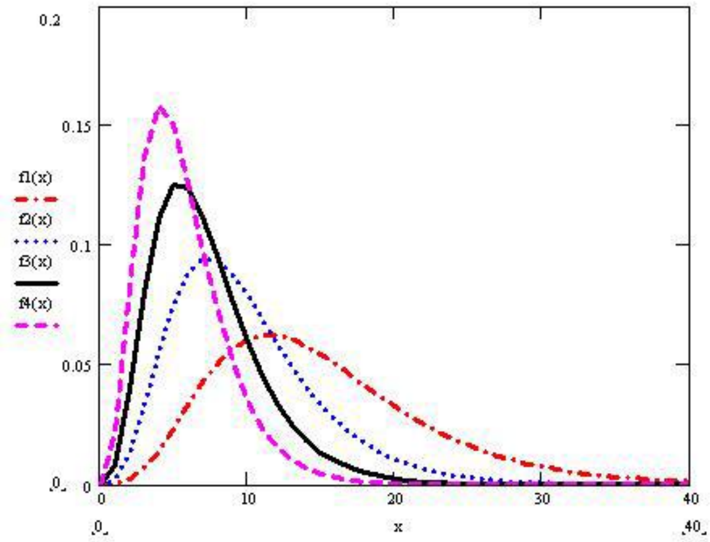

Figure 3.1. Probability density function of Generalized Lindley distribution.

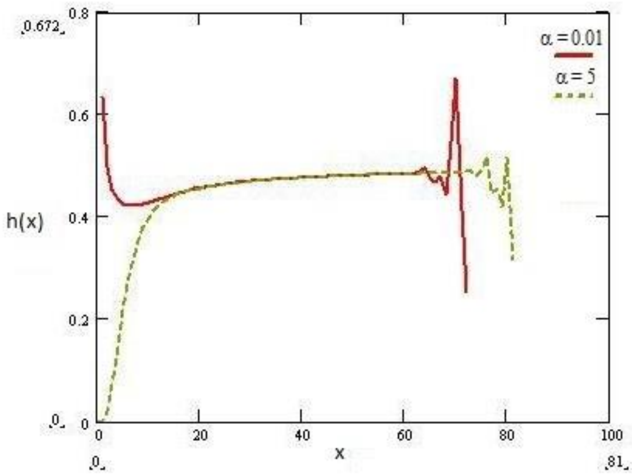

Figure 3.2. Failure rate function of Generalized Lindley distribution

IV. Exponentiated Weibull Distribution

We consider the Exponentiated Weibull (EW) distribution which has a scale parameter and two shape parameters. The Weibull family and the Exponentiated Exponential (EE) family are found to be particular cases of this family. The cumulative distribution function of the Exponentiated Weibull distribution is given by

$$F(x) = \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\lambda}, \lambda > 0, \alpha > 0, \beta > 0.$$

Here λ and ◎ denote the shape parameters and β is the scale parameter. For When λ= 1, the distribution reduces to the Weibull Distribution with parameters. When β = 1, ◎=1 it represents the (EE) family. Thus, EW is a generalization of EE family as well as the Weibull family.

Then the corresponding density function is

$$f(x) = \left(\frac{\alpha\theta}{\sigma}\right)[1-\exp\{-(x/\sigma)^{\alpha}\}]^{\theta-1}\exp\{-(\frac{x}{\sigma})^{\alpha}\}(\frac{x}{\sigma})^{\alpha-1}, x \geq 0.$$

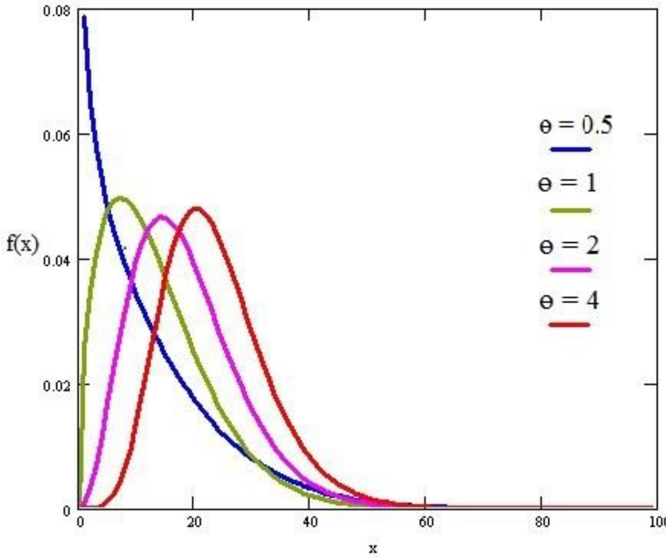

Figure 4.1 Probability density function of Exponentiated Weibull distribution

$$f(x) = \frac{\left(\frac{\alpha\theta}{\sigma}\right)\left[1-\exp\left\{-\left(\left[\!\left[\frac{x}{\sigma}\right]\!\right]\right)^{\alpha}\right\}\right]^{\theta-1}\exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}\left(\frac{x}{\sigma}\right)^{\alpha-1}}{1-\left[1-\exp\left\{-\left(\left[\!\left[\frac{x}{\sigma}\right]\!\right]\right)^{\alpha}\right\}\right]^{\theta}},$$

$$x \geq 0, \qquad \alpha, \theta, \sigma > 0.$$

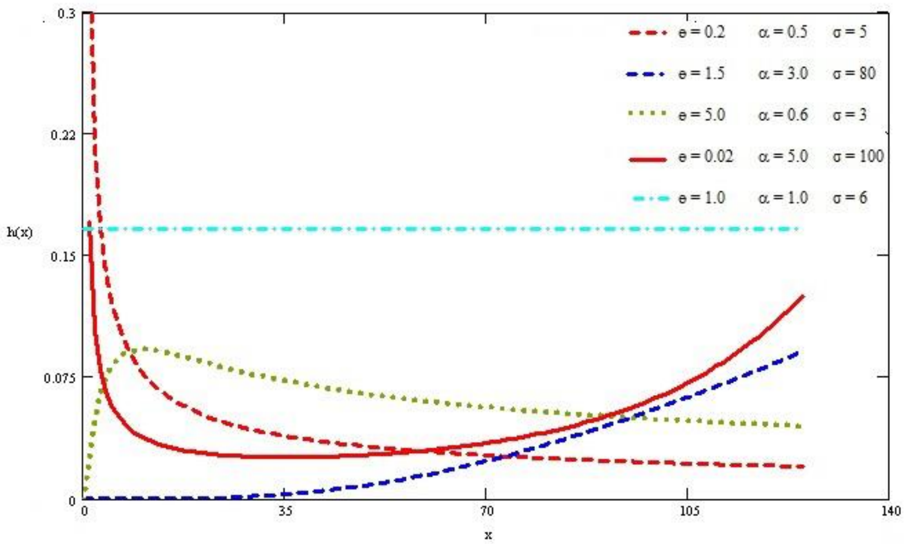

Figure 4.2: Plot of the failure rate function of EW distribution

The EW distribution is constant for ◉= 1 and = 1. The EW distribution is IFR for > 1 and ◉≥1. The EW distribution is DFR for ◉< 1 and ◉≤1. The EW distribution is BT(Bathtub) for ◉> 1 and ◉< 1. The EW distribution is UBT (Upside down Bathtub) for ◉ < 1 and ◉ > 1.

V. Exponentiated Gamma Distribution

The Gamma distribution is the most popular model for analyzing skewed data and hydrological processes. This model is flexible enough to accommodate both monotonic as well as non-monotonic failure rates. The Exponentiated Gamma (EG) distribution is one of the important families of distributions in lifetime tests. The Exponentiated Gamma distribution has been introduced as an alternative to Gamma and Weibull distributions.

The Cumulative Distribution function of the Exponentiated Gamma distribution is given by

$$G(x) = [1 - \exp\{-\lambda x\}\,(1 + \lambda x)]^{\theta}, x > 0, \lambda, \theta > 0.$$

where λ and ◉ are scale and shape parameters respectively.

Then the corresponding probability density function (pdf) is given by

$$g(x) = \theta\lambda^2 x \exp\{-\lambda x\}([1 - \exp\{-\lambda x\}\,(1 + \lambda x)]^{\theta - 1}, x > 0, \lambda, \theta > 0.$$

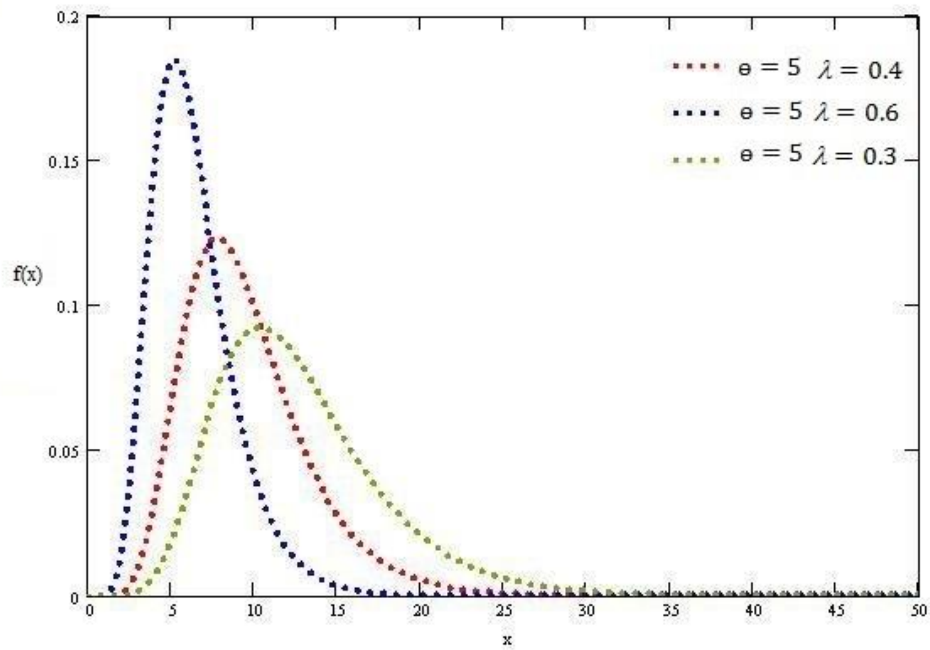

Figure 5.1. Probability density function of EG distribution.

The failure rate function is

$$h(x) = \frac{\theta\lambda\text{^}2\ x \exp\{-\lambda x\}([1 - \exp\{-\lambda x\}\,(1 + \lambda x)\,]\text{^}(\theta - 1)}{1 - [1 - \exp\{-\lambda x\}\,(1 + \lambda x)\,]\text{^}\theta}, x > 0, \lambda, \theta > 0.$$

Then the other advantage is that it has various shapes of failure function for different values of . It has increasing failure function when ◉≥ 1/2 and its failure function takes Bath-tub shape for

θ<1/2.

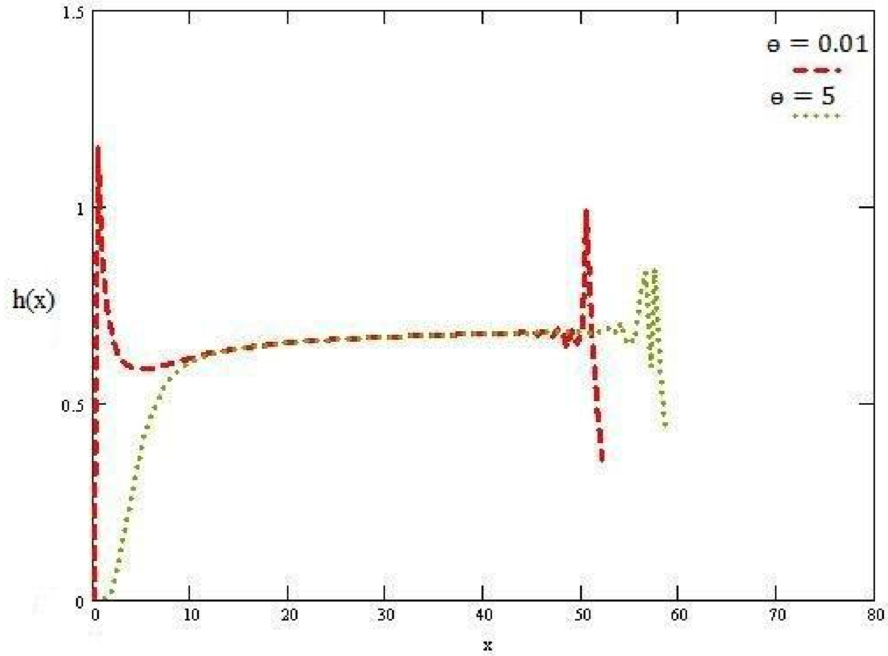

Figure 5.2: Failure rate function of EG distribution.

VI. x-Exponential Distribution

We introduce a new distribution, call it as x-Exponential, as an alternative to Generalized Lindley, Generalized Gamma and Exponentiated Weibull distributions. It is a very simple model than these GL, GG, EW distributions.

A life time random variable X is called x-Exponential distribution if its cumulative distribution function is

$$F(x) = (1 - (1 + \lambda x^2)e\hat{}(-\lambda x))^{\alpha}, x > 0, \lambda > 0.$$

Clearly F(0)=0, F(∞) = 1, F is non-decreasing and right continuous. More over F is absolutely continuous.

The probability density function (pdf) of a x-Exponential random variable X, with scale parameter λ is given by

$$f(x) = \alpha\lambda(1 - (1 + \lambda x^2)e^{-\lambda x})^{\alpha-1}.[\lambda x^2 - 2x + 1]e^{-\lambda x}, \qquad x > 0, \alpha > 0, \lambda > 0$$

It is positively skewed distribution.

Failure rate function of x-Exponential distribution is

$$h(x) = \frac{\alpha(1 - (1 + \lambda x^2)e^{-\lambda x})^{\alpha-1}.[\lambda x^2 - 2x + 1]\lambda e^{-\lambda x}}{1 - (1 - (1 + \lambda x^2)e^{-\lambda x})^{\alpha}}, x > 0, \alpha > 0, \lambda > 0$$

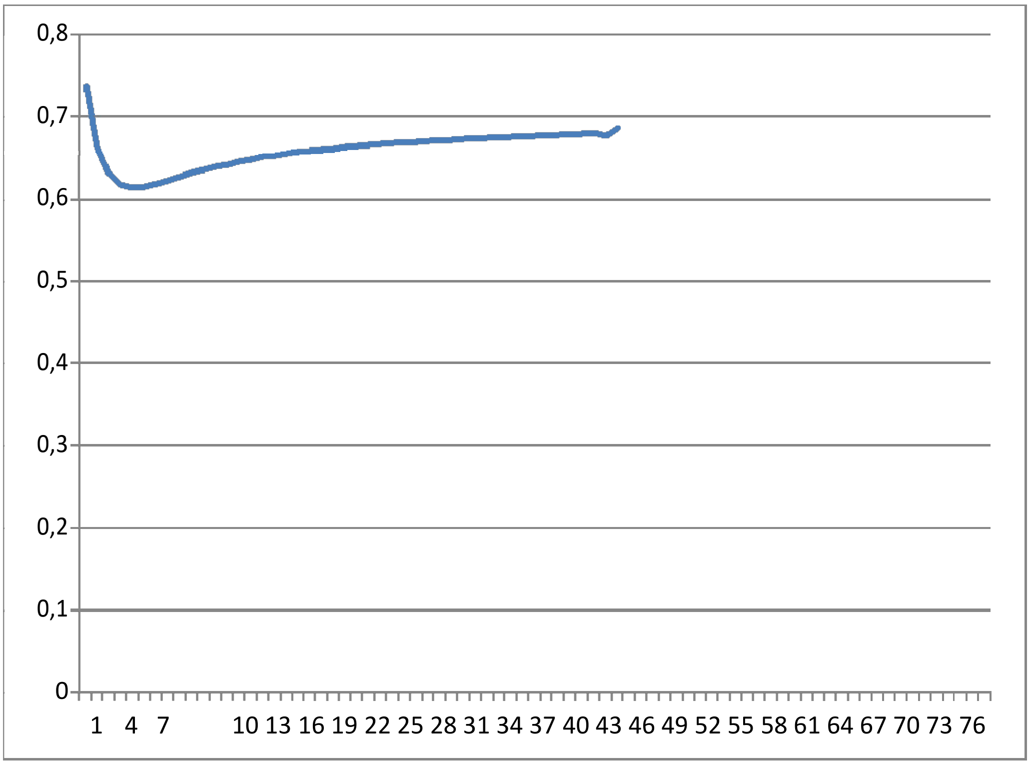

Figure 6.1. Failure rate function of x-Exponential distribution for $\boldsymbol{\alpha}$ =0.01 and λ= .6

VII. Conclusions

There are many distributions in reliability which exhibit Bathtub shaped failure rate model, but most of them are complicated in finding the moments, reliability etc. Moreover the increased number of parameters make complication and difficulty in estimation process. The proposed model is similar to Generalized Lindley, so all the computational procedures are like GL distribution. So I am not trying to provide a rigorous proof for that. This distribution can be viewed as distribution of Min(X_1, ... , X_n) where X_i is having i.i.d distribution with d.f. $1 - (1 + \lambda x^2)e^{-\lambda x}$, X>0, λ>0, a linear failure rate model.

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