NUMBER OF CALLS IN A CYCLIC-WAITING SYSTEM

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Abstract

A single-server cyclic-waiting system, i.e. a queuing system, in which a call gets service either at the time of arrival or after some cycle time T, has been studied in the article. The distribution functions of the intervals between arrivals and service times have exponential form; the service discipline is FCFS. Analytical results for the mean number of calls in the system are indicated. Numerical and graphical results have been given.

Keywords: cyclic-waiting system, queuing system, FCFS-service.

1 INTRODUCTION

Queuing systems (Gnedenko 1989) have several applications in different technical and biological systems, especially in telecommunication (Pustova 2010) and network systems. In order to adequately describe a real queuing system, one should consider different aspects of the real system, such as number of servers, availability of buffer and its type, order and duration of service, etc. Retrial queuing systems (Artalejo & Gomez-Corral 2008, Kovalenko & Koba 2010) are one of important tools in research and optimization of characteristics of different systems. Such systems take into account not only the initial flow of calls, but also the repeated ones. Retrial queuing systems have the following behavior. If all servers are busy upon call's arrival the call doesn't get lost, but goes to the virtual queue (an orbit), and after some time (a cycle) repeats its attempt to get service. Ignoring the retrial effect can cause significant accuracy errors in research (Aguir et al. 2004, Aguir et. al. 2008).

Retrial queuing systems may also take into account the service discipline: random service, FCFS (first come first served), LCFS (last come first served), etc. The analysis of such disciplines might increase the complexity of research, so it becomes hard to obtain analytical results.

We consider a so-called cyclic-waiting system (Lakatos 2010) having a special feature comparing to the common retrial systems, namely, calls get service in the order of arrival: if there is any call in the orbit, it will be served earlier than the arriving new ones. The calls from the orbit are served after some time, multiple to a given cycle time T.

The process of airplane landing initiated research of cyclic-waiting systems (Lakatos 1994 & 1998). An airplane lands on a given route. If a runway is available the airplane starts the landing procedure immediately. If there is a threat to the non-compliance of the separation standards or there already are some planes in the waiting area (in the circle), waiting for permission to land, the aircraft is sent to the queue (to the circle). While on circle, the aircraft waits for permission to land. After the permission is given, the airplane has to get to the corresponding geometric point on the circle route, and start the landing procedure from there. The time period from the moment of landing clearance to the moment until the plane reaches the corresponding geometric position is called "idle": the system is ready to take a plane, but the landing is possible only on reaching the corresponding point on the circle. Thereby, the airplane landing process can be described as a cyclic-waiting system.

Kovalenko (2002) considered the multichannel retrial queuing system with a constant cycle orbit, and gave estimation of the loss probability in light traffic. Rogiest et al. (2006) applied cyclic-waiting system to the research of optical buffers. Lakatos & Zbaganu (2007) investigated different generalizations of the initial model. Karasz (2008) studied some cyclic-models with two different types of calls. Kovalenko (2015) started to study a new class of multiple-cyclic waiting systems, namely, considered a two-cyclic queuing system.

2 ANALYTICAL MODELING

2.1 Problem statement

We consider a single server retrial queuing system with exponentially distributed intervals between arrivals and service times, with corresponding parameters λ and μ (Fig. 1). If the server is free, the arrived call gets service immediately. Otherwise, if the server is occupied, an arrived call goes to the orbit (i.e. a virtual queue), in which it retries its requests with a determined cycle time $T > 0$. The service is defined by FCFS (first come first served) discipline. It means that if there's any call cycling in the orbit, the call from the initial flow doesn't get service immediately, but goes to the orbit instead and after some time, the multiple of T, repeats its attempt to get service.

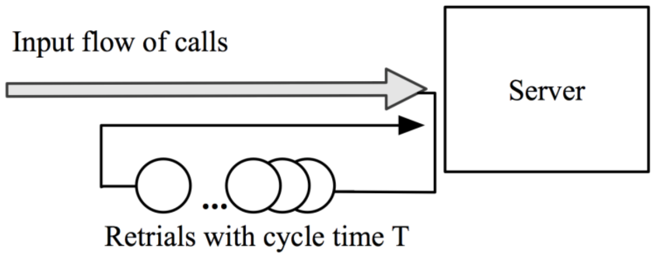

Figure 1. A cyclic-waiting system structure.

2.2 Ergodicity conditions

We will use results, obtained in (Lakatos 1994 & 2010) to find some characteristics of the system.

Let t_n, $n \geq 0$ denote the moment of the service beginning of the n^{th} call. Let's introduce an embedded Markov chain, the states of which are defined as the number of calls in the system at moments $(t_n - 0)$. The matrix of the chain's transition probabilities is defined as

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b_0 & b_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}, \tag{1}$$

the elements of which are defined by generating functions

$$A(z)=\sum_{i=0}^{\infty} a_i z^i = \frac{\mu}{\lambda+\mu} + \frac{\lambda z}{\lambda+\mu}\frac{(1-e^{-\mu T})e^{-\lambda(1-z)T}}{1-e^{-[\lambda(1-z)+\mu]T}}, \tag{2}$$

(where

$$\frac{\mu}{\lambda+\mu} = \int_0^{\infty} e^{-\lambda x}\mu e^{-\mu x}\,dx$$

is the probability of calls' non-arrival, and multiplier z in (2) represents the mandatory occurrence of a call), and

$$B(z)=\sum_{i=0}^{\infty} b_i z^i = \frac{1}{(1-e^{-\lambda T})[1-e^{-[\lambda(1-z)+\mu]T}]}\cdot\left\{\frac{1}{2-z}\left(1-e^{-\lambda(2-z)T}\right)\left(1-e^{-[\lambda(1-z)+\mu]T}\right)-\right.$$
$$\left.-\frac{\lambda}{\lambda(2-z)+\mu}\left(1-e^{-[\lambda(2-z)+\mu]T}\right)\left(1-e^{-\lambda(1-z)T}\right)\right\}. \tag{3}$$

Let p_i, $i = 0,1,\ldots$ be the ergodic probabilities of the chain. Hence, the generating function for the number of calls in the system is

$$P(z)=\sum_{i=0}^{\infty} p_i z^i = p_0\frac{(\lambda z+\mu)B(z)-(\lambda+\mu)zA(z)}{\mu[B(z)-z]}, \tag{4}$$

where $A(z)$ and $B(z)$ are defined by (3) and (4) respectively, and p_0 is defined by

$$p_0 = 1-\frac{\lambda}{\lambda+\mu}\frac{1-e^{-(\lambda+\mu)T}}{e^{-\lambda T}(1-e^{-\mu T})}. \tag{5}$$

Then the ergodicity condition is

$$\frac{\lambda}{\mu}\frac{1-e^{-\lambda T}}{e^{-\lambda T}(1-e^{-\mu T})}<1. \tag{6}$$

2.3 Mean number of calls in the system

Let us determine mean number of calls in the system, in other words $P'(1)$. Let us denote

$$g=\frac{p_0}{\mu},$$
$$N(z)=B(z)-z,$$
$$S(z)=\lambda z B(z)+\mu B(z)-\lambda z A(z)-\mu z A(z),$$

then (4) can be written as

$$P(z)=g\frac{S(z)}{N(z)}.$$

If $z = 1$ then

$$N(1)=1-1=0,$$
$$S(1)=\lambda+\mu-\lambda-\mu=0.$$

Let us obtain $P'(z)$:

$$P'(z)=\left(\frac{S(z)}{N(z)}\right)' = \frac{S'(z)N(z)-S(z)N'(z)}{N^2(z)}. \tag{7}$$

If $z = 1$ in (7) the indeterminate form of 0 / 0 occurs.

By applying L'Hôpital's rule to (7) we get

$$\frac{(S'(z)N(z)-S(z)N'(z))'}{(N^2(z))'} = \frac{S''(z)N(z)+S'(z)N'(z)-S'(z)N'(z)-S(z)N''(z)}{2N(z)N'(z)} =$$
$$=\frac{S''(z)N(z)-S(z)N''(z)}{2N(z)N'(z)}. \tag{8}$$

Similarly, if $z = 1$ in (8) the indeterminate form of 0 / 0 occurs. Analogously, for (8) we obtain

$$\frac{(S''(z)N(z)-S(z)N''(z))'}{(2N(z)N'(z))'}=\frac{S'''(z)N(z)+S''(z)N'(z)-S'(z)N''(z)-S(z)N'''(z)}{2[(N'(z))+N(z)N''(z)]}. \tag{9}$$

If $z = 1$ (9) is transformed to

$$P'(1)=\frac{S''(1)N'(1)-S'(1)N''(1)}{2[N'(1)]^2}. \tag{10}$$

Let us find the unknown derivatives in (10):

$$N'(z)=B'(z)-1, \tag{11}$$

$$N''(z)=B''(z), \tag{12}$$

$$S'(z)=\lambda B(z)+\lambda zB'(z)+\mu B'(z)-\lambda A(z)-\lambda zA'(z)-\mu A(z)-\mu zA'(z), \tag{13}$$

$$S''(z)=\lambda B'(z)+\lambda B'(z)+\lambda zB''(z)+\mu B''(z)-$$
$$-\lambda A'(z)-\lambda A'(z)-\lambda zA''(z)-\mu A'(z)-\mu A'(z)-\mu zA''(z). \tag{14}$$

If $z = 1$ the (11) – (14) are transformed to

$$N'(1)=B'(1)-1, \tag{15}$$

$$N''(1)=B''(1), \tag{16}$$

$$S'(1)=B'(1)(\lambda+\mu)-A'(1)(\lambda+\mu)-\mu, \tag{17}$$

$$S''(1)=2\lambda B'(1)+\lambda B''(1)+\mu B''(1)-2\lambda A'(1)-\lambda A''(1)-2\mu A''(1)=$$
$$=2\lambda B'(1)+B''(1)(\lambda+\mu)-2(\lambda+\mu)A'(1)-(\lambda+\mu)A''(1). \tag{18}$$

After some computations the derivatives for $A(z)$ and $B(z)$ that are part of the (15) – (18) can be written as

$$A'(1)=\frac{\lambda(1-e^{-\mu T}+\lambda T)}{(\lambda+\mu)(1-e^{-\mu T})}, \tag{19}$$

$$A''(1)=\frac{\lambda^2T(2-2e^{-\mu T}+\lambda T+\lambda Te^{-\mu T})}{(\lambda+\mu)(1-e^{\mu T})^2}, \tag{20}$$

$$B'(1)=\frac{1}{(\lambda+\mu)(1-e^{-\lambda T})(1-e^{-\mu T})}\Big(\lambda+\mu+\lambda^2T+\lambda\mu Te^{-(\lambda+\mu)T}-(\lambda+\mu)(e^{-\lambda T}+e^{-\mu T})+$$
$$+(\lambda+\mu)e^{-(\lambda+\mu)T}-(\lambda+\mu)\lambda Te^{-\lambda T}\Big), \tag{21}$$

$$B''(1)=\frac{2\lambda^2T^2e^{-2\mu T}}{(1-e^{-\mu T})^2}+\frac{2\lambda Te^{-\mu T}}{(1-e^{-\lambda T})(1-e^{-\mu T})^2}\Big((1-e^{-\lambda T})(1-e^{-\mu T})-$$
$$-\lambda Te^{-\lambda T}(1-e^{-\mu T})-\lambda Te^{-\mu T}(1-e^{-\lambda T})+\frac{\lambda^2T(1-e^{-(\lambda+\mu)T})}{\lambda+\mu}\Big)+$$
$$+\frac{\lambda^2T^2e^{-\mu T}}{1-e^{-\mu T}}+\frac{1}{(1-e^{-\lambda T})(1-e^{-\mu T})}\Big(2(1-e^{-\lambda T})(1-e^{-\mu T})-$$
$$-2\lambda Te^{-\lambda T}(1-e^{-\mu T})-2\lambda Te^{-\mu T}(1-e^{-\lambda T})-\lambda^2T^2e^{\lambda T}(1-e^{-\mu T})+$$
$$+2\lambda^2T^2e^{-(\lambda+\mu)T}-\lambda^2T^2e^{-\mu T}(1-e^{-\lambda T})+\frac{2\lambda^3T(1-e^{-(\lambda+\mu)T})}{(\lambda+\mu)^2}-$$
$$-\frac{2\lambda^3T^2e^{-(\lambda+\mu)T}}{\lambda+\mu}+\frac{\lambda^3T^2(1-e^{-(\lambda+\mu)T})}{\lambda+\mu}\Big). \tag{22}$$

Therefore, formulas (11) – (22) define expression (10) entirely. Due to cumbersome form, the final expression of $P'(z)$ (as well as $P'(1)$) isn't given here, though it can be easily obtained with the help of formulas (10), (11) – (22).

2.3 Analysis of the modelling results

Let us denote the left side of (6) as H (we will use this notation for graphical results):

$$H = \frac{\lambda}{\mu}\frac{1-e^{-\lambda T}}{e^{-\lambda T}(1-e^{-\mu T})}.$$

According to formula (6) we have obtained graphical representation for egrodicity conditions of the system under consideration (see Figs 2-3): everything below the line H = 1 qualifies for ergodicity condition (6).

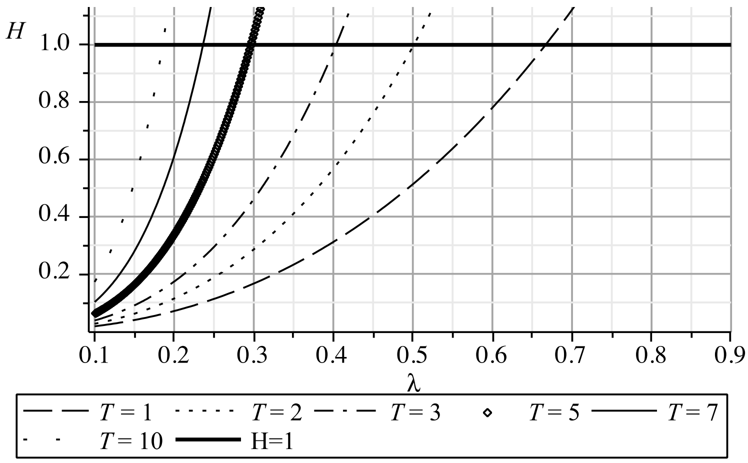

Figure 2. Graphical representation for ergodicity conditions, where T = {1, 2, 3, 5, 7, 10}, λ = [0.1;0.9], μ = 1.0.

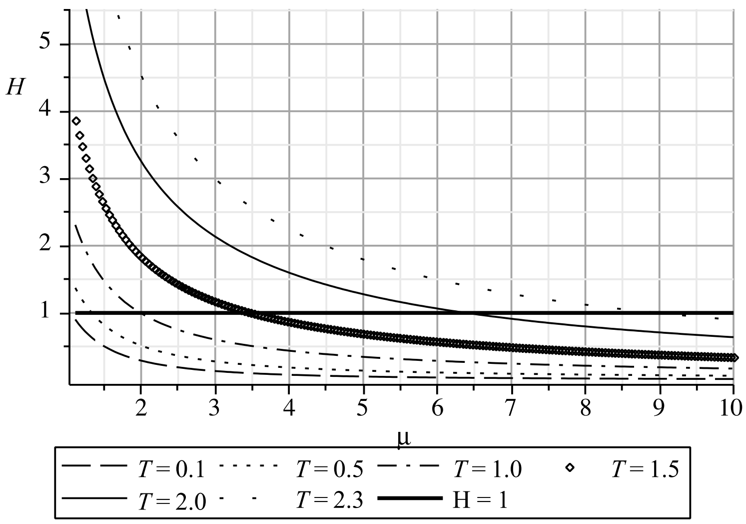

Figure 3. Graphical representation for ergodicity conditions, where T = {0.1, 0.5, 1.0, 1.5, 2.0, 2.3}, λ =1.0, μ = [1.11;10.0].

The values of λ and μ for the results on Figures 2-3 have been chosen so that the ratio λ / μ = [0.1; 0.9]. From Figure 2 one may see that with the increase of the value of cycle time T the left part of the (6) increases sharply. The same situation persists on Figure 3.

Using results from Figures 2-3, we have also obtained a graphical dependence for the mean number of calls $P'(1)$ from the mean service time $1 / \mu$ and cycle time T (see Fig. 4), as well as from the intensity of arrivals λ and cycle time T (see Fig. 5).

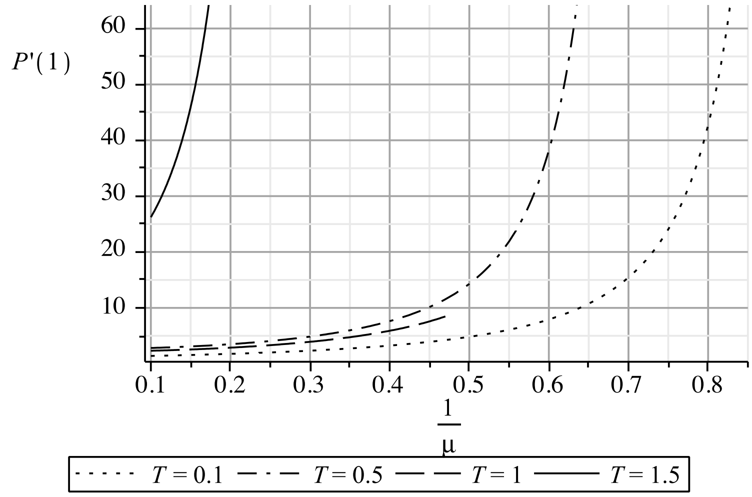

Figure 4. Graph for the mean number of calls ($\lambda = 1.0$).

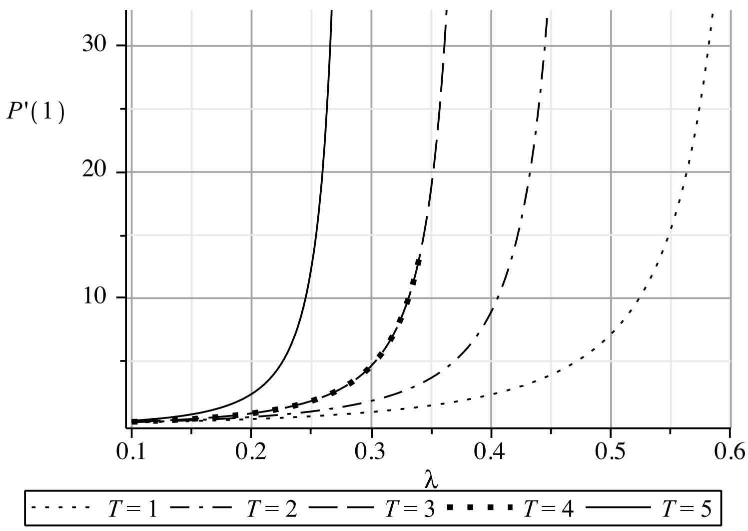

Figure 5. Graph for the mean number of calls ($\mu = 1.0$).

From Figures 4-5 one may see that the mean number of calls rises with the increasing service time as well as with the increasing intensity of arrivals. On both graphs (see Figs 4-5) $P'(1)$ increases with the increasing cycle time T.

3 CONCLUSIONS

In the article we have obtained analytical and graphical results for the mean number of calls in the single-channel cyclic-waiting system. Ergodicity conditions have been shown. The analytical result for $P'(1)$ can be used, for example, to obtain the average time a call spends in a system: from Little's law $W = P'(1) / \lambda$.

The cyclic-waiting system under consideration may have different practical applications (Koba & Pustova 2012, Pustova 2016), such as an airplane landing process, a simple call center, a telephone answering machine, an optical fiber buffer (Rogiest et al. 2006, Langenhorst et al. 1996), an optical delay line, a microring resonator (Bogaerts et al. 2012), a compact optical buffer with ring resonators (IBM, 2006). All these systems have a common feature: a new call, which can't get service until there is at least another one in the system, goes to the orbit and cycles there.

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