RELIABILITY PERFORMANCE MEASURES OF SYSTEMS WITH LOCATION-SCALE GENERALIZED ABSOLUTELY CONTINUOUS MULTIVARIATE EXPONENTIAL FAILURETIME DISTRIBUTION

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Abstract

This paper deals with the equal marginal location-scale Generalized Absolutely Continuous Multivariate Exponential model. The distributional properties and applications of the location-scale model arising out of the k-parameter Generalized Absolutely Continuous Multivariate Exponential distribution are studied. Standby, parallel, series and relay systems of order k with location-scale Generalized Absolutely Continuous Multivariate Exponential failuretimes are discussed and their performance measures are obtained. The optimal estimators of the meantime before failure times are also derived.

Keywords: Equivariant estimation, location-scale, multivariate exponential, performance measures

1. Introduction

Though, there is an extensive literature on the reliability aspects of systems with independent failure times, not much work seems to have been carried out on systems with dependent component failure times. Rau (1970) discusses reliability analysis of systems with independent components. Chandrasekar and Paul Rajamanickam (1996), Paul Rajamanickam and Chandrasekar (1997, 1998a, 1998b), Paul Rajamanickam (1999) discuss repairable systems with dependent structures mainly assuming Marshall - Olkin type of joint distributions for the system component failure and repair times. Recently Chandrasekar and Sajesh (2013) and Chandrasekar and Amala Revathy (2016) discussed reliability applications of location-scale equal marginal absolutely continuous bivariate and multivariate exponential distributions respectively.

By considering location-scale Generalized Absolutely Continuous Multivariate Exponential (GACMVE) failuretime distribution, for k unit systems, we derive the reliability performance measures and obtain optimal estimators. In Section 2, we propose the probability density function for the location- scale GACMVE model. In Section 3, we derive some important distributional results required for further discussion. In Section 4, we consider a k unit standby system and obtain the mean time before failure (MTBF) and the reliability function of the system. Further the minimum risk equivariant estimator (MREE) and the uniformly minimum variance unbiased estimator (UMVUE) of the MTBF are derived. Similar results for parallel, series and relay systems are presented in Sections 5, 6 and 7 respectively.

2. Generalized Absolutely Continuous Multivariate Exponential location scale model

The joint pdf of GACMVE is

$$f(x_1,x_2\ldots,x_k)=\frac{1}{k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\exp\left[-\lambda_1\sum_{i=1}^{k}x_i-\lambda_2\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_i\vee x_j\right)-\ldots..-\lambda_k\left(x_1\vee x_2\vee\ldots.\vee x_k\right)\right]$$

$$x_i\geq 0\ \forall i;\ \lambda_1>0,\ \lambda_i\geq 0,\ i=2,3\ldots.k \quad \ldots.(2.1)$$

Here $x_1\vee x_2\vee\ldots..\vee x_k=\max\{x_1,x_2,\ldots..x_k\}$.

Let X be a random variable (vector) with the distribution function $F_{\xi,\tau}(.),\xi\in R,\tau>0$.

Let $\{F_{\xi,\tau};\xi\in R,\tau>0\}$ be a location-scale family, so that $F_{\xi,\tau}(x)=F\left(\frac{x-\xi}{\tau}\right)$ for some distribution function F.

The location-scale GACMVE has the pdf

$$f_{\xi,\tau}(x_1,x_2\ldots,x_k)=\frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}$$

$$\exp\left\{-\frac{1}{\tau}\left[\lambda_1\sum_{i=1}^{k}x_i+\lambda_2\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_i\vee x_j\right)\ldots..+\lambda_k\left(x_1\vee x_2\vee\ldots.\vee x_k\right)-\sum_{p=1}^{k}\binom{k}{p}\lambda_p\xi\right]\right\}$$

$$x_i>\xi\ \forall i,\ \xi\in R,\ \tau>0,\ \lambda_1>0,\ \lambda_2\geq 0 \quad \ldots(2.2)$$

For fixed $(\lambda_1,\lambda_2,\ldots.,\lambda_k)$, the distribution of $\left(\frac{X_1-\xi}{\tau},\frac{X_2-\xi}{\tau},\ldots.,\frac{X_k-\xi}{\tau}\right)$ does not depend on $(\xi,\tau)'$. Therefore the above family is a location – scale family with the location – scale parameter $(\xi,\tau)'$. Let us refer to the distribution as location-scale GACMVE. When τ = 1, the resulting family is the location GACMVE family. When ξ = 0, the resulting family is the scale GACMVE family. Since we are interested in the location-scale parameter, it is assumed that the parameters $\lambda_1,\lambda_2,\ldots.,\lambda_k$ are known.

3. Distributional properties

Theorem 3.1

Let $\left(X_1, X_2, \ldots X_k\right) \sim$ GACMVE distribution given in (2.1), and $Y_1, Y_2 \ldots, Y_k$ denote the order statistics based on $X_1, X_2 \ldots, X_k$. Define $W_1 = Y_1$, $W_2 = Y_2 - Y_1$, …… $W_k = Y_k - Y_{k-1}$. Then $B_0 W_1$, $B_1 W_2$,…, $B_{k-1} W_k$ are independent and identical standard exponential random variables, where $B_l = \sum_{i=l}^{k-1} A_i$, $l = 0,1,2,\ldots,k-1$ and $A_i = \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1}$; $i = 0,1,2,\ldots,k-1$.

Proof

The joint pdf of $\left(X_1, X_2 \ldots, X_k\right)$ is

$$f(x_1, x_2 \ldots, x_k) = \frac{1}{k!} \prod_{l=0}^{k-1} \sum_{i=l}^{k-1} \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1}$$

$$\exp\left[-\lambda_1 \sum_{i=1}^{k} x_i - \lambda_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(x_i \vee x_j \right) - \ldots - \lambda_k \left(x_1 \vee x_2 \vee \ldots \vee x_k \right) \right]$$

$$x_i \geq 0 \, \forall i; \; \lambda_1 > 0, \lambda_i \geq 0, i = 2,3 \ldots k$$

The pdf of $\left(Y_1, Y_2 \ldots, Y_k\right)$ is

$$g(y_1, y_2 \ldots, y_k) = \left(\prod_{l=0}^{k-1} B_l \right) \exp\left[-\lambda_1 \sum_{i=1}^{k} y_i - \lambda_2 \left(y_2 + 2y_3 + 3y_4 + \ldots \overline{k-1} y_k \right) \right.$$
$$\left. - \lambda_3 \left(y_3 + \binom{3}{2} y_4 + \binom{4}{2} y_5 + \ldots \binom{k-1}{2} y_k \right) \ldots - \lambda_k y_k \right]$$
$$y_1 < y_2 < \ldots y_k; \; \lambda_1 > 0, \lambda_i \geq 0, i = 2,3,\ldots k.$$

Consider the pdf of $\left(Y_1, Y_2 \ldots, Y_k\right)$

$$g\left(y_1, y_2, \ldots, y_k\right) = B_0 B_1 \ldots B_{k-1} \exp\left\{ -\lambda_1 y_1 - \left(\lambda_1 + \lambda_2\right) y_2 \right\} \exp\left\{ -\left(\lambda_1 + \binom{2}{1} \lambda_2 + \lambda_3 \right) y_3 \right\}$$

$$\exp\left\{ -\left(\lambda_1 + \binom{3}{1} \lambda_2 + \binom{3}{2} \lambda_3 + \lambda_4 \right) y_4 \right\} \ldots \exp\left\{ -\left(\lambda_1 + \binom{k-1}{1} \lambda_2 + \ldots + \lambda_k \right) y_k \right\}.$$

$$= B_0 B_1 \ldots B_{k-1} \exp\left\{ -A_0 y_1 - A_1 y_2 \ldots - A_{k-1} y_k \right\}$$

In order to find the distribution of $\left(W_1, W_2 \ldots, W_n\right)$, consider the transformation

$w_i = y_i - y_{i-1}$, $i = 1,2,\ldots k$, with $y_0 \equiv 0$.

Then $y_j = w_1 + w_2 + \ldots + w_j$, j = 1,2,3,…,k.

Note that the Jacobian of the transformation is 1.

The joint pdf of $\left(W_1, W_2 \ldots, W_k\right)$ is

$$h(w_1, w_2, \ldots w_k) = B_0\, B_1 \ldots B_{k-1} \exp\left\{-A_0\, w_1 - A_1\left(w_1 + w_2\right) - \ldots - A_{k-1}\sum_{i=1}^{k} w_i\right\}$$

$$= B_0\, B_1 \ldots B_{k-1} \exp\left\{-B_0\, w_1 - B_1\, w_2 - \ldots - B_{k-1}\, w_k\right\}$$

Hence $B_0 W_1$, $B_1 W_2$ ….. $B_{k-1} W_k$ are independent and identical E(0,1) random variables.

Sufficient statistic

Let $X_p = \left(X_{1p}, X_{2p} \ldots, X_{kp}\right)'$; p = 1,2,….n be a random sample of size n from (2.2).

The joint pdf of $\left(X_{1p}, X_{2p} \ldots, X_{kp}\right)$; j = 1,2,….n is

$$p(x;\xi,\tau) = \left\{\frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}^n$$

$$\exp\left\{-\frac{1}{\tau}\sum_{p=1}^{n}\left[\lambda_1\sum_{i=1}^{k} x_{ip} + \lambda_2 \sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip} \vee x_{jp}\right) \ldots + \lambda_k\left(x_{1p} \vee x_{2p} \vee \ldots \vee x_{kp}\right) - \sum_{m=1}^{k}\binom{k}{m}\lambda_m \xi\right]\right\}$$

$$\min_{1\le p\le n}\left(x_{1p} \wedge x_{2p} \wedge \ldots \wedge x_{kp}\right) > \xi$$

Let $U_p = \left(X_{1p} \wedge X_{2p} \wedge \ldots \wedge X_{kp}\right)$; *and* $U_{(1)} = \min_{1\le p\le n} U_p$.

$$p(x;\xi,\tau) = \left\{\frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}^n$$

$$\exp\left\{-\frac{1}{\tau}\sum_{p=1}^{n}\left[\lambda_1\sum_{i=1}^{k} x_{ip} + \lambda_2 \sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip} \vee x_{jp}\right) + \ldots + \lambda_k\left(x_{1p} \vee x_{2p} \vee \ldots \vee x_{kp}\right) - \sum_{m=1}^{k}\binom{k}{m}\lambda_m U_{(1)} + \sum_{m=1}^{k}\binom{k}{m}\lambda_m\left(U_{(1)} - \xi\right)\right]\right\}$$

$$= g_{\xi,\tau}\left(t_1, t_2\right) h\left(\underline{x}\right),$$

where, $T_1^* = U_{(1)}$ and

$$T_2^* = \sum_{p=1}^{n}\left\{\lambda_1\sum_{i=1}^{k} x_{ip} + \lambda_2 \sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip} \vee x_{jp}\right) \ldots + \lambda_k\left(x_{1p} \vee x_{2p} \vee \ldots \vee x_{kp}\right)\right\}.$$

By factorization theorem, $T^* = \left(T_1^*, T_2^*\right)$ is a sufficient statistic.

Theorem 3.2

(i) $T_1^* \sim E\left[\xi, \frac{\tau}{n B_0}\right]$

(ii) $T_2^* \sim G\left(nk-1, \tau\right)$ and

(iii) T_1^* and T_2^* are independent.

Proof

(i) Let $X_p = \left(X_{1p}, X_{2p} \ldots, X_{kp}\right)'$; p = 1,2,….n be a random sample of size n from (2.2).

The joint pdf of $\left(X_{1p}, X_{2p} \ldots, X_{kp}\right)$; j = 1,2,….n is

$$p(x;\xi,\tau) = \left\{\frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}^n$$

$$\exp\left\{-\frac{1}{\tau}\sum_{p=1}^{n}\left[\lambda_1\sum_{i=1}^{k}x_{ip} + \lambda_2 \sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip} \vee x_{jp}\right) \ldots . + \lambda_k\left(x_{1p} \vee x_{2p} \vee \ldots . \vee x_{kp}\right) - \sum_{m=1}^{k}\binom{k}{m}\lambda_m \xi\right]\right\}$$

$$min\left(x_{1p} \wedge x_{2p} \wedge \ldots \wedge x_{kp}\right) > \xi$$

Let $U_p = \left(X_{1p} \wedge X_{2p} \wedge \ldots \wedge X_{kp}\right)$; *and* $U_{(1)} = min\ U_p$.

Then $U_{(1)} > \xi$ and $\frac{n B_0}{\tau}\left(U_{(1)} - \xi\right) \sim E(0,1)$

Therefore $U_{(1)} \sim E\left(\xi, \frac{\tau}{n B_0}\right)$

(ii) Let $Y_{1j}, Y_{2j,} \ldots, Y_{kj}$ denote the order statistics based on $\left(X_{1j}, X_{2j} \ldots, X_{kj}\right)$, j = 1,2,….n. Note that $Y_{1j} = U_j$, j = 1,2,….n,. Define $W_{rj} = Y_{rj} - Y_{(r-1)j}$, r = 1,2,3…..k; j = 1,2,3,….n.
$Y_{0j} = 0$ for all j.
Consider

$$T_2{}^* = \sum_{p=1}^{n}\left\{\lambda_1\sum_{i=1}^{k}X_{ip} + \lambda_2 \sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(X_{ip} \vee X_{jp}\right) \ldots . + \lambda_k\left(X_{1p} \vee X_{2p} \vee \ldots . \vee X_{kp}\right)\right\}$$

$$= \sum_{p=1}^{n}\left\{\lambda_1\sum_{i=1}^{k}Y_{ip} + \lambda_2\sum_{i=2}^{k}(i-1)Y_{im} + \lambda_3\sum_{i=3}^{k}(i-2)Y_{im} + \ldots . + \lambda_k Y_{km} - \sum_{m=1}^{k}\binom{k}{m}\lambda_m U_{(1)}\right\}$$

$$=\sum_{p=1}^{n}\Big\{\lambda_1\left(kW_{1p}+(k-1)W_{2p}+.....+W_{kp}\right)$$

$$+\lambda_2\left(W_{1p}\sum_{i=1}^{k-1}i+W_{2p}\sum_{i=1}^{k-1}i+W_{3p}\sum_{i=1}^{k-1}i.....+(k-1)W_{kp}\right)$$

$$+\lambda_3\left(W_{1p}\sum_{i=1}^{k-2}i+W_{2p}\sum_{i=1}^{k-2}i+W_{3p}\sum_{i=1}^{k-2}i.+W_{4p}\sum_{i=2}^{k-2}i+W_{5p}\sum_{i=3}^{k-2}i.+....+(k-2)W_{kp}\right)+......$$

$$+\lambda_k\sum_{i=1}^{k}W_{im}-\sum_{m=1}^{k}\lambda_m\binom{k}{m}U_{(1)}\Big\}$$

$$=\sum_{p=1}^{n}\Big\{W_{1p}\left(\lambda_1 k+\lambda_2\sum_{i=1}^{k-1}i+\lambda_3\sum_{i=1}^{k-2}i+\lambda_4\sum_{i=1}^{k-3}i+....+\lambda_k\right)+$$

$$W_{2p}\left(\lambda_1(k-1)+\lambda_2\sum_{i=1}^{k-1}i+\lambda_3\sum_{i=1}^{k-2}i+\lambda_4\sum_{i=1}^{k-3}i+....+\lambda_k\right)+$$

$$W_{3p}\left(\lambda_1(k-2)+\lambda_2\sum_{i=1}^{k-1}i+\lambda_3\sum_{i=1}^{k-2}i+\lambda_4\sum_{i=1}^{k-3}i+....+\lambda_k\right)+......+$$

$$W_{kp}\left(\lambda_1 k+\lambda_2(k-1)+\lambda_3(k-2)+\lambda_4(k-3)+....+\lambda_k\right)+\sum_{m=1}^{k}\lambda_m\binom{k}{m}U_{(1)}\Big\}$$

$$=\sum_{p=1}^{n}\left\{W_{1p}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+W_{2p}\sum_{i=1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+......W_{kp}\sum_{i=k-1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+\sum_{m=1}^{k}\lambda_m\binom{k}{m}U_{(1)}\right\}$$

$$=\sum_{p=1}^{n}\left\{\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\left(W_{1m}-U_{(1)}\right)+W_{2p}\sum_{i=1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+......+W_{kp}\sum_{i=k-1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}$$

$$=\sum_{p=1}^{n}\left\{\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\left(U_{(m)}-U_{(1)}\right)+W_{2p}\sum_{i=1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+......+W_{kp}\sum_{i=k-1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}$$

Since $U_{(1)},U_{(2)}....U_{(n)}$ are order statistics from $E\left(\xi,\ \frac{\tau}{B_0}\right)$, it follows that the first term on the right hand side follows $G(n-1,\tau)$.

By Theorem 3.1, each of the other (k-1) terms on the right hand side follows $G(n,\tau)$. Since $W_{1j},W_{2j},\ldots,W_{kj}$ are independent for each j, the k random variables on the right hand side are independent.

Hence $T_2^{*}\sim G(nk-1,\tau)$.

(iii) For fixed τ, the joint distribution of $\left(X_{1j},X_{2j}\ldots,X_{kj}\right)$, j = 1,2,….n, belongs to a location family with the location parameter ξ. The statistic T_2^* is ancillary and T_1^* is complete sufficient. Hence T_1^* and T_2^* are independent (Basu, 1955).

The following theorem will help us in obtaining the reliability performance measures of standby and parallel syste

Theorem 3.3

Let $\left(T_1, T_2, \ldots, T_k\right)$ follow $GACMVE\left(\lambda_1, \lambda_2, \ldots \lambda_k; \xi, \tau\right)$ with pdf given in equation (2.2). Then

(i) $\sum_{i=1}^{k} T_i - k\xi \underset{=}{d} V_1 + V_2 + \ldots + V_k$, where V1, V2,…. Vk are independent and

$$V_l \sim E\left(0, \frac{\left\{k-(l-1)\tau\right\}}{B_{l-1}}\right), \text{ for all } l = 1, 2, .., k. \quad \ldots(3.1)$$

(ii) $\left(T_1 \vee T_2 \vee \ldots T_k\right) - k\ \xi \underset{=}{d} V_1^* + V_2^* + \ldots\ldots V_k^*$, where V*1, V*2,…. V*k are independent and

$$V_j^* \sim E\left(0, \frac{\tau \sum_{i=l}^{k-1} \binom{i}{k-1}}{\sum_{i=l}^{k-1} \binom{i}{k-1} \lambda_k}\right), \text{ for all } j = 1, 2, .., k. \quad \ldots(3.2)$$

Proof:

The MGF of $\left(\sum_{i=1}^{k} T_i, \sum_{i<j=1}^{k} \sum_{j=1}^{k} T_i \vee T_j, \ldots, T_1 \vee T_2 \vee \ldots T_k\right)$ at $\left(U_1, U_2, \ldots, U_k\right)$ is

$$M(u_1, u_2 \ldots, u_k) = \int_{\xi}^{\infty}\int_{\xi}^{\infty} \ldots \int_{\xi}^{\infty} \frac{1}{\tau^k k!} \prod_{l=0}^{k-1} \sum_{i=l}^{k-1} \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1}$$

$$\exp\left\{u_1 \sum_{i=1}^{k} t_i + u_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(t_i \vee t_j\right) + \ldots\ldots + u_k\left(t_1 \vee t_2 \vee \ldots. \vee t_k\right)\right\}$$

$$\exp\left\{-\frac{1}{\tau}\left[\lambda_1 \sum_{i=1}^{k} t_i + \lambda_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(t_i \vee t_j\right) + \ldots\ldots + \lambda_k\left(t_1 \vee t_2 \vee \ldots. \vee t_k\right)\right] - \sum_{p=1}^{k} \binom{k}{p} \lambda_p \xi\right\}$$

$$dt_1\, dt_2 \ldots\ldots. dt_k$$

$$M(u_1,u_2\ldots,u_k)=\int_\xi^\infty\int_\xi^\infty\ldots\int_\xi^\infty \frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\exp\left\{-\frac{1}{\tau}\left[(\lambda_1-\tau u_1)\sum_{i=1}^{k}t_i+\right.\right.$$

$$(\lambda_2-\tau u_2)\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(t_i\vee t_j\right)+\ldots..+(\lambda_k-\tau u_k)(t_1\vee t_2\vee\ldots.\vee t_k)$$

$$\left.\left.-\sum_{p=1}^{k}\binom{k}{p}\lambda_p\xi\right]\right\}\;dt_1\,dt_2\ldots\ldots.dt_k$$

$$=\prod_{l=0}^{k-1}\frac{\exp\left(-\sum_{p=1}^{k}\binom{k}{p}u_p\,\xi\right)}{\left(1-\frac{\tau\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}u_{j+1}}{B_l}\right)}$$

(i) $$M(u_1,0,\ldots,0)=\prod_{l=0}^{k-1}\frac{\exp(-k\,u_1\,\xi)}{\left(1-\frac{(k-l)\tau u_1}{B_l}\right)}$$

$$\therefore\sum_{i=1}^{k}T_i-k\,\xi\underset{=}{d}\sum_{i=1}^{k}V_i\,,$$

where Vi's are independent and

$$V_l\sim E\left[0,\frac{(k-l)\tau}{B_l}\right],\quad l=1,2,\ldots.k.$$

(ii) $$M(0,0,\ldots,u_k)=\prod_{l=0}^{k-1}\frac{\exp(-u_k\,\xi)}{\left(1-\frac{\tau\sum_{i=l}^{k-1}\binom{i}{k-1}u_k}{B_l}\right)}$$

$$\therefore(T_1\vee T_2\vee\ldots T_k)-k\,\xi\underset{=}{d}V_1^*+V_2^*+\ldots..V_k^*\,,$$

where Vi*'s are independent and $V_l^*\sim E\left[0,\frac{\tau\sum_{i=l}^{k-1}\binom{i}{k-1}}{B_l}\right],\quad i=0,1,\ldots.k-1.$

The following Lemma helps us in finding the reliability function.

Lemma 3.1

Let $M(u)=\left[\prod_{j=1}^{k}\left(1-\alpha_j u\right)\right]^{-1}$; $u<\dfrac{1}{\alpha_j}\ \forall\ j$.

Then $M(u)=\sum_{j=1}^{k}\dfrac{w_j}{\left(1-\alpha_j u\right)}$, where $w_j=\dfrac{\alpha_j^{\,k-1}}{\prod_{\substack{r=1\\ \neq j}}^{k}\left(\alpha_j-\alpha_r\right)}$, and $\sum_{j=1}^{k}w_j=1$.

Proof

$$M(u)=\frac{1}{(1-\alpha_1 u)(1-\alpha_2 u)......(1-\alpha_k u)}$$

Resolving into partial fractions,

$$M(u)=\frac{w_1}{(1-\alpha_1 u)}+\frac{w_2}{(1-\alpha_2 u)}+......+\frac{w_k}{(1-\alpha_k u)}=$$

$$\frac{w_1(1-\alpha_2 u).....(1-\alpha_k u)+w_2(1-\alpha_1 u)(1-\alpha_3 u)....(1-\alpha_k u)+....+w_k(1-\alpha_1 u)....(1-\alpha_{k-1} u)}{\prod_{j=1}^{k}\left(1-\alpha_j u\right)}$$

$$=\frac{w_1\prod_{j=2}^{k}\left(1-\alpha_j u\right)+w_2\prod_{\substack{j=1\\ \neq 2}}^{k}\left(1-\alpha_j u\right)+....+w_k\prod_{j=1}^{k-1}\left(1-\alpha_j u\right)}{\prod_{j=1}^{k}\left(1-\alpha_j u\right)}$$

Thus, for j = 1,2,3,…,k, we get $w_j=\dfrac{\alpha_j^{\,k-1}}{\prod_{\substack{r=1\\ \neq j}}^{k}\left(\alpha_j-\alpha_r\right)}$.

Corollary 3.1

The survival function corresponding to M(u) is

$$\overline{G}(u)=\sum_{j=1}^{k}w_j\exp\left(-\frac{1}{\alpha_j}u\right),\ u>0.$$

4 Standby system

Consider a k unit standby system with component failure times $T_1,T_2,\ldots,T_k$ having location-scale GACMVE distribution.

Then the system failure time is $T=\sum_{i=1}^{k}T_i$.

The MTBF of the system is

MTBF = E (T)

$$= E(V_1 + V_2 + ... + V_k) + k\xi$$

$$= \sum_{l=1}^{k} \frac{\{k-l\}\tau}{B_{l-1}} + k\,\xi \text{, in view of (3.1).}$$

Following the arguments of Chandrasekar and Amala Revathy (2016), the MREE of η = α ξ +β τ, α, β ϵ R, is given by

$$\delta^* = \alpha\,\delta_{01} + \frac{1}{kn}\left[\beta - \frac{\alpha}{n\,B_0}\right]\delta_{02}$$

Define,

$$\delta_{01} = \min_{1\le p\le n}\left\{X_{1p} \wedge X_{2p} \wedge \ldots \wedge X_{kp}\right\} \text{ and}$$

$$\delta_{02} = \sum_{p=1}^{n}\left\{\lambda_1 \sum_{i=1}^{k} X_{ip} + \lambda_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k}\left(X_{ip} \vee X_{jp}\right) + ... + \lambda_k\left(X_{1p} \vee X_{2p} \vee .... \vee X_{kp}\right)\right\}$$

By taking α = k and $\beta = \sum_{l=1}^{k} \frac{(k-1)}{B_{l-1}}$, the MREE of the MTBF is given by

$$k\,\delta_{01} + \frac{1}{kn}\left[\sum_{l=1}^{k} \frac{(k-1)}{B_{l-1}} + \frac{\alpha}{n\,B_0}\right]\delta_{02}$$

Reliability function of the standby system is

R(t) = P(T > t)

$$= P\left(\sum_{i=1}^{k} T_i - k\,\xi > t\right),\ t > 0$$

$$= P\left(\sum_{i=1}^{k} V_i > t\right),\ t > 0$$

$$= \sum_{l=1}^{k} \beta_l \exp\left(-\frac{1}{\alpha_l}t\right)$$

in view of Lemma 3.1.

Here $\alpha_l = \frac{(k-l)\tau}{B_{l-1}}$ $\quad \forall\, l = 1, 2, ..., k$, and $\beta_l = \frac{\alpha_l^{k-1}}{\prod_{\substack{r=1\\ \neq l}}^{k}(\alpha_l - \alpha_r)}$, and $\sum_{j=1}^{k} \beta_j = 1$.

Therefore,

$$R(t) = \sum_{l=1}^{k} \frac{\left(\frac{(k-l)\tau}{B_{l-1}}\right)^{k-1}}{\prod_{\substack{r=1\\ \neq l}}^{k}\left(\frac{(k-l)\tau}{B_{l-1}} - \frac{(k-r)\tau}{B_{r-1}}\ _r\right)} \exp\left(-\frac{1}{\frac{(k-l)\tau}{B_{l-1}}}t\right)$$

5 Parallel system

Consider a k unit parallel system with component failure times $T_1, T_2, \ldots, T_k$ having the GACMVE distribution. Then the system failure time is $T = \underset{1\le i\le k}{Max}\, T_i$.

MTBF = E (T)

$$= E\left(V_1^* + V_2^* + \ldots + V_k^*\right) + k\xi$$

$$= \tau \sum_{l=0}^{k-1} \left[\frac{\sum_{i=l}^{k-1} \binom{i}{k-1}}{B_l} \right] + k\,\xi, \text{ in view of (3.2)}$$

When η = α ξ +β τ, α, β ϵ R, the MREE of η is given by $\delta^* = \alpha\,\delta_{01} + \frac{1}{kn}\left[\beta - \frac{\alpha}{n\,B_0}\right]\delta_{02}$

By taking α = k and $\beta = \sum_{l=0}^{k-1} \left[\frac{\sum_{i=l}^{k-1} \binom{i}{k-1}}{B_l} \right]$ the MREE of the MTBF is given by

$$\delta^* = k\,\delta_{01} + \frac{1}{kn}\left[\sum_{l=0}^{k-1} \left[\frac{\sum_{i=l}^{k-1} \binom{i}{k-1}}{B_l} \right] - \frac{k}{n\,B_0} \right]\delta_{02}$$

Reliability function

R (t) = P(T > t)

$$= P\left(\sum_{i=1}^{k} V_i^* > t\right),\; t > 0$$

$$= \sum_{l=1}^{k} w_l \exp\left(-\frac{1}{\alpha_l} t\right)$$

Here $\alpha_l = \frac{\sum_{i=l}^{k-1} \binom{i}{k-1}}{B_{l-1}}$ $\forall\, l = 0,1,\ldots,k-1$, and $w_l = \frac{\alpha_l^{k-1}}{\prod_{\substack{r=1\\ \neq l}}^{k} (\alpha_1 - \alpha_r)}$ $\forall\, l = 0,1,\ldots,k-1$

6 Series system

Consider a k unit series system with component failure times $T_1, T_2, \ldots, T_k$ having the GACMVE distribution.

Then the system failure time is $T = \underset{1\le i\le k}{Min}\, T_i$.

From Theorem 3.2, $\underset{1\le l\le k}{Min}\, T_l \sim E\left[\xi, \frac{\tau}{B_0}\right]$

Thus, MTBF = $\frac{\tau}{B_0} + \xi$

When η = α ξ +β τ, α, β ϵ R, the MREE of η is given by $\delta^* = \alpha\,\delta_{01} + \frac{1}{kn}\left[\beta - \frac{\alpha}{n\,B_0}\right]\delta_{02}$

By taking α = 1 and $\beta = \frac{1}{B_l}$, the MREE of the MTBF is given by $\delta^* = \delta_{01} + \frac{1}{kn}\left[\frac{1}{B_l} - \frac{1}{n\,B_0}\right]\delta_{02}$

Reliability function

R(t) = P(T > t)

$$= \exp\left[\frac{B_0}{\tau}(t-\xi)\right],\ t>\xi$$

7 Relay system

Consider a k unit relay system with component failure times $T_1, T_2, \ldots, T_k$ having the GACMVE distribution. A relay system of order k operates if the first component and anyone of the remaining (k-1) components operate. Therefore, the failure time of the system is $T = T_1 \wedge (T_2 \vee T_3 \vee \ldots \vee T_k)$.

The reliability function of the system is

$$R(t) = P(T>t)$$
$$= \sum_{r=2}^{k} (-1)^r \binom{k-1}{r-1} \overline{F}_r(t,t,\ldots,t,0,\ldots,0),$$

using distributive law and routine arguments.

Here $\overline{F}_r(t,t,\ldots,t,0,\ldots,0)$ represents $P(X_1>t, X_2>t,\ldots,X_r>t, X_{r+1}>0,\ldots,X_k>0)$.

Let us discuss in detail the case when k = 3.

Here

$$R(t) = P(T>t)$$
$$= \sum_{r=2}^{3} (-1)^r \binom{k-1}{r-1} \overline{F}_r(t,t,0)$$
$$= 2\exp\left\{-\frac{(2\lambda_1+\lambda_2)}{\tau}(t-\xi)\right\} - \exp\left\{-\frac{(3\lambda_1+3\lambda_2+\lambda_3)}{\tau}(t-\xi)\right\}.$$

The MTBF is given by

$$\text{MTBF} = \frac{2\tau}{(2\lambda_1+\lambda_2)} - \frac{\tau}{(3\lambda_1+3\lambda_2+\lambda_3)} + \xi$$
$$= \left[\frac{(4\lambda_1+5\lambda_2+2\lambda_3)}{(2\lambda_1+\lambda_2)(3\lambda_1+3\lambda_2+\lambda_3)}\right]\tau + \xi$$

When η = α ξ +β τ, α, β ϵ R, the MREE of η is given by $\delta^* = \alpha\,\delta_{01} + \frac{1}{kn}\left[\beta - \frac{\alpha}{n B_0}\right]\delta_{02}$.

By taking α = 1 and $\beta = \left[\frac{(4\lambda_1 + 5\lambda_2 + 2\lambda_3)}{(2\lambda_1 + \lambda_2)(3\lambda_1 + 3\lambda_2 + \lambda_3)}\right]$, in the above equation, we get the MREE of the MTBF.

Therefore, MREE of the MTBF is

$$\delta^* = \delta_{01} + \frac{1}{kn}\left[\left(\frac{(4\lambda_1 + 5\lambda_2 + 2\lambda_3)}{(2\lambda_1 + \lambda_2)(3\lambda_1 + 3\lambda_2 + \lambda_3)}\right) - \frac{1}{n B_0}\right]\delta_{02}$$

Remark 7.1

From Theorem 3.2, we can obtain the UMVUE's of ξ and τ, and hence obtain the UMVUE of α ξ +β τ:

$$\delta^{**} = \alpha\,\delta_{01} + \frac{1}{kn-1}\left[\beta - \frac{\alpha}{n B_0}\right]\delta_{02}$$

Hence one can obtain the UMVUE's of the MTBF in each of the four systems discussed in this chapter.

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