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RELIABILITY OF A k-OUT-OF-n SYSTEM WITH REPAIR BY A SINGLE SERVER EXTENDING SERVICE TO EXTERNAL CUSTOMERS WITH PRE-EMPTION

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Abstract

In this paper we study the reliability of a k-out-of-n system, with a single technician, who also renders service to external customers besides repairing the failed components in the system. For optimizing the revenue from external service without compromising the system reliability, we introduce the N-policy in which the repair of the internal customers (failed components) starts only on accumulation of N failed components. The service to external customers is of preemptive nature in the sense that their service can be interrupted on accumulation of N failed components. It is assumed that an external customer, who finds the server busy with an external customer at the epoch of its arrival joins a queue of infinite capacity; whereas an external customer who finds the server busy with an internal customer leaves the system forever. The failure times of the components of the k-out-of-n system follow an exponential distribution; the arrival of external customers is according to a Poisson process and the service times of the internal and external customers follow non-identical phase-type distributions. Using matrix-analytic methods, we discuss the system stability and steady state distribution. A special case of the model where the underlying distributions are all exponential has been considered, to obtain an expression for the stability condition and a product form solution for the steady state have been obtained for this case. Also several system performance measures have been obtained explicitly. Analysis of a cost function indicates that N-policy does help to optimize the system revenue maintaining high system reliability.

1. Introduction

A k-out-of-n system can be defined as an n-component system, which works if and only if at least k of the n components are operational. The literature on k-out-of-n systems is vast (see Chakravarthy *et al.* [1] and the references therein). In a highly competitive world organizations pay high attention on giving service to external customers in addition to their internal customers. One main intention behind this is the additional income collected through external service. In addition, it may be expected that the expertise of the server be improved by attending jobs that are more diverse. The main drawback of providing service to external customers is that this decreases the attention on the internal customers. Also there is a chance of the service facility getting overloaded with too much of work. Hence keeping a proper balance between the internal and external services is much needed and at the same time much harder a task. In this context studying the reliability of a k-out-of-n system where the server attends external customers also could be of great value. There had been a few studies by Dudin *et al.* [2], Krishnamoorthy *et al.* [3, 4] in this area. In [2], the external customers are sent to an orbit from where they can try to access the idle server. Once selected for service, an external customer is assumed to get a non-preemptive service. Through numerical illustrations they show that providing service to external customers in this fashion is economical to the system in comparison with the decrease in the reliability caused due to external service. In [3] it is assumed that the external customers, finding the service station busy on arrival, are directed to a pool of infinite capacity. They also assume that if the size of the buffer of internal customers is less than L, a pooled customer is selected for service with probability p. In [4], a finite pool and an orbit of infinite capacity accommodate the external customers in such a manner that external customers join the orbit with some probability and from there try to enter the pool. The external customers are selected for service from the pool. The internal customers (failed components) are served based on an N-policy In addition they assume that the on-going service of an external customer is not pre-empted on accumulation of N-failed components. Under this assumption, numerical illustrations on [3, 4] indicate a decrease in the server idle probability, and an increase in the overall system revenue as in [2].

In the present paper we study a k-out-of-n system, where the sever also offers service to external customers for additional income. For optimizing revenue by way of providing external service, maintaining a high system reliability, we introduce an N-policy in which the service of the failed components starts on accumulation of N failed components at the beginning of each cycle (a cycle starts with the server being switched over to service of the failed components of the system on accumulation of N components until all of them, and the subsequent failed components get repaired. In other words the moment all failed components of the k-out of-n system are repaired, the server switches over to serve external customers; the service to external customers continue until the next epoch at which N failed components of the system again get accumulated). The service to the external customers is of preemptive nature in the sense that their service is interrupted on accumulation of N failed components. The external customers join a queue of infinite capacity on finding a busy server, provided the customer in service is an external arrival. The current study differs from that in [4] in that the pool (waiting space) of external customers is of infinite capacity and here there is no orbit of retrying customers. Also in contrast to [4], in the present work the service of external customers is assumed to be preemptive in nature. Under these stronger assumptions we obtain an explicit steady state distribution of the underlying Markov chain has been obtained.

This paper is arranged as follows: In section 2, we perform the Stochastic Modeling of the above problem and in section 3, we perform the steady state analysis of the underlying Markov chain after finding a necessary and sufficient condition for the stability of the system. Section 4, discusses a special case of the model discussed in Section 2, where the service time distributions are assumed to follow exponential distribution. In section 5 we conduct a numerical study of the model discussed in

Section 4 and compares it with a model in which no external customers are allowed. Section 6 concludes the discussion.

2. Modeling and Analysis

In this paper we study the reliability of a k-out-of-n system with repair by a single repair facility which also provides service to external customers. The system consists of two parts.

(1) A main queue consisting of customers (failed components of the k-out-of-n system) and
(2) A queue of external customers.

A k-out-of-n system is in the up state (working state) as long as at least k components are in operational state. Otherwise the system is in the down state.

The arrival process.

Arrival of main customers have inter-occurrence time exponentially distributed with parameter λ_i when the number of operational components of the k-out-of-n system is i. By taking $\lambda_i = \frac{\lambda}{i}$ we notice that the failure rate is a constant λ. Arrival of external customers have inter-occurrence time exponentially distributed with parameter $\overline{\lambda}$. Arrival of external customers is temporarily halted while serving the main customers (the failed components of the k-out -of-n system).

The service process.

Commencement of service to the failed components of the main system is governed by the N-policy, that is at the epoch the system starts with all components operational, the server starts attending one by one the customers from the queue of external customers (if there is any waiting). At the epoch when the accumulated number of failed components of the main system reaches N, the external customer in service will get pre-empted and the server is switched on to the service of main customers. Service times of main customers and external customers follow phase-type distributions with representations (α, S) and (β, T) of orders m_1 and m_0 respectively.

Objective.

To maximize the reliability of a k-out-of-n system with repair by a single server, who provides service to external customers also, based on N-policy.

The Markov Chain.

Let $X_1(t)$ denotes at time t number of external customers in the system including the one getting service (if any) ,

$X_2(t)$ denotes the server status at time t defined as;

$$X_2(t) = \begin{cases} 0, & \text{if the server is idle or serving an external customer} \\ 1, & \text{if the server is busy with a failed component.} \end{cases}$$

$X_3(t)$ denotes number of main customers in the system at time t including the one getting service (if any). $X_4(t)$ denotes the phase of the service process.

Let $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$ then $\{X(t),\ t \geq 0\}$ is a continuous time Markov chain on the state space whose levels are designated

$$\begin{aligned} l(0) &= \{(0, 0, j_1)/0 \leq j_1 \leq N-1\} \cup \{(0, 1, j_1, j_2)/1 \leq j_1 \leq n-k+1, 1 \leq j_2 \leq m_1\}, \\ l(i) &= l(i, 0) \cup l(i, 1), \\ l(i, 0) &= \{(i, 0, j_1, j_2)/0 \leq j_1 \leq N-1, 1 \leq j_2 \leq m_0\} \\ l(i, 1) &= \{(i, 1, j_1, j_2)/1 \leq j_1 \leq n-k+1, 1 \leq j_2 \leq m_1\}. \end{aligned}$$

In the sequel,

(i) I_n denotes the identity matrix of order n;
(ii) I denotes an identity matrix of appropriate size;
(iii) e_n denotes a $n \times 1$ column matrix of 1's
(iv) e denotes a column matrix of 1's of appropriate order;
(v) E_n denotes a square matrix of order n defined as

$$E_n(i, j) = \begin{cases} -1, & \text{if } i = j; 1 \leq i \leq n \\ 1, & \text{if } j = i+1; 1 \leq i \leq n-1 \\ 0, & \text{otherwise} \end{cases}$$

(vi) E'_n = Transpose of E_n

(vii) $r_n(i)$ denotes a $1 \times n$ row matrix whose i^{th} entry is 1 and all other entries are zeros

(viii) $C_n(i)$ = Transpose of $r_n(i)$

(ix) $\otimes$ denotes Kronecker product of matrices

(x) $S^0 = -Se$, $T^0 = -Te$.

The infinitesimal generator matrix of $\{X(t)\}$ is given by

$$Q = \begin{bmatrix} \widetilde{A}_1 & \widetilde{A}_0 & & & \\ \widetilde{A}_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \end{bmatrix}, \text{ where } \widetilde{A}_1 = \begin{bmatrix} \widetilde{A}_{00} & \widetilde{A}_{01} \\ \widetilde{A}_{10} & \widetilde{A}_{11} \end{bmatrix}$$

$$\widetilde{A}_{00} = \lambda E_N - \bar{\lambda} I_N, \widetilde{A}_{01} = [C_N(N) \otimes r_{n-k+1}(N)] \otimes \lambda\alpha, \widetilde{A}_{10} = [C_{n-k+1}(1) \otimes r_N(1)] \otimes S^0,$$

$$\widetilde{A}_{11} = I_{n-k+1} \otimes S + (E'_{n-k+1} + I_{n-k+1}) \otimes \left(S^0\alpha\right)$$
$$+ [E_{n-k+1} + C_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1)] \otimes \lambda I_{m_1};$$

$$A_1 = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix};$$

$$A_{00} = E_N \otimes \lambda I_{m_0} + I_N \otimes \left(T - \bar{\lambda} I_{m_0}\right), A_{01} = [C_N(N) \otimes r_{n-k+1}(N)] \otimes (\lambda e_{m_0}\alpha);$$

$$A_{10} = [C_{n-k+1}(1) \otimes r_N(1)] \otimes (S^0\beta), \quad A_{11} = \widetilde{A}_{11};$$

$$\widetilde{A}_0 = \begin{bmatrix} I_N \otimes (\bar{\lambda}\beta) & 0 \\ 0 & 0 \end{bmatrix}, \widetilde{A}_2 = \begin{bmatrix} I_N \otimes T^0 & 0 \\ 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} I_N \otimes (\bar{\lambda} I_{m_0}) & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} I_N \otimes (T^0\beta) & 0 \\ 0 & 0 \end{bmatrix}$$

3. Steady State Analysis

3.1. Stability condition.

Let $A = A_0 + A_1 + A_2$ and π be the steady state vector of A. That is π satisfies the equations

$$\boldsymbol{\pi} A = 0 \quad \text{and} \tag{3.1}$$

$$\boldsymbol{\pi}\, \boldsymbol{e} = 1. \tag{3.2}$$

Partitioning $\boldsymbol{\pi}$ as $\boldsymbol{\pi} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1})$, equation (3.1) gives

$$\boldsymbol{\pi_0}\left[E_N \otimes \lambda I_{m_0} + I_N \otimes (T + T^0\beta)\right] + \boldsymbol{\pi_1} A_{10} = 0 \tag{3.3}$$

$$\boldsymbol{\pi_0} A_{01} + \boldsymbol{\pi_1} A_{11} = 0. \tag{3.4}$$

From equation (3.4), $\boldsymbol{\pi_1} = -\boldsymbol{\pi_0} A_{01} A_{11}^{-1}$.
Substituting in equation (3.3), we get

$$\boldsymbol{\pi_0}\left[E_N \otimes \lambda I_{m_0} + I_N \otimes (T + T^0\beta)\right] - \boldsymbol{\pi_0} A_{01} A_{11}^{-1} A_{10} = 0 \tag{3.5}$$

We notice that $A_{10} = (-A_{11}e)(r_N(1) \otimes \beta)$ and therefore $-A_{11}^{-1}A_{10} = e(r_N(1) \otimes \beta)$

$$-A_{01}A_{11}^{-1}A_{10} = (C_N(N) \otimes \lambda e_{m_0})\,(r_N(1) \otimes \beta)\,. \tag{3.6}$$

Thus equation (3.5) reduce to

$$\boldsymbol{\pi_0}\left[E_N \otimes \lambda I_{m_0} + (C_N(N) \otimes r_N(1)) \otimes (\lambda e_{m_0}\beta) + I_N \otimes (T + T^0\beta)\right] = 0. \tag{3.7}$$

Further partitioning $\boldsymbol{\pi_0} = (\pi_{0,0}, \pi_{0,1}, \ldots, \pi_{0,N-1})$, equation (3.7) give rise to the following set equations

$$\pi_{0,0}\left(T + T^0\beta - \lambda I_{m_0}\right) + \pi_{0,N-1}\lambda e_{m_0}\beta = 0 \tag{3.8}$$

$$\pi_{0,i}\lambda I_{m_0} + \pi_{0,i+1}\left(T + T^0\beta - \lambda I_{m_0}\right) = 0, 0 \le i \le N-1. \tag{3.9}$$

Postmultiply both sides of equation (3.8) and (3.9) by the column vector e, we get

$$\pi_{0,0}\left(T + T^0\beta - \lambda I_{m_0} + \lambda e_{m_0}\beta\right) = 0 \tag{3.10}$$

$$\pi_{0,i}\boldsymbol{e} = \pi_{0,i+1}\boldsymbol{e}, 0 \le i \le N-1. \tag{3.11}$$

And this gives

$$\pi_{0,0} = a\eta \tag{3.12}$$

where η is the steady state vector of the generator matrix $T + T^0\beta - \lambda I_{m_0} + \lambda e_{m_0}\beta$ and 'a' is a constant.

Now equation (3.9) gives

$$\pi_{0,i} = (-1)^i a\lambda^i \eta \left(T + T^0\beta - \lambda I_{m_0}\right)^{-i}, 0 \leq i \leq N-1. \tag{3.13}$$

Equation (3.13) determines the vector π_0 up to the multiplicative constant.
It follows from equations (3.11) and (3.13) that

$$\begin{aligned}
\boldsymbol{\pi} A_0 \boldsymbol{e} &= \bar{\lambda} \boldsymbol{\pi_0} \, \boldsymbol{e} \\
&= \bar{\lambda} \, aN \\
\boldsymbol{\pi} A_2 \boldsymbol{e} &= \sum_{i=0}^{N-1} \pi_{0,i} T^0 \\
&= a \sum_{i=0}^{N-1} (-1)^i \lambda^i \eta \left(T + T^0\beta - \lambda I_{m_0}\right)^{-i} T^0.
\end{aligned}$$

Here $\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e}$ becomes

$$N\bar{\lambda} < \sum_{i=0}^{N-1} (-1)^i \lambda^i \eta \left(T + T^0\beta - \lambda I_{m_0}\right)^{-i} T^0.$$

This leads to the following theorem for the stability of the system.

Theorem 3.1. *The Markov chain $\{X(t)\}$ is stable if and only if*

$$N\bar{\lambda} < \sum_{i=0}^{N-1} (-1)^i \lambda^i \eta \left(T + T^0\beta - \lambda I_{m_0}\right)^{-i} T^0.$$

3.2. **Steady State Vector.**

The steady state vector $\boldsymbol{x}$ is partitioned as $\boldsymbol{x} = (x_0, x_1, x_2, \ldots)$ satisfies the equations

$$\begin{aligned}
x_0 \widetilde{A}_1 + x_1 \widetilde{A}_2 &= 0 \\
x_0 \widetilde{A}_0 + x_1 A_1 + x_2 A_2 &= 0 \\
x_i A_0 + x_{i+1} A_1 + x_{i+2} A_2 &= 0, i \geq 1.
\end{aligned}$$

Matrix theoretic approach (See Neuts [5]*) gives*

$$x_i = x_1 R^{i-1}, \ i \geq 1 \tag{3.14}$$

where R is the minimal non negative solution of the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0. \tag{3.15}$$

It then follows that

$$x_1 = -x_0 \widetilde{A}_0 (A_1 + RA_2)^{-1} \tag{3.16}$$

and that x_0 satisfies the system of equations

$$x_0 \left(\widetilde{A}_1 - \widetilde{A}_0 \left(A_1 + RA_2 \right)^{-1} \widetilde{A}_2 \right) = 0. \tag{3.17}$$

From the structure of the matrix A_0, it follows that the R matrix has the form

$$R = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \tag{3.18}$$

where R_1 is a square matrix of order Nm_0 and R_2 is a matrix of order $Nm_0 \times (n-k+1)m_1$.

$$R^2 = \begin{bmatrix} R_1^2 & R_1 R_2 \\ 0 & 0 \end{bmatrix}$$

Equation (3.15) *then reduces to the following equations*

$$R_1^2 \left(I_N \otimes T^0 \beta \right) + R_1 A_{00} + R_2 A_{10} + I_N \otimes \bar{\lambda} I_{m_0} = 0 \tag{3.19}$$

$$R_1 A_{01} + R_2 A_{11} = 0 \tag{3.20}$$

$$\textit{Equation } (3.20) \textit{ gives} \quad R_2 = -R_1 A_{01} A_{11}^{-1} \tag{3.21}$$

which when substituted in Equation (3.19) *gives*

$$R_1^2 \left(I_N \otimes T^0 \beta \right) + R_1 A_{00} - R_1 A_{01} A_{11}^{-1} A_{10} + \bar{\lambda} I_{Nm_0} = 0$$

$$\text{i.e., } R_1^2 \left(I_N \otimes T^0 \beta \right) + R_1 \left(A_{00} - A_{01} A_{11}^{-1} A_{10} \right) + \bar{\lambda} I_{Nm_0} = 0.$$

Using equation (3.6), *the above equation can be rewritten as*

$$R_1^2\left(I_N \otimes T^0\beta\right) + R_1\left[A_{00} + (C_N(N) \otimes r_N(1)) \otimes (\lambda e_{m_0}\beta)\right] + \bar{\lambda} I_{Nm_0} = 0. \tag{3.22}$$

Solving equation (3.22), *we get* R_1 *and hence the steady state vector of* $\{X(t)\}$. *For Solving equation* (3.22) *we use Logarithmic reduction algorithm (refer Latouche and Ramaswami* [6]*).*

4. A Special Case

We now concentrate on a special case of the problem discussed in Section 2 where the service time distributions of main and external customers follow exponential distributions with parameters μ and $\bar{\mu}$ respectively. As expected, this resulted in arriving at explicit expression for the stability condition, steady state distribution and several performance measures.

4.1. The Markov Chain Model.

With $X_1(t), X_2(t)$ and $X_3(t)$ having same definition as in section 2, $\widetilde{X}(t) = (X_1(t), X_2(t), X_3(t))$ is a continuous time Markov chain on the state space

$$\{(j_1, 0, j_2)|j_1 \geq 0; 0 \leq j_2 \leq N-1\} \cup \{(j_1, 1, j_2)|j_1 \geq 0; 0 \leq j_2 \leq n-k+1\}.$$

Arranging the states lexicographically and then partitioning the state space into levels i, where each level i corresponds to the collection of states with number of external customers in the system including the one getting service (if any) at time t as i. We get the infinitesimal generator of the above chain as

$$\widetilde{Q} = \begin{bmatrix} F_{10} & F_0 & & & & \\ F_2 & F_1 & F_0 & & & \\ & F_2 & F_1 & F_0 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \end{bmatrix} \tag{4.1}$$

The entries of the matrix are described below.

The transition from level i to level $i+1$ is represented by the matrix

$$F_0 = \begin{bmatrix} \bar{\lambda} I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1)\times N} & 0_{(n-k+1)\times(n-k+1)} \end{bmatrix}.$$

The transition from level i to level $i-1$ is represented by the matrix

$$F_2 = \begin{bmatrix} \bar{\mu} I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1)\times N} & 0_{(n-k+1)\times(n-k+1)} \end{bmatrix}.$$

The transition within level 0 to level 0 is represented by the matrix

$$F_{10} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where $\quad B_1 = \lambda E_N - \bar{\lambda} I_N;$

B_2 is a $N \times (n-k+1)$ matrix whose $(N, N)^{\text{th}}$ entry is λ and all other entries are zeroes.
B_3 is a $(n-k+1) \times N$ matrix whose $(1, 1)^{\text{th}}$ entry is μ and all other entries are zeroes.

$$B_4 = \lambda E_{n-k+1} + \mu E'_{n-k+1} + \lambda C_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1).$$

The transitions within level i, $i \geq 1$, is represented by matrix

$$F_1 = \begin{bmatrix} D_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where $D_1 = \lambda E_N - (\bar{\lambda} + \bar{\mu}) I_N$.

4.2. **Steady State Analysis.** First we derive the condition for stability of the system.

4.2.1. Stability condition.

Consider the generator matrix

$$F = F_0 + F_1 + F_2 = \begin{bmatrix} H_1 & H_2 \\ H_3 & B_4 \end{bmatrix}, \quad \text{where} \quad H_1 = \lambda E_N.$$

H_2 is a $N \times (n-k+1)$ matrix whose $(N,N)^{\text{th}}$ entry is λ and all other entries are zeroes.
H_3 is a $(n-k+1)\times N$ matrix whose $(1,1)^{\text{th}}$ entry is μ and all other entries are zeroes.

The stationary probability vector $\widetilde{\Pi} = (\widetilde{\pi}_{(0,0)}, \widetilde{\pi}_{(0,1)}, \cdots, \widetilde{\pi}_{(0,N-1)}, \widetilde{\pi}_{(1,1)}, \cdots, \widetilde{\pi}_{(1,N)} \cdots, \widetilde{\pi}_{(1,n-k+1)})$ of the generator matrix A satisfies the equations $\widetilde{\Pi}F = 0$ and $\widetilde{\Pi}\boldsymbol{e} = 1$.
$\widetilde{\Pi}F = 0$ gives the following equations

$$\widetilde{\pi}_{(0,i)} = \widetilde{\pi}_{(0,0)}, 1 \le i \le N-1 \quad \text{and}$$

$$\widetilde{\pi}_{(1,i)} = \begin{cases} \alpha_i \widetilde{\pi}_{(0,0)}, & \text{where } \alpha_i = \sum_{j=1}^{i} (\lambda/\mu)^j, i = 1,2,\ldots N \\ \beta_i \widetilde{\pi}_{(0,0)}, & \text{where } \beta_i = \sum_{j=1-N+1}^{i} (\lambda/\mu)^j, i = N+1,\ldots n-k+1 \end{cases}$$

The normalizing condition $\widetilde{\Pi}\boldsymbol{e} = 1$ gives $\widetilde{\pi}_{(0,0)} = \frac{1}{\varphi - \psi}$, where

$$\varphi = N + \frac{\left(\mu^{N-2} - \lambda^{N-2}\right)\lambda}{(\mu-\lambda)\mu^N} \left\{ N + \frac{\lambda\left(\mu^{n-k+1-N} - \lambda^{n-k+1-N}\right)}{\mu^{n-k+1-N}(\mu-\lambda)} \right\} \quad \text{and}$$

$$\psi = \frac{(\mu-\lambda)\left(\mu^{N-1} - (N-1)\lambda^N\right) + \lambda\mu\left(\mu^{N-2} - \lambda^{N-2}\right)}{\mu^{N-1}(\mu-\lambda)}$$

Thus we arrive at the following

Theorem 4.1. *The process $\{\widetilde{X}(t), t \ge 0\}$ is positive recurrent if and only if $\overline{\lambda} < \overline{\mu}$.*

Proof. It is well known (see Neuts [5]) that the Markov chain with infinitesimal generator $\widetilde{Q}$ is stable if and only if $\widetilde{\pi}F_0e < \widetilde{\pi}F_2e$, that is if and only if the left drift rate exceeds that to the right.
We have $\widetilde{\pi}F_0e = N\overline{\lambda}\widetilde{\pi}_{(0,0)}$ and $\widetilde{\pi}F_2e = N\overline{\mu}\widetilde{\pi}_{(0,0)}$. Thus $\{\widetilde{X}(t), t \ge 0\}$ is positive recurrent if and only if $\overline{\lambda} < \overline{\mu}$. □

4.2.2. Steady State Distribution.

Here using the steady state vector $\widetilde{\Pi}$ of the generator matrix F, we proceed construct the steady state vector $\widetilde{X} = (\widetilde{X}(0), \widetilde{X}(1), \widetilde{X}(2), \ldots)$ of the Markov chain $\{\widetilde{X}(t), t \ge 0\}$ by defining, $\widetilde{X}(i) = \eta\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$, for $i \ge 0$, where η is a positive constant to be found out.

First we will prove that $\widetilde{X}$ satisfies the equation $\widetilde{X}\widetilde{Q} = 0$. For this, notice that we can decompose the infinitesimal generator matrix $\widetilde{Q}$ as $\widetilde{Q} = \widetilde{Q}_1 + \widetilde{Q}_2$, where

$$\widetilde{Q}_1 = \begin{bmatrix} F & & & & \\ & F & & & \\ & & F & & \\ & & & \cdots & \\ & & & & \cdots \end{bmatrix} \quad \text{and}$$

$$\widetilde{Q}_2 = \begin{bmatrix} -F_0 & F_0 & & & & \\ F_2 & \overline{F_1} & F_0 & & & \\ & F_2 & \overline{F_1} & F_0 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \end{bmatrix},$$

where each entry is a square matrix of order $N + n - k + 1$ listed as:

$$\overline{F}_1 = \begin{bmatrix} -(\overline{\lambda} + \overline{\mu})I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1)\times N} & 0_{(n-k+1)\times(n-k+1)} \end{bmatrix}.$$

Since $\widetilde{\Pi}F = 0$ and $\widetilde{X}(i) = \eta\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$, we have

$$\widetilde{X}\widetilde{Q}_1 = 0. \tag{4.2}$$

Now,

$$\widetilde{X}\widetilde{Q}_2 = \left[\widetilde{X}(0)(-F_0) + \widetilde{X}(1)F_2, \widetilde{X}(0)F_0 + \widetilde{X}(1)\overline{F_1} + \widetilde{X}(2)F_2, \widetilde{X}(1)F_0 + \widetilde{X}(2)\overline{F_1} + \widetilde{X}(3)F_2, \cdots\right].$$

Notice that $(-F_0) + \frac{\overline{\lambda}}{\overline{\mu}}F_2 = 0$ and

$$F_0 + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)\overline{F_1} + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^2 F_2 = F_0 + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\overline{F_1} + \frac{\overline{\lambda}}{\overline{\mu}}F_2\right)$$

$$= F_0 - \frac{\overline{\lambda}}{\overline{\mu}} F_2$$
$$= 0,$$

which leads us to $\widetilde{X}(0)(-F_0) + \widetilde{X}(1)F_2 = 0$ and

$$\widetilde{X}(i)F_0 + \widetilde{X}(i+1)\overline{F_1} + \widetilde{X}(i+1)F_2 = \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{X}(0)\left[F_0 + \frac{\overline{\lambda}}{\overline{\mu}}F_1 + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^2 F_2\right]$$
$$= 0,\ i = 0, 1, 2, 3, \ldots.$$

Hence

$$\widetilde{X}Q_2 = 0. \tag{4.3}$$

From (4.2) and (4.3), we have $\widetilde{X}\widetilde{Q_1} + \widetilde{X}\widetilde{Q_2} = 0$, which implies that $\widetilde{X}\widetilde{Q} = 0$.
Finally, $\widetilde{X}\boldsymbol{e} = 1$ gives the unknown constant $\eta = \frac{(\overline{\mu}-\overline{\lambda})}{\overline{\mu}}$.

Hence, $\widetilde{X} = (\widetilde{X}(0), \widetilde{X}(1), \widetilde{X}(2)\cdots)$, where $\widetilde{X}(i) = \left(\frac{\overline{\mu}-\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$ is the steady state vector for the matrix $\widetilde{Q}$ and we have the following theorem;

Theorem 4.2. *Let* $\widetilde{\Pi} = (\widetilde{\pi}_{(0,0)}, \widetilde{\pi}_{(0,1)}, \cdots, \widetilde{\pi}_{(0,N-1)}, \widetilde{\pi}_{(1,1)}, \cdots, \widetilde{\pi}_{(1,N)}, \cdots \widetilde{\pi}_{(1,n-k+1)})$ *be the steady state vector for the matrix F, where*

$$\widetilde{\pi}_{(0,i)} = \widetilde{\pi}_{(0,0)}, 1 \le i \le N-1 \quad \textit{and}$$

$$\widetilde{\pi}_{(1,i)} = \begin{cases} \alpha_i \widetilde{\pi}_{(0,0)}, & \textit{with } \alpha_i = \sum\limits_{j=1}^{i} (\lambda/\mu)^j, i = 1, 2, \ldots N \\ \beta_i \widetilde{\pi}_{(0,0)}, & \textit{for } \beta_i = \sum\limits_{j=1-N+1}^{i} (\lambda/\mu)^j, i = N+1, \ldots n-k+1 \end{cases}.$$

Further $\widetilde{\pi}_{(0,0)} = \frac{1}{\varphi - \psi}$, *where*

$$\varphi = N + \frac{\left(\mu^{N-2} - \lambda^{N-2}\right)\lambda}{(\mu-\lambda)\mu^N}\left\{N + \frac{\lambda\left(\mu^{n-k+1-N} - \lambda^{n-k+1-N}\right)}{\mu^{n-k+1-N}(\mu-\lambda)}\right\} \quad \textit{and}$$

$$\psi = \frac{(\mu-\lambda)\left(\mu^{N-1} - (N-1)\lambda^N\right) + \lambda\mu\left(\mu^{N-2} - \lambda^{N-2}\right)}{\mu^{N-1}(\mu-\lambda)}.$$

Then $\widetilde{X} = (\widetilde{X}(0), \widetilde{X}(1), \widetilde{X}(2)\cdots)$, *where* $\widetilde{X}(i) = \left(1 - \frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$ *is the steady state probability vector for the Markov chain* $\{\widetilde{X}(t), t \ge 0\}$.

4.3. **Performance Measures.**

Here we derive certain important performance measures of the system under study.

4.3.1. Busy period of the server with the failed components of the main system.

The busy period of the server with failed components starts the instant when N failed components accumulate and it ends when no failed components are left in the system. Let $T_N(i)$, for $i \geq 0$, denote the server busy period with failed components, which starts with i external customers in the system. Note that, the number of external customers does not affect the busy period of the server with the failed components. Hence, $T_N(i) = T_N$, for $i \geq 0$. For analyzing the time T_N, we consider the Markov chain $\{Y(t)\}$ with state space $\{0, 1, 2, \dots, N, N+1, \dots, n-k+1\}$ and infinitesimal generator given by:

$$B_N = \begin{bmatrix} 0 & 0 \\ -\widehat{B}_N e & \widehat{B}_N \end{bmatrix}, \quad \text{where}$$

$$\widehat{B}_N = \lambda E_{n-k+1} + \mu E'_{n-k+1}.$$

Note that $Y(t)$ denotes the number of failed components of the main system and $Y(t) = 0$ is considered as an absorbing state; so that the busy period T_N is the time until absorption in the Markov chain $\{Y(t)\}$, assuming that it starts at the state N. Hence, the busy period T_N has a phase type distribution with representation $(\omega, \widehat{B}_N)$, where the probability vector $\omega = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 appearing in the N^{th} position. The expected value of T_N is therefore given by $ET_N = -\omega(B_N^{-1})\boldsymbol{e}$ where $\boldsymbol{e}$ is a column vector with $n-k+1$ elements all equal to 1. Now for finding ET_N, let us partition the column vector $(\widehat{B}_N^{-1})\boldsymbol{e}$ as $(t_1, t_2, \dots, t_{n-k+1})^T$. Then the identity $\widehat{B}_N(\widehat{B}_N^{-1})\boldsymbol{e} = \boldsymbol{e}$ leads us to the following equations:

$$-(\lambda + \mu)t_1 + \lambda t_2 = 1$$

$$\mu t_{i-1} - (\lambda + \mu)t_i + \lambda t_{i+1} = 1, \quad \text{for } 2 \leq i \leq n-k$$

$$\mu t_{n-k} - \mu t_{n-k+1} = 1.$$

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The above equations give

$$t_i - t_{i+1} = \frac{1}{\mu}\sum^{n-k-i} j = 0(\lambda/\mu)^j, 1 \leq i \leq n-k$$

$$t_{n-k} - t_{n-k+1} = \frac{1}{\mu} \quad \text{and} \quad -\mu t_1 = \sum_{j=0}^{n-k}(\lambda/\mu)^j.$$

Hence

$$ET_N = -t_N = \frac{1}{\mu}\left(N\sum_{j=0}^{n-k-N+1}(\lambda/\mu)^j + \sum_{j=n-k-N+2}^{n-k}(n-k+1-j)(\lambda/\mu)^j\right). \tag{4.4}$$

The expected value of the busy period of the server with failed components, which starts with an arbitrary number of external customers is given by

$$E_B = ET_N\sum_{j_1=0}^{\infty}\widetilde{x}(j_1, 0, N-1)$$

$$= \frac{1}{(\varphi-\psi)}\frac{1}{\mu}\left(N\sum_{j=0}^{n-k-N+1}(\lambda/\mu)^j + \sum_{j=n-k-N+2}^{n-k}(n-k+1-j)(\lambda/\mu)^j\right). \tag{4.5}$$

We sum up the above results in

Theorem 4.3. *The busy period of the server with the repair of the components of the k-out-of-n system has phase type distribution with representation* $(\omega, \widehat{B}_N)$. *The expected length of the busy period is given by* (4.5).

4.3.2. Expected number of pre-emptions of an external customer who is taken for service.

Consider the Markov process $X_p(t) = (N_p(t), J(t))$, where $N_p(t)$ is the number of pre-emptions occurred upto time t (measured from the time he is taken for service) of a particular external customer who is taken for service and $J(t)$ is the number of failed components of the main system. Then $X_p(t)$ has the state space

$$\left\{(j_1, j_2)\big/ j_1 = 0, 1, 2, \ldots, 0 \leq j_2 \leq N-1\right\} \cup \{\Delta\}$$

where Δ is an absorbing state which denotes the service completion of the external customer. The infinitesimal generator of this process is

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \widetilde{T}^0 & \widetilde{T} & \widehat{A}_0 & 0 & \cdots & \cdots & \cdots \\ \widetilde{T}^0 & 0 & \widetilde{T} & \widehat{A}_0 & \cdots & \cdots & \cdots \\ \widetilde{T}^0 & 0 & 0 & \widetilde{T} & \widehat{A}_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } \quad \widetilde{T}^0 = \overline{\mu} e_N$$

$$\widetilde{T} = \lambda E_N - \overline{\mu} I_N$$

and $\widehat{A}_0$ is an $N \times N$ matrix whose $(N, 1)^{\text{th}}$ entry is λ.
If p_{k_i} is the probability for k pre-emptions of an external customer who starts service with i failed components, then $p_{0_i} = \left(-\widetilde{T}^{-1}\widetilde{T}^0\right)_i = 1 - \left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N-i}$, $0 \leq i \leq N-1$ and for $k \geq 1$,

$$\begin{aligned} p_{k_i} &= \left(\left(-\widetilde{T}^{-1}\widehat{A}^0\right)\left(-\widetilde{T}^{-1}\widetilde{T}^0\right)\right)_i \\ &= \left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N-i}\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N(k-1)}\left(1-\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^N\right) \\ &= \left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{Nk-i}\left(1-\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^N\right). \end{aligned}$$

Expected number of pre-emptions of an external customer, starting service with i failed components

$$= \sum_{k=0}^{\infty} k p_{k_i} = \left(1-\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^N\right)^{-1}\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N-i}.$$

4.3.3. Expected waiting time of an external customer.

For computing the expected waiting time of an external customer who joins as the r^{th} customer in the queue of external customers, we consider the Markov process $X_w(t) = (J_1(t), S(t), J_2(t))$, where $J_1(t)$ is the rank of the external customer, $S(t) = 0$ if the server is busy with external customers and $S(t) = 1$ if the server is busy with a main customer. $J_2(t)$ is the number of main customers in the system. The rank

$J_1(t)$ of an external customer is assumed to be 'l' if it finds $l-1$ external customers ahead of it. The rank of an external customer may decrease by 1 if an external customer ahead of it leaves the system after completing the service. Now consider the Markoov process $X_w(t)$ for a tagged external customer who finds $l-1$ external customers ahead of it while joining the system. The state space for this process is given by $\{*\} \cup \{\{1,2,\ldots,l\} \times (\{0\} \times \{0,1,\ldots,N-1\} \cup \{1\} \times \{1,2,\ldots,n-k+1\})\}$, where $*$ is an absorbing state, which denotes the service completion of the tagged customer. The infinitesimal generator Q_w of this process is $Q_w = \begin{bmatrix} 0 & 0 \\ W_l^0 & W_l \end{bmatrix}$, where

$$W_l = \begin{bmatrix} w_{11} & & & & \\ w_{22} & w_{12} & & & \\ & w_{23} & w_{13} & & \\ & & & \ddots & \\ & & & w_{2l} & w_{1l} \end{bmatrix}$$

with
$$w_{1i} = F_1 + F_0; 1 \le i \le l$$
$$w_{2i} = F_2; 1 \le i \le l$$
$$w_l^0 = C_l(1) \otimes (F_2 e)$$

The waiting time of the tagged customer is the time until absorption in the Markov process $X_w(t)$. Let $E_W^{(i)}(l)$ denote the expected waiting time of a tagged customer who joins the system with rank l, who finds 'i' failed components. Defining the row vector $\widetilde{\theta}_i$ as $\widetilde{\theta}_i = r_l(l) \otimes r_{N+n-k+1}(i+1)$, $0 \le i \le N-1$. Then $E_W^{(i)}(l) = -\widetilde{\theta}_i W_l^{-1} \boldsymbol{e}$, $0 \le i \le N-1$. Let $E_W(l)$ be the $N \times 1$ column matrix whose $(i,1)^{\text{th}}$ entry is $E_W^{(i-1)}(l)$. Taking the probability that an external customer see i external customers, j failed components and server busy with external customers on its arrival as $\left(1-\frac{\bar{\lambda}}{\bar{\mu}}\right)\left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \frac{1}{\varphi-\psi}$, the expected waiting time of an arbitrary external customers is given by

$$\sum_{i=0}^{\infty}\left(1-\frac{\bar{\lambda}}{\bar{\mu}}\right)\left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \frac{1}{\varphi-\psi}\sum_{j=0}^{N-1} E_W^{(j)}(i+1).$$

4.4. **Other Performance measures.**

(1) Fraction of time the system is down is given by,

$$P_{down} = \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\lambda^{n-k+2-N}\left(\mu^N - \lambda^N\right)}{\mu^{n-k+1}(\mu - \lambda)(\varphi - \psi)}.$$

(2) System reliability defined as the probability that at least k components are operational

$$P_{rel} = 1 - P_{down} = 1 - \frac{\lambda^{n-k+2-N}(\mu^N - \lambda^N)}{\mu^{n-k+1}(\mu - \lambda)(\varphi - \psi)}.$$

(3) Average number of external units waiting in the queue is given by,

$$N_q = \sum_{j_1=0}^{\infty} j_1 \sum_{j_3=1}^{n-k+1} X_{(j_1,1,j_3)} + \sum_{j_1=2}^{\infty}(j_1 - 1) \sum_{j_3=1}^{N-1} X_{(j_1,0,j_3)}$$

$$= \bar{\lambda}\left[\frac{1}{\bar{\mu} - \bar{\lambda}} - \frac{N}{\bar{\mu}(\varphi - \psi)}\right]$$

(4) Average number of failed components of the main system,

$$N_{fail} = \sum_{j_3=0}^{N-1} J_3 \left(\sum_{j_1=0}^{\infty} X_{(j_1,0,j_3)}\right) + \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} X_{(j_1,1,j_3)}\right)$$

$$= \frac{1}{(\varphi - \psi)} \left\{ \frac{N(N-1)}{2} + \sum_{i=1}^{N-1} i \left(\sum_{j=1}^{i} (\lambda/\mu)^j \right) + \frac{\lambda(\mu^N - \lambda^N)}{\mu^N(\mu - \lambda)} \left(\sum_{i=N}^{n-k+1} i(\lambda/\mu)^{i-N} \right) \right\}$$

(5) Average number of failed components waiting when the server is busy with external customers

$$= \sum_{j_3=0}^{N-1} j_3 \left(\sum_{j_1=1}^{\infty} x_{(j_1,0,j_3)} \right)$$

$$= \frac{N(N-1)\bar{\lambda}}{2\bar{\mu}(\varphi - \psi)}$$

(6) Expected number of external customers joining the system,

$$\theta_3 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right)$$
$$= N \frac{\bar{\lambda}}{(\varphi - \psi)}.$$

(7) Expected number of external customers, on arrival, getting service directly

$$= \bar{\mu} \sum_{j_3=0}^{N-1} x_{(0,0,j_3)}$$
$$= N \frac{(\bar{\mu} - \bar{\lambda})}{(\varphi - \psi)}.$$

(8) Fraction of time the server is busy with external customers,

$$P_{ex.busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right) = \frac{N \cdot \bar{\lambda}}{\bar{\mu}(\varphi - \psi)}.$$

(9) Probability that the server is found idle,

$$P_{idle} = \sum_{j_3=0}^{N-1} x_{(0,0,j_3)} = N \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}(\varphi - \psi)}.$$

(10) Probability that the server is found busy,

$$P_{busy} = 1 - P_{idle} = 1 - N \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}(\varphi - \psi)}.$$

(11) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} x_{(j_1,1,j_3)} \right) = \bar{\lambda} \left(1 - \frac{N}{(\varphi - \psi)} \right).$$

(12) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)}$$
$$= \frac{N \bar{\mu}}{(\varphi - \psi)}.$$

(13) Expected number of external customers in the system when the server is busy with external customers

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right) = \frac{N\overline{\lambda}}{(\overline{\mu} - \overline{\lambda})(\varphi - \psi)}.$$

4.5. **Another Special case.** Next we consider second special case of the problem discussed in section 4.1, where we take $N = 1$; that is the case where no special policy has been applied for providing service to external customers. Notice that in this case, at most importance is given to the failed components and an external customer can get service only when there are no failed components in the system. Further, an ongoing external customer's service may be pre-empted if a component of the system fails during the service of the former. Since in this case, knowing the number of external as well as the failed components is enough for determining the server status, the Markov chain becomes $\widehat{X}(t) = (X_1(t), X_3(t))$, with state space $\widetilde{S} = \{(j_1, j_2) | j_1 \geq 0, 0 \leq j_2 \leq n - k + 1\}$ and infinitesimal generator

$$\widehat{Q} = \begin{bmatrix} \widetilde{A}_{10} & \widetilde{A}_0 & & & \\ \widetilde{A}_2 & \widetilde{A}_1 & \widetilde{A}_0 & & \\ & \widetilde{A}_2 & \widetilde{A}_1 & \widetilde{A}_0 & \\ & & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots \end{bmatrix}, \text{where}$$

$$\widetilde{A}_{10} = \lambda E_{n-k+2} + \lambda C_{n-k+2}(n - k + 2) \otimes r_{n-k+2}(n - k + 2)$$
$$+ \mu E'_{n-k+2} + (\mu - \overline{\lambda}) C_{n-k+2}(1) \otimes r_{n-k+2}(1);$$

$\widetilde{A}_0$ is a $(n - k + 2) \times (n - k + 2)$ matrix whose $(1, 1)$ entry is $\overline{\lambda}$ and all other entries are zeroes;

$\widetilde{A}_2$ is a $(n - k + 2) \times (n - k + 2)$ matrix whose $(1, 1)$ entry is $\overline{\mu}$ and all other entries are zeroes;

$$\widetilde{A}_1 = \widetilde{A}_{10} - \overline{\mu} C_{n-k+2}(1) \otimes r_{n-k+2}(1).$$

Let $\widetilde{A} = \widetilde{A}_0 + \widetilde{A}_1 + \widetilde{A}_2$; then

$\widetilde{A} = \lambda E_{n-k+2} + \lambda C_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2) + \mu E_{n-k+2} + \mu C_{n-k+2}(1) \otimes r_{n-k+2}(1)$

The stationary probability vector $\widehat{\Pi} = (\widehat{\pi}_{(0,0)}, \widehat{\pi}_{(0,1)}, \ldots \widehat{\pi}_{(0,N-1)}, \widehat{\pi}_{(1,1)}, \ldots \widehat{\pi}_{(1,N)}, \ldots \widehat{\pi}_{(1,n-k+1)})$ of the generator matrix $\widetilde{A}$ is given by $\widehat{\pi}_{(1,i)} = \left(\frac{\lambda}{\mu}\right)^i \widehat{\pi}_{(0,0)}$, $i = 1, 2, \ldots n-k+1$, where

$$\widehat{\pi}_{(0,0)} = \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

Here again, from the condition $\widetilde{\pi}\widetilde{A}_0 e < \widetilde{\pi}\widetilde{A}_2 e$, it can be easily verified that the necessary and sufficient condition for the stability of the Markov chain $\widehat{X}(t)$ is $\overline{\lambda} < \overline{\mu}$.
Applying the same technique as in section 4.2.2, we can easily prove that the vector $\widehat{X} = \left(\widehat{X}(0), \widehat{X}(1), \widehat{X}(2), \ldots\right)$, with $\widehat{X}(i) = \left(1 - \frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widehat{\Pi}$, is the steady state probability vector for the matrix $\widehat{Q}$.

Performance Measures for the case $N = 1$

(1) Fraction of time the system is down,

$$P_{down} = \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\lambda^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(2) System reliability,

$$P_{rel} = 1 - P_{down} = 1 - \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\mu\left(\mu^{n-k+1} - \lambda^{n-k+1}\right)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(3) Average number of customers waiting in the queue,

$$N_q = \sum_{j_1=2}^{\infty} X_{(j_1,0,1)} + \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=1}^{n-k+1} x(j_1, 1, j_3)\right)$$

$$= \frac{\overline{\mu}}{(\overline{\mu} - \overline{\lambda})} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})} \left[\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^2 + \frac{\lambda\left(\mu^{n-k+1} - \lambda^{n-k+1}\right)}{\mu^{n-k+1}(\mu - \lambda)}\right]$$

(4) Average number of failed components,

$$N_{fail} = \sum_{j_3=1}^{n-k+1} J_3 \left(\sum_{j_1=0}^{\infty} X_{(j_1,1,j_3)} \right) = \frac{\lambda \mu^{n-k+2}}{(\mu - \lambda)\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(5) Expected number of external customers joining the system in unit time,

$$\theta_3 = \bar{\lambda} \sum_{j_1=0}^{\infty} x_{(j_1,0,0)} = \frac{\bar{\lambda} \mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(6) Expected number of external customers, on arrival, getting service directly

$$= \bar{\mu} x_{(0,0,0)}$$
$$= \frac{(\bar{\mu} - \bar{\lambda})\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(7) Fraction of time the server is busy with external customers,

$$P_{ex.busy} = \sum_{j_1=0}^{\infty} x_{(j_1,0,0)}$$
$$= \frac{\bar{\lambda} \mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(8) Probability that the server is idle,

$$P_{idle} = x_{(0,0,j_3)} = \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}} \frac{\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(9) Probability that the server is found busy,

$$P_{busy} = 1 - P_{idle} = 1 - \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}} \frac{\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(10) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} x_{(j_1,1,j_3)} \right) = \bar{\lambda} \frac{\mu\left(\mu^{n-k+1} - \lambda^{n-k+1}\right)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(11) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} x_{(j_1,0,0)} = \bar{\mu} \frac{\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(12) Expected number of external customers in the system when the server is busy with external customers

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 x_{(j_1,0,0)} = \bar{\lambda} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

5. Numerical illustrations

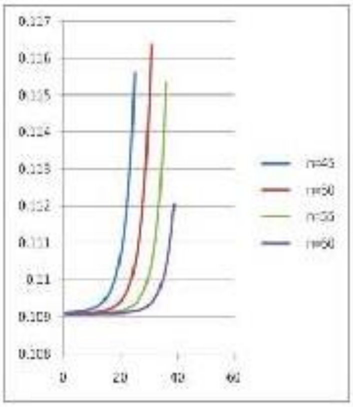
Here, we perform a numerical study on the effect of the N-policy on the system performance. Unless otherwise stated, the parameter values for the numerical study are the following: $\bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$.

5.1. Effect of the N-policy on the probability that server is busy with external customers.

While studying a k-out-of-n system, where the server provides service to external customers also, the main purpose of N-policy is to provide improved attention to external customers for optimizing the system revenue. According to the N-policy considered here, the moment the number of failed components of the main system reaches N, the external customer's service ('if there is any') is pre-empted to attend the failed components. Hence, an increase in the value of N will extend the time during which external customers can get service and so it is expected that the probability that the server is busy with external customers increases with an increase in the value of N. The column wise increase in Table 1 supports this intuition. The high service rate for the external customers, as compared to their arrival rate can be considered as the reason for the slow increase in the above probability. The row wise decrease in Table 1 points to the decrease in the probability that the server is busy with external customers with an increase in the total number of components in the system. We have the following reasoning for this behavior: With an increase in the total number of components n in the system, there can be more number of failed components in the system for a fixed N, which leads to an increase in the probability that the server is attending failed components, resulting in a decrease in the probability $P_{ex.busy}$. A closer scrutiny of Table 1 shows that, by increasing the policy level N with an increase in the number of components n, the same value for the fraction $P_{ex.busy}$ can be achieved as that when

TABLE 1. Dependence of the probability $P_{ex\,busy}$ on the N-policy level

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$
1	0.10910	0.10909	0.10909	0.10909
2	0.10910	0.10910	0.10909	0.10909
3	0.10912	0.10910	0.10909	0.10909
6	0.10914	0.10910	0.10909	0.10909
7	0.10915	0.10910	0.10909	0.10909
10	0.10922	0.10912	0.10910	0.10909
11	0.10925	0.10912	0.10910	0.10909
12	0.10929	0.10913	0.10910	0.10910
15	0.10952	0.10918	0.10911	0.10910
18	0.11002	0.10928	0.10913	0.10910
21	0.11118	0.10952	0.10918	0.10911
22	0.11185	0.10965	0.10921	0.10912
23	0.11275	0.10982	0.10925	0.10913
24	0.11397	0.11006	0.10929	0.10914
25	0.11562	0.11037	0.10935	0.10915
26		0.11078	0.10944	0.10917
27		0.11134	0.10955	0.10919
28		0.11209	0.10970	0.10922
29		0.11310	0.10989	0.10926
30		0.11448	0.11016	0.10932
31		0.11638	0.11051	0.10939



n has a lesser value. For example, when $n = 45$ and $N = 7$, $P_{ex.busy} = 0.10915$ and $P_{ex.busy} = 0.10909$, when $n = 60$ with the same N. Now with $n = 60$ and when N is increased to 25, we see that $P_{ex.busy} = 0.10915$. This suggests that, when n increases, the N-policy level can be adjusted in favor of the external customers, which was our objective while introducing the N policy. However, when N increases, it is probable that the server spends more time for failed components, once he starts attending them, which leads to a loss of the external customers who finds the server busy with internal customers. In Table 1, one can see that the probability $P_{ex.busy}$ has a lesser value when $n = 60$, $N = 30$ than in the case when $n = 45$, $N = 15$, which points to the loss of external customers. Another challenge here is that, while increasing the N-policy level, the system reliability is not affected significantly.

5.2. **Effect of the N-policy on the system reliability.**

In the previous section, we discussed how N-policy helps in longer duration of attention to external customers and the challenge there is the possibility of a decrease in the system reliability. Here we discuss how the N-policy level affects the system reliability P_{rel}. We study two cases with $\frac{\lambda}{\mu} < 1$ and $\frac{\lambda}{\mu} > 1$ respectively, results of which are given in Table 2(a) and (b) respectively. While studying the impact of the N-policy on the system reliability, a decrease in P_{rel} is expected with an increase in value of N. Hence, the purpose of the Tables 2(a) and (b) is to show the magnitude of this impact. Table 2(a) shows that when $\frac{\lambda}{\mu} < 1$, $n = 45$ and when N increased from 3 to 25, there is a decrease in reliability of magnitude equal to 0.02. As the total number of components n increases, the magnitude of decrease in reliability reduces. This is because, when n increases, k being fixed, $n - k + 1$ increases; as a result, once the server starts attending the failed components on accumulation of N of them, he spends more time for the failed components, which maintains a high system reliability even when N increases. In Table 1 we have seen that as n increases, the probability $P_{ex.busy}$ decreases and that increasing the N-policy level can remedy this to some extent; Table 2(a) shows that the reliability of the system is not much affected by increasing the N-policy level. However, the magnitude of drop in the system reliability increases with the increase in N-policy level. Table 2(b) studies the system reliability when the failure rate of the components λ is larger than their repair rate μ. As expected, there is a drop in the system reliability compared to the case $\lambda < \mu$. Other behaviour of the system reliability are similar to that in Table 2(a).

TABLE 2. **(a)**: Dependence of the system reliability on the N-policy level in the $\lambda < \mu$ case $\lambda = 4$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.999930799	0.999985933	0.999997139	0.999999404	0.999999881
3	0.999901652	0.999979973	0.999995947	0.999999166	0.999999821
5	0.999855518	0.999970615	0.999994040	0.999998808	0.999999762
9	0.999660194	0.999930918	0.999985933	0.999997139	0.999999404
13	0.999121249	0.999821544	0.999963701	0.999992609	0.999998510
17	0.997560024	0.999506116	0.999899626	0.999979556	0.999995828
21	0.992828071	0.998562694	0.999708474	0.999940693	0.999987960
25	0.977587163	0.995647013	0.999122441	0.999821782	0.999963760
26		0.994222760	0.998838782	0.999764323	0.999952078
29		0.986251056	0.997281969	0.999450147	0.999888241
31		0.974976659	0.995165646	0.999026358	0.999802291
34			0.984254420	0.996900022	0.999531090
35			0.978649259	0.995844364	0.999373376
38				0.989870846	0.998496175
39				0.986294508	0.979825020
40				0.981382251	0.972903130
41					0.996356070
45					0.987866700
46					0.983495116

TABLE 2. **(b)**: Dependence of the system reliability on the N-policy level in the $\lambda > \mu$ case $\lambda = 6$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.907874525	0.911180377	0.913196325	0.914452970	0.915247083
3	0.907009840	0.910661936	0.912876606	0.914252222	0.915119767
5	0.906079888	0.910108566	0.912536800	0.914039671	0.914985061
9	0.904014528	0.908894181	0.911796451	0.913578153	0.914693415
11	0.902873158	0.908231616	0.911395609	0.913329482	0.914536774
13	0.901655436	0.907531500	0.910974264	0.913069129	0.914373279
17	0.898979187	0.906016290	0.910070777	0.912513614	0.914025128
21	0.895960152	0.904344857	0.909087002	0.911913455	0.913650930
25	0.892570674	0.902514517	0.908024848	0.911270797	0.913252294
26		0.902032018	0.907747209	0.911103785	0.913149118
29		0.900522947	0.906886399	0.910588324	0.912831187
31		0.89946568	0.906289339	0.910232842	0.912612915
34			0.905359924	0.909682870	0.912276387
35			0.905041218	0.909495234	0.912169460
38				0.908919990	0.911812007
39				0.908724248	0.911693335
40				0.908526540	0.911573648
41				0.908326924	0.911453009
45					0.910961330
46					0.910836279

5.3. **Cost analysis.**

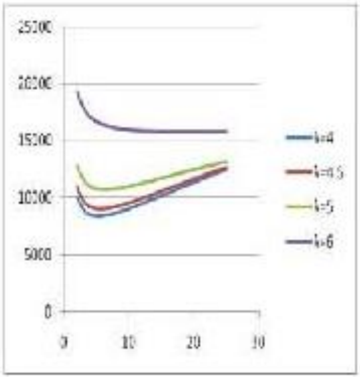
In sections 1.5.1 and 1.5.2, we have seen that by increasing N, we can provide uninterrupted service over a long duration to more external customers and without compromising the system reliability significantly. However, the magnitude of decrease in the system reliability increases with N. Hence, it is worth finding whether there exists an optimal value for the N-policy level. For this, we construct the following cost function. Let C_1 be the cost per unit time incurred if the system is down; C_2, the holding cost per unit time per external customer in the queue; C_3 is the cost incurred towards set up (instantaneous) of the server to serve main customers; C_4 be the cost due to loss of an external customer, C_5, be the holding cost per unit time of one failed component and C_6 be the cost per unit idle time.

Expected Cost per unit time $= C_1 \cdot P_{down} + C_2 \cdot N_q + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \left(\frac{C_3}{E_B}\right) + C_6 \cdot P_{idle}.$

Table 3 studies the variation of cost function as N varies. We study the cost function for different failure rates of the components. In all the 4 cases studied, for the various costs assumed, we get a concave nature for the cost curve, which gives an optimal value for N. Table 3 shows that when $\lambda < \mu$, the optimal values for N are 5,6 and 6 when λ equal to 4, 4.5 and 5 respectively; whereas when $\lambda = 6 > 5.5 = \mu$, we get a much higher optimal value 18 for N. This is as expected, since when λ is greater than μ, there will be a heavier traffic of failed components so that the server has to spend more time attending the failed components. Hence, the policy level N needs to be increased to a much higher value than in the $\lambda < \mu$ situation, for the system to earn maximum profit. Also note that the optimal value of the cost function is much higher in the $\lambda > \mu$ case, when compared to the opposite situation.

TABLE 3. Variation in the cost function $n = 50$, $k = 20$, $C_1 = 2000$, $C_2 = 1000$, $C_3 = 1600$, $C_4 = 1000$, $C_5 = 500$, $C_6 = 100$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 6$
2	10139.47	10923.82	12783.28	19330.75
3	8910.199	9663.817	11496.69	17827.49
4	8489.626	9199.57	10981.16	17095.61
5	**8370.844**	9038.382	10764.71	16671.58
6	8396.2	**9024.268**	**10694.5**	16401.28
7	8500.631	9092.232	10706.09	16218.57
8	8652.372	9210.307	10767.47	16090.46
12	9474.447	9919.942	11245.17	15847.97
14	9939.594	10337.5	11542.35	15805.31
16	10416.93	10769	11849.9	15786.79
17	10657.6	10986.45	12003.95	15783.35
18	10898.62	11203.53	12156.71	**15782.6**
19	11139.36	11419.23	12307.2	15783.87
20	11379.19	11632.57	12454.51	15786.61
23	12085.88	12248.03	12868.57	15799.16
24	12313.97	12441.01	12994.22	15803.64
25	12536.01	12625.49	13111.97	15807.79

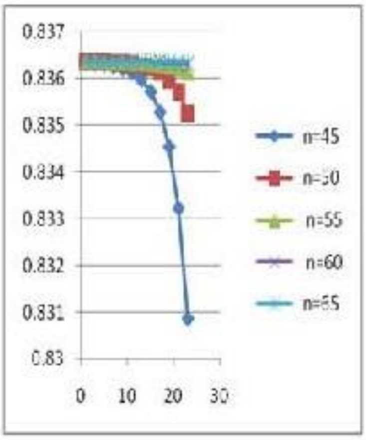


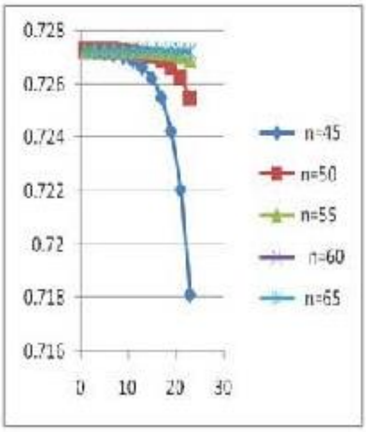
5.4. Comparison with a k-out-n system where no external customers are serviced.

Here we compare the model discussed above with another model where no external customers are allowed but N-policy is maintained. Notice that because of the assumption of the preemption of service of an external customer on accumulation of N failed components, the two systems will have the same reliability. The nature of the steady state distribution obtained in Theorem 4.2 further substantiates this claim. Hence, it can be concluded that the external customers when allowed as in this study, utilizes the server idle time without affecting the performance of the k-out-of-n system. In Table 4, we present the results of the numerical study conducted for comparing the increase in the server busy probability, when external customers are allowed. In that Table, case 1 refers to the model discussed above and case 2 stands for k-out-of-n system where no external customers are allowed. Table 4 shows that when external customers are allowed, there is an increase, of magnitude 0.11, in the server busy probability.

Table 4. Variation in the server busy probability

Case 1 $\lambda = 4$						Case 2 $\lambda = 4$					
N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$	N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.83633	0.83635	0.83636	0.83636	0.83636	1	0.72722	0.72726	0.72727	0.72727	0.72727
3	0.83632	0.83635	0.83636	0.83636	0.83636	3	0.7272	0.72726	0.72727	0.72727	0.72727
5	0.8363	0.83635	0.83636	0.83636	0.83636	5	0.72717	0.72725	0.72727	0.72727	0.72727
7	0.83626	0.83634	0.83635	0.83636	0.83636	7	0.72711	0.72724	0.72727	0.72727	0.72727
9	0.83621	0.83633	0.83635	0.83635	0.83635	9	0.72703	0.72722	0.72726	0.72727	0.72727
11	0.83612	0.83631	0.83634	0.83635	0.83635	11	0.72688	0.72719	0.72726	0.72727	0.72727
13	0.83597	0.83627	0.83634	0.83635	0.83635	13	0.72663	0.72714	0.72725	0.72727	0.72727
15	0.83572	0.83622	0.83632	0.83635	0.83635	15	0.72622	0.72706	0.72723	0.72726	0.72727
17	0.83528	0.83613	0.83631	0.83634	0.83635	17	0.7255	0.72691	0.7272	0.72726	0.72727
19	0.83453	0.83598	0.83627	0.83633	0.83634	19	0.72425	0.72666	0.72715	0.72725	0.72727
21	0.83322	0.83572	0.83622	0.83632	0.83634	21	0.72206	0.72623	0.72706	0.72723	0.72726
23	0.83086	0.83526	0.83613	0.8363	0.83634	23	0.71814	0.72546	0.72691	0.7272	0.72726





6. Conclusion

Rendering service to external customers could be an effective idea for utilizing the server idle time and thereby earning more revenue to the system. However, in the case of a system, where a minimum number of working components is necessary for its operation, the external service should be carefully managed so that it does not affect the system reliability considerably. In the present paper, we have adopted N-Policy for managing the external service. Precisely, we assume that the server starts attending failed system components only on the accumulation of N of them. During this idle period, he/she renders service to external customers (if there is any). This scenario has been modeled using a continuous time Markov chain. Further, we make the reasonable assumption that the external service is pre-empted on accumulation of N failed components and also that the external arrivals which finds the server busy with failed components of the main system, are blocked from entering the system. These assumptions lead us to a product form solution the system in steady state. For this purpose, we employed a novel matrix decomposition approach. Though we have an explicit expression for the system reliability, due to the complex involvement of the parameters in the same, we studied the effect of the N-policy on the system reliability, numerically. This study reveals that by introducing the N-policy, we can maximize the system revenue, by rendering service to external customers, maintaining high system reliability. In future, we plan to study the effect of pre-emption on the system reliability as well as on the product form nature of the system steady state under the N-policy. Another extension is to allow external customers to join the system even when the server is busy with internal customers.

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