Reliability Of Heterogeneous *(k,r)-out-of-(n,m)* System

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Abstract

We consider a coherent binary system consisting of m components of a-type and n components of b-type. The a-type and b-type components have i.i.d lifetimes with cdf $F_a(t)$ and $F_b(t)$, respectively. The a-type and b-type components are stochastically independent. Our system is UP if at least k a-type components are up and at least r components of b-type are up.We present a simple formula for this system lifetime cumulative distribution function.

Keywords: heterogeneous system; k-out-of-n system; survival signature.

We consider the following generalization of a"standard" *k-out-of-n* system. Our system has two types of components: n components of a-type and m components of b-type. The components of each type have iid lifetimes denoted as $F_a(t)$ and $F_b(t)$, respectively. The a-type and b-type components are stochastically independent. The standard $k-out-of-n$ system is operational (i.e. in state UP) iff at least k of its components are operational, i.e. are up. Our system is defined to be operational if and only if at least k components of a-type **and** at least r of b-type components are up. Formally, our system can be viewed as a series connection of two $k-out-of-n$-type subsystems.

Suppose, without loss of generality, we number the a-type components by numbers $1,2,\dots,m$ and components of b-type by numbers $m+1,\dots,m+n$. System state is therefore a binary vector $x=(x_1,\dots,x_m,\dots,x_{m+n},$ where $x_i=1$ or $x_i 0$ if component i is *up* or *down*, respectively.

System state is a binary function $\varphi(x)$ which takes values 1 or 0 if the system is UP or *DOWN*, respectively.

If $\varphi(x_1)=1$, then x_1 is called an *UP-vector* or an *UP-set.*If the state vector x is not an *UP-vector*, we call it a *DOWN-vector* or *DOWN-set*

According to the above description of our system, an *UP-set* must have at least k ones on the first m positions of vector x and at least r ones on the last n positions. For example, for $m=5$ and $n=6$, $k=4, r=4$, the vector $x=(0,1,1,1,1;1,1,1,0,1,0)$ is an *UP-vector.*

Denote by $NU(v,w)$ the number of *UP-vectors* which have exactly v ones on the first m positions and w ones on the last n positions. Obviously, $v\geq k, w\geq r$.

The following Lemma follows from the above description:

Lemma

$$NU(v,w) = C_m^v \cdot C_n^w = \frac{m!n!}{v!(m-v)!w!(n-w)!}. \quad (1)$$

The proof is obvious: there are C_m^v ways to locate v ones on the first n positions of the state vector x and C_n^w ways to locate w ones on the last n positions of this vector.#

For sake of brevity, an *UP-vector* with v and w components of a-type and b-type, will be called an (v,w)-*UP-vector*.

Now everything is ready to write the formula for system *UP* probability. Let us take an arbitrary time instant t and denote by $q_a = F_a(t)$ the probability that an a-type component is *down* at time instant t.Smilarly, $q_b = F_b(t)$ is the *down* probability that component of b-type is *down* at time instant t. Denote $p_a = 1 - q_a$ and $p_b = 1 - q_b$.

By independence of all a-type and b-type components and by independence of a-type components from b-type components,the probability that a state vector x is an (v,w)-*UP-vector* equals

$$P(U(v,w)) = p_a^v q_a^{n-v} p_b^w q_b^{m-w}. \quad (2)$$

Now we arrive at

Theorem 1

$$P(\,system\ is\ UP\ at\ time\ t\,) = \sum_{v\geq k, w\geq r} NU(v,w) \cdot P(U(v,w). \quad (3)$$

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If the system is *UP* at time instant t its lifetime τ_s is greater or equal t, Therefore,

$$P(\tau_s \geq t) = \sum_{v\geq k, w\geq r} C_m^v \cdot C_n^w \cdot [F_a(t)]^v [1 - F_a(t)]^{n-v} [F_b(t)]^w [1 - F_b(t)]^{m-w}.\# \quad (4)$$

Remark 1. The central role for deriving formula (4) is played by the expression for $UN(v,w)$, see (1). Let us note that$NU(v,w)$ depend only on system structure function and they are, therefore, system *structural invariants*. It is quite obvious how to generalize the above derivation for the case when the system has more than two, say $K > 2$ types of components. By definition, this system is *UP* iff it has at least v_i *up* components of each type, $i = 1,2, \ldots K$.#

Remark 2. A system consisting of several $k - out - of - n$ subsystems is, to the best of our knowledge, the only lucky case where we can find in a simple form (like in (1)) an explicit formula for the number of system *UP-state* vectors having exactly v_i components of i-th type in *up* state, $i = 1,2, \ldots, K$.

Samaniego and Navarro suggested to call the collection of all $NU(v,w)$ values *survival signature*, see [1]. If $ND(u,w)$ is the number of system *DOWN* states with exactly v a-components and w b-components *down*, then it would be natural to call the collection of all $ND(v,w)$ *failure signature. #*

Remark 3. There is a simple relationship between the values of $ND(v,w)$ and $NU(v,w)$:

$$ND(v,w) + NU(n-v, m-w) = \frac{m!n!}{v!(m-v)!w!(n-w)!}. \quad (5)$$

Indeed, let us chose v components of a-type and w components of b-type and let them be *down*. Then we will obtain either a *DOWN* state or an *UP* state vector for the system. But having v, w components *down*, means having the remaining components *up*, which proves (5). From practical point of view, (5) shows that the knowledge of the survival signature provides us the knowledge its dual failure signature.#

Remark 4. Let us return to coherent binary systems consisting of one type iid components. Crucial role in its reliability evaluations play so-called *signature* $f = (f_1, f_2, \dots, f_n)$, see [2]. Let $F(j) = \sum_{k=1}^{j} f_j$, $j = 1,2,\dots,n$ be the so-called cumulative signature or system D-spectrum [3,4]. $F(j)$ is the probability that the system is $DOWN$ if j of its components are *down*, i.e. the probability that system failure appeared after x components have failed, $x = 1,2,\dots,j$. If we know the D-spectrum of the system, we can find the number $ND(r)$-the number of system failure or *DOWN* states with exactly r components *down* and $n - r$ components *up*, by using the following simple formula, see [3,4]:

$$ND(r) = F(r)n!/(r!\,(n-r)!). \quad (6)$$

For systems of real size, having $n > 8 - 10$ components, there are efficient Monte Carlo algorithms for fast and accurate estimation of $F(j)$, see [4]

In our opinion, in case of coherent systems having two types of independent and identical components, reliability calculations must be based on the knowledge of a two-dimensional analogue of the cumulative D-spectrum. It should be a function $G(k,r)$ expressing the probability that a random permutation of n and m components of both types contains a failure set with k and r *down* components of a- and b-type, respectively.#

References

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